Lecture Notes on First Order PDE (NTHU EMI PDE COURSE)

Dong-Ho Tsai

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First order PDE.

Assume $u(x,y) \in C^1(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$ is an open set. If $u(x,y)$ satis. an eq. of the form

$$F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0, \quad \forall \ (x,y) \in \Omega, \tag{1}$$

(in abbreviation $F(x, y, u, u_x, u_y) = 0, \ \forall \ (x,y) \in \Omega$), we say it satis. a **first order PDE** (for a **2-varibale** func.) on Ω. Here $F = F(x, y, z, p, q) : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a given **conti. func.**. In case $u(x_1, x_2, \ ..., x_n)$ is a C^1 func. of n variables def. on some open set $\Omega \subseteq \mathbb{R}^n$, then (1) becomes

$$F(x_1, \ ... \ , x_n, u_{x_1}(x_1, \ ... \ , x_n), \cdots, u_{x_n}(x_1, \ ... \ , x_n)) = 0.$$

First order PDE.

Conversely, if we are given an eq. of the form

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in \Omega \subseteq \mathbb{R}^2 \tag{2}$$

and you want to **"solve"** the above first order PDE, that means you want to find a C^1 **func.** $u(x, y)$ **def. at least on some open set** $\tilde{\Omega} \subseteq \Omega$ such that (2) is satis. for all $(x, y) \in \tilde{\Omega}$. In general, the sol. $u(x, y)$ you find may not be def. on the whole Ω.
In case the dom. $\Omega \subset \mathbb{R}^2$ in (2) is not specified, which is often the case, then we usually pick a "**natural dom.**" for it and we want to find a C^1 sol. $u(x, y)$ def. on some **open set** $\tilde{\Omega} \subseteq \Omega$. For example, if

$$F(x, y, z, p, q) = \log\left(\frac{y}{x}\right) + z^2 + pq,$$

then the natural dom. for (x, y) is $\left\{(x, y) \in \mathbb{R}^2 : x \neq 0 \text{ and } \frac{y}{x} > 0\right\}$.

First order PDE.

In the above example of $F(x, y, z, p, q)$, the first order PDE $F(x, y, u, u_x, u_y) = 0$ is given by

$$u_x(x, y)\, u_y(x, y) + u^2(x, y) = -\log\left(\frac{y}{x}\right),$$

where $(x, y) \in \left\{(x, y) \in \mathbb{R}^2 : x \neq 0 \text{ and } \frac{y}{x} > 0\right\}$.

Definition

If the func. $F(x, y, z, p, q)$ is **linear** in the three variables $z,\ p,\ q$ (but may not be linear in $x,\ y$), we say the PDE is **linear**, otherwise it is **nonlinear**.

First order linear PDE.

Therefore, a general first order linear PDE for a func. $u(x,y)$ has the form:

$$a(x,y)\,u_x(x,y)+b(x,y)\,u_y(x,y)+c(x,y)\,u(x,y)=f(x,y), \qquad (3)$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$ are given conti. func. on a common dom. $\Omega\subseteq\mathbb{R}^2$.

Solving a PDE is, in general, very difficult, even for a first order linear PDE. In this elementary course, in most cases, we will focus on simple first and second order **linear PDE (with const. coeff. most of the time)**. For simplicity of discussions, we will focus on a func. $u(x,y)$ with two variables, but most theory of sol. method can be easily generalized to a func. $u(x_1, \ldots, x_n)$ with n variables.

First order linear PDE with const. coeff..

We first look at the simple case when $a(x, y)$, $b(x, y)$, $c(x, y)$ in (3) are **const.** with $a \neq 0$ and $b \neq 0$. Now we have the PDE

$$au_x(x, y) + bu_y(x, y) + cu(x, y) = f(x, y), \quad (x, y) \in \Omega \subseteq \mathbb{R}^2 \quad (4)$$

and we want to find a C^1 sol. (or to find the **general sol.** if possible) $u(x, y)$ def. at least on some open subset $\tilde{\Omega}$ of Ω.

REMARK: If $a \neq 0$, $b = 0$ or $a = 0$, $b \neq 0$, then (4) is just an ODE with a **parameter**. Hence we assume $a \neq 0$ and $b \neq 0$.

First order linear PDE with const. coeff..

Note that if $u(x,y)$ and $v(x,y)$ are two sol. of (4) def. on a common dom. $\tilde{\Omega} \subseteq \Omega$, their difference $w(x,y) = u(x,y) - v(x,y)$ will satis. the **homogeneous** eq. on $\tilde{\Omega}$:

$$aw_x(x,y) + bw_y(x,y) + cw(x,y) = 0. \tag{5}$$

By this, the general sol. $(u(x,y))$ of (4) can be decomposed as a **particular sol.** $\mathbf{(}v(x,y)\mathbf{)}$ of the nonhomogeneous eq. (4) plus the **general sol.** $\mathbf{(}w(x,y)\mathbf{)}$ of the **homogeneous eq**. (5).

First order linear PDE with const. coeff..

Since eq. (4) is **linear with const. coeff**., the idea of solving it is to use a **linear change of variables** (ch. of var.) to convert it into an ODE and solve it.

Let (w, z) be the new variables given by

$$\begin{pmatrix} w \\ z \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}, \quad J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{6}$$

where $A,\ B,\ C,\ D$ are const. **to be determined later on** with $\det J = AD - BC \neq 0$ (condition for ch. of var.).
Under the ch. of var., the func. $u(x, y)$ becomes a new func. $U(w, z)$. Their relation is given by

$$u(x, y) = U(Ax + By, Cx + Dy).$$

First order linear PDE with const. coeff..

By the chain rule (since we seek for a C^1 sol. $u(x,y)$, it is differentiable and the chain rule holds) we have

$$\begin{cases} u_x(x,y) = U_w(w,z)\,A + U_z(w,z)\,C, \\ u_y(x,y) = U_w(w,z)\,B + U_z(w,z)\,D, \end{cases} \tag{7}$$

which, in terms of column vector notation, is the same as the following **gradient vector relation:**

$$\underbrace{\nabla u = J^T \nabla U}, \quad J^T \text{ is the transpose of } J \tag{8}$$

or the **first order differential operator relation**:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial z} \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} w \\ z \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix}. \tag{9}$$

First order linear PDE with const. coeff..

For later use, we also need to know the **second-order operator relation**:

$$\begin{cases} \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(A\frac{\partial}{\partial w} + C\frac{\partial}{\partial z}\right)(\cdots) = A^2\frac{\partial^2}{\partial w^2} + 2AC\frac{\partial^2}{\partial w \partial z} + C^2\frac{\partial^2}{\partial z^2} \\ \frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \left(A\frac{\partial}{\partial w} + C\frac{\partial}{\partial z}\right)(\cdots) = \begin{cases} AB\frac{\partial^2}{\partial w^2} \\ +(AD+BC)\frac{\partial^2}{\partial w \partial z} \\ +CD\frac{\partial^2}{\partial z^2} \end{cases} \\ \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(B\frac{\partial}{\partial w} + D\frac{\partial}{\partial z}\right)(\cdots) = B^2\frac{\partial^2}{\partial w^2} + 2BD\frac{\partial^2}{\partial w \partial z} + D^2\frac{\partial^2}{\partial z^2}. \end{cases}$$

First order linear PDE with const. coeff..

In terms of matrix notation we have

$$\begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} \frac{\partial^2}{\partial w^2} & \frac{\partial^2}{\partial w \partial z} \\ \frac{\partial^2}{\partial w \partial z} & \frac{\partial^2}{\partial z^2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$= J^T \begin{pmatrix} \frac{\partial^2}{\partial w^2} & \frac{\partial^2}{\partial w \partial z} \\ \frac{\partial^2}{\partial w \partial z} & \frac{\partial^2}{\partial z^2} \end{pmatrix} J$$

or equivalently, the **Hessain matrix relation**:

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial^2 U}{\partial w^2} & \frac{\partial^2 U}{\partial w \partial z} \\ \frac{\partial^2 U}{\partial w \partial z} & \frac{\partial^2 U}{\partial z^2} \end{pmatrix} J, \quad J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and we will denote the above identity as

$$\underbrace{\nabla^2 u = J^T \left(\nabla^2 U \right) J,} \tag{10}$$

where $\nabla^2 u$ is the **Hessian matrix** of u, etc.

First order linear PDE with const. coeff..

Back to the PDE, by (7), we conclude

$$\underbrace{au_x(x,y)+bu_y(x,y)}+cu(x,y)$$
$$=\underbrace{(\underline{aA+bB})\,U_w(w,z)+(\underline{aC+bD})\,U_z(w,z)}+cU(w,z),$$

which indicates that the new eq. for $U(w,z)$ will become an **ODE** if and only if one of the following is satisfied (**but not both !!!, otherwise we will have** $\det J=0$):

$$\text{either}\quad \langle (A,B),(a,b)\rangle=0 \quad \text{or} \quad \langle (C,D),(a,b)\rangle=0. \tag{11}$$

By (11), we can choose (A,B) so that $(A,B)\cdot(a,b)=0$ and choose (C,D) so that $(C,D)\cdot(a,b)\neq 0$ (or the other way around). In doing so, we automatically have $AD-BC\neq 0$. Hence (6) is indeed a ch. of var.

First order linear PDE with const. coeff..

We can choose $A = b,\ B = -a,\ C = 0,\ D = 1$ to make $(A, B) \cdot (a, b) = 0,\ (C, D) \cdot (a, b) = b \neq 0$. Now the ch. of. var. is

$$\begin{cases} w = bx - ay, \\ z = y, \end{cases} \tag{12}$$

and we obtain the **linear ODE** in z for $U(w, z)$ (view w as a parameter):

$$\underbrace{bU_z(w, z) + cU(w, z)} = f\left(\frac{w + az}{b}, z\right), \quad b \neq 0, \quad (w, z) \in \Sigma, \tag{13}$$

where $\Sigma \subseteq wz$-plane is the image of Ω in xy-plane under the linear map in (6) and the relation between u and U is:

$$u(x, y) = U(bx - ay, y), \quad U(w, z) = u\left(\frac{w + az}{b}, z\right).$$

First order linear PDE with const. coeff..

By ODE theory we can solve (13) to get its general sol. def. on Σ :

$$\begin{aligned} U(w,z) &= e^{-\frac{c}{b}z}\left(\int\left[e^{\frac{c}{b}z}\frac{1}{b}f\left(\frac{w+az}{b},z\right)\right]dz + C(w)\right) \\ &= \underbrace{\frac{e^{-\frac{c}{b}z}}{b}\cdot\int e^{\frac{c}{b}z}f\left(\frac{w+az}{b},z\right)dz}+e^{-\frac{c}{b}z}C(w) \\ &:= \underbrace{I(w,z)}+e^{-\frac{c}{b}z}C(w), \quad w = bx-ay, \quad z = y, \end{aligned} \tag{14}$$

where $C(w)$ is an arbitrary C^1 func. of w ($C(w)$ is "**integ. const. func.**" for the integral $\int dz$).

First order linear PDE with const. coeff..

We note that $U(w,z) = I(w,z) + e^{-\frac{c}{b}z}C(w)$ is the sum of a particular sol. $I(w,z)$ of $bU_z + cU = f((w+az)/b, z)$ and the general sol. $e^{-\frac{c}{b}z}C(w)$ of $bU_z + cU = 0$. **Moreover, we see that** $I(w,z)$ **is def. on** Σ. **However, the dom. of** $e^{-\frac{c}{b}z}C(w)$ **depends on how you choose** $C(w)$. Back to $u(x,y)$, we will have

$$u(x,y) = \underbrace{I(bx-ay,y) + e^{-\frac{c}{b}y}C(bx-ay)}, \tag{15}$$

where $I(bx-ay,y)$ is a **particular sol.** of $au_x + bu_y + cu = f$ and is **def. on** Ω (since $I(w,z)$ is defined on Σ) and $e^{-\frac{c}{b}y}C(bx-ay)$ is the **general sol.** of $au_x + bu_y + cu = 0$. Its dom. **depends on how you choose** $C(w)$.

Definition

The sol. in (15) describes **all possible sol.** of (4) and is called the **general sol.** of the eq. (4).

First order linear PDE with const. coeff..

Definition

Any line L in xy-plane of the form

$$bx - ay = \lambda, \quad \lambda \text{ is a const.} \tag{16}$$

is called a **characteristic line** (**char. line**) of the PDE (4). A char. line $bx - ay = \lambda$ in xy-plane corresponds to a **coordinate line** $w = \lambda$ in the wz-plane (recall that $w = bx - ay$). **We use char. lines to do ch. of var.**

The **char. lines** play an important role in solving the **first order** PDE (4). They are the **coordinate lines** for the new variable (w, z). **Without using the char. lines, we cannot convert the PDE (4) for** $u(x, y)$ **into an ODE for** $U(w, z)$. **This is also true for first order linear PDE with variable coeff..** Draw a picture on blackboard for this ....

First order linear PDE with const. coeff..

The most important terms of (4) are the leading first order derivative terms $au_x(x,y) + bu_y(x,y)$. One can view it as

$$au_x(x,y) + bu_y(x,y) = \langle V(x,y), \nabla u(x,y)\rangle, \tag{17}$$

where $V(x,y) = (a,b)$ is a **vector field** on $\mathbb{R}^2$ (xy-plane). By definition, the **integral curves** (also known as **sol. curves**) $\mathbf{x}(t) = (x(t), y(t))$ of the vector field $V(x,y)$ on $\mathbb{R}^2$ satis. the ODE

$$\mathbf{x}'(t) = V(\mathbf{x}(t)) = (a,b) \quad \left(\text{same as } \frac{dx}{dt} = a, \quad \frac{dy}{dt} = b\right) \tag{18}$$

and we get $x(t) = at + x_0$, $y(t) = bt + y_0$, $t \in (-\infty, \infty)$. **Integral curves (with parametric eq.) of $V(x,y)$ are precisely the char. lines (with algebraic eq.)**. We will need this observation later on for PDE with **variable** coeff..

Some examples.

Example

If $f(x,y) \equiv 0$ in (4), then the ODE for $U(w,z)$ is $bU_z(w,z) + cU(w,z) = 0$ and its general sol. is given by

$$U(w,z) = e^{-\frac{c}{b}z} C(w),$$

where $C(w)$ is an arbitrary C^1 func. of w. Back to $u(x,y)$, the general sol. of $au_x + bu_y + cu = 0$ is

$$u(x,y) = e^{-\frac{c}{b}y} C(bx - ay), \quad b \neq 0, \tag{19}$$

which can also be expressed as (multiply $e^{-\frac{c}{b}y} C(bx - ay)$ by $e^{\lambda(bx-ay)}$, where $\lambda = -c/ab$).

$$u(x,y) = e^{-\frac{c}{a}x} C(bx - ay), \quad a \neq 0. \tag{20}$$

In fact, you can also get the sol. (20) by the ch. of var. (see (12)): $w = bx - ay,\ z = x$.

Some examples.

Example

(**Continued.**) Moreover, if, we also have $c = 0$ (we still assume $f(x, y) \equiv 0$ in (4)), the general sol. is

$$u(x, y) = C(bx - ay). \tag{21}$$

If we choose $C(w) = w^2,\ w \in (-\infty, \infty)$, we get the sol. $u(x, y) = (bx - ay)^2$ and it is def. on $\mathbb{R}^2$; if $C(w) = \sqrt{w},\ w \in (0, \infty)$, then $u(x, y) = \sqrt{bx - ay}$ is def. only on the open half-plane $bx - ay > 0$. If $C(w) = 1/w,\ w \neq 0 \in \mathbb{R}$, then $u(x, y) = 1/(bx - ay)$ is def. on the $\mathbb{R}^2 \setminus \{bx - ay = 0\}$.

Some examples.

Example

Find the general sol. of the PDE

$$3u_x - 2u_y + u = x, \quad u = u(x, y). \tag{22}$$

Solution: To make the ch. of var. and the ODE look better, we rewrite the eq. as $-3u_x + 2u_y - u = -x$. According to the method, we introduce the ch. of var.

$$w = bx - ay = 2x + 3y, \qquad z = y, \tag{23}$$

and the func. $u(x, y)$ becomes $U(w, z)$, where by (13) we have

$$\begin{aligned} &-3u_x(x, y) + 2u_y(x, y) - u(x, y) \\ &= 2U_z(w, z) - U(w, z) = -x = -\frac{w - 3z}{2}, \end{aligned}$$

and the ODE (in the variable z) for $U(w, z)$ is

$$U_z(w, z) - \frac{1}{2}U(w, z) = -\frac{1}{4}(w - 3z).$$

Some examples.

The above ODE has the sol.

$$U(w,z) = e^{\frac{z}{2}}\left(\int e^{-\frac{z}{2}}\left(-\frac{1}{4}(w-3z)\right)dz + C(w)\right)$$
$$= e^{\frac{z}{2}}\left(-\frac{w}{4}\int e^{-\frac{z}{2}}dz + \frac{3}{4}\int ze^{-\frac{z}{2}}dz + C(w)\right),$$

where, by $\int ze^{-\frac{z}{2}}dz = -2ze^{-\frac{z}{2}} - 4e^{-\frac{z}{2}}$, we get

$$U(w,z) = \frac{w}{2} - \frac{3}{2}z - 3 + e^{\frac{z}{2}}C(w).$$

As a result, the general sol. of the original eq. is $(w = 2x+3y,\ z = y)$

$$u(x,y) = x - 3 + e^{\frac{y}{2}}C(2x+3y),$$

where $C(\cdot)$ is an arbitrary C^1 func.. We see that $x-3$ is a particular sol. of (22)) and $e^{\frac{y}{2}}C(2x+3y)$ is the general sol. of $3u_x - 2u_y + u = 0$. □

Some examples.

Example

Find the general sol. of the PDE

$$u_x + u_y + u = e^{x+2y}, \quad u = u(x, y). \tag{24}$$

Solution: We do the ch. of var. $w = bx - ay = x - y$, $z = y$ and get the **linear ODE** for $U(w, z)$:

$$U_z(w, z) + U(w, z) = e^{x+2y} = e^{(w+z)+2z} = e^{w+3z},$$

which we can solve it to get

$$U(w, z) = e^{-z}\left(\int e^z \cdot e^{w+3z} dz + C(w)\right) = \frac{1}{4}e^{w+3z} + e^{-z}C(w).$$

Therefore, the general sol. of (24) is given by

$$u(x, t) = U(x - y, y) = \frac{1}{4}e^{x+2y} + e^{-y}C(x - y).$$

We note that the **particular sol.** $e^{x+2y}/4$ is def. on $\mathbb{R}^2$ and the dom. of $e^{-y}C(x - y)$ depends on your choice of $C(w)$.

ODE along a char. line.

Note that if $c = 0$ and $f(x, y) \equiv 0$ in (4), then the eq. is $au_x(x, y) + bu_y(x, y) = 0$, where $a \neq 0$ and $b \neq 0$ and its general sol. is given by

$$u(x, y) = C(bx - ay), \quad C(w) : I \subseteq \mathbb{R} \to \mathbb{R}$$

where $C(w)$ is an arbitrary C^1 func. def. on $I \subseteq \mathbb{R}$. In this situation, $u(x, y)$ is **const. along each char. line** $bx - ay = \lambda$.

ODE along a char. line.

On the other hand, if $c \neq 0$ or $f(x,y) \not\equiv 0$ is not a zero func., then $u(x,y)$ is in general **no longer a const. func. along each char. line**. However, if we know the value of $u(x,y)$ at **a point** $p_0 = (x_0, y_0) \in L$, where L is the **char. line passing through** p_0 with eq. $bx - ay = bx_0 - ay_0$ (denote it as λ_0), then the func. $u(x,y)$ will satis. an ODE on L (with respect to the **parameter** of the line L, where you can use either x, or y, or t as parameter) and by solving the ODE we can know all values of $u(x,y)$ on L.

ODE along a char. line.

More precisely, we can parametrize L as (can also use other parameters like x or t)

$$L = \{(x, y) : bx - ay = \lambda_0\} = \left\{(x, y) : (x, y) = \left(\frac{\lambda_0 + ay}{b}, y\right), \quad y \in (-\infty, \infty)\right\}$$

and let

$$P(y) = u\left(\frac{\lambda_0 + ay}{b}, y\right), \quad P(y_0) = u(x_0, y_0) \text{ is known.}$$

From the PDE $au_x + bu_y + cu = f$ (for simplicity we assume both $u(x, y)$ and $f(x, y)$ are def. on $\mathbb{R}^2$), we can derive

$$P'(y) = \frac{a}{b}u_x\left(\frac{\lambda_0 + ay}{b}, y\right) + u_y\left(\frac{\lambda_0 + ay}{b}, y\right) = \frac{1}{b}f\left(\frac{\lambda_0 + ay}{b}, y\right) - \frac{c}{b}P(y), \quad y \in (-\infty, \infty).$$

ODE along a char. line.

Therefore, $P(y)$ satis. the following **ODE along** L (which is similar to (13)):

$$bP'(y) + cP(y) = f\left(\frac{\lambda_0 + ay}{b}, y\right), \quad y \in (-\infty, \infty), \tag{25}$$

where $P(y_0) = u(x_0, y_0)$ is known. By ODE theory, **one can know** $P(y)$ **for all** $y \in (-\infty, \infty)$ (since (25) is a linear eq., there is a sol. formula for it, and we also know the initial condition $P(y_0)$). We can summarize the following: **if** $u(x, y)$ **satis. the PDE (4) on** $\mathbb{R}^2$, **then it satis. the ODE (25) on any char. line** L (or: if $u(x, y)$ satis. the PDE (4) on dom. $\Omega \subseteq \mathbb{R}^2$, then it satis. the ODE (25) on any char. line L lying inside Ω).

The physical meaning of the eq. $cu_x + u_t = 0$.

Let $c > 0$ be a const.. The eq.

$$cu_x + u_t = 0, \quad u = u(x, t),$$

is called a **simple transport eq.**. Here $x \in \mathbb{R}$ represents space (one-dimensional) coordinate and $t \in \mathbb{R}$ represents time. Assume we have a fluid (like sea water or juice ..., etc.), moving to the right with **const. speed** c, along a horizontal **thin pipe** which we view it as a one-dimensional space with coordinate $x \in \mathbb{R}$. Let $u(x, t)$ be the **concentration** of some substance suspended in the water. The amount of the substance in the interval $[0, x_0]$ (assume $x_0 > 0$) at time t_0 is given by (here we view the space dimension as 1)

$$M = \int_0^{x_0} u(x, t_0)\, dx.$$

The physical meaning of the eq. $cu_x + u_t = 0$.

A small time $h > 0$ later, the same substance (previously lies in the interval $[0, x_0]$) has moved to the right by distance $c \cdot h$. Hence we have the identity

$$M = \int_0^{x_0} u(x, t_0)\, dx = \int_{ch}^{x_0+ch} u(x, t_0 + h)\, dx. \tag{26}$$

Note that the above identity is valid **for all** $x_0 > 0$ **and all** $h > 0$. Differentiation with respect to x_0 gives

$$u(x_0, t_0) = u(x_0 + ch, t_0 + h), \quad \forall\ x_0 > 0, \quad \forall\ h > 0, \tag{27}$$

which says that u is **const. in** $h \in (0, \infty)$ **along the ray** $(x_0 + ch, t_0 + h)$ for all $h > 0$. Differentiation with respect to h and letting $h = 0$ gives

$$cu_x(x_0, t_0) + u_t(x_0, t_0) = 0, \quad \forall\ x_0 > 0, \quad t_0 \in \mathbb{R}. \tag{28}$$

The physical meaning of the eq. $cu_x + u_t = 0$.

The same analysis on the interval $[x_0, 0]$, $x_0 < 0$, also gives us the same eq.. Hence we conclude

$$cu_x(x_0, t_0) + u_{t_0}(x_0, t_0) = 0, \quad \forall\ x_0 \in \mathbb{R}, \quad t_0 \in \mathbb{R}. \tag{29}$$

Note that for fixed (x_0, t_0), $u(x, t)$ is **const. along the line** $x - ct = x_0 - ct_0$ (it has a simple physical interpretation due to the identity $u(x_0, t_0) = u(x_0 + ch, t_0 + h)$ for any $h \in \mathbb{R}$). The general sol. of the eq. is given by $u(x, t) = F(x - ct)$ for arbitrary C^1 func. $F(\cdot)$ with dom. $I \subseteq \mathbb{R}$. Physically, we also call $u(x, t) = F(x - ct)$ a **traveling wave sol.** with initial profile $u(x, 0) = F(x)$, $x \in I$. Draw a picture on blackboard for this ....

First order linear PDE with const. coeff. and side condition.

Consider eq. (4). We know that it has infinitely many sol. if we do not impose any condition on the sol.. The purpose of the extra "**side condition**" is to ensure that the sol. is **unique**. A general side condition has the form $u|_C = g$, where C is a given **curve** in the plane $\mathbb{R}^2$, **which intersects each char. line L "transversally" at exactly one point (draw a picture on blackboard)**, and g is a given func. on C. In most cases, we will just consider the case when $C = \tilde{L}$ is a **line** (or a **line segment**) in $\mathbb{R}^2$. Now the side condition has the form

$$u(x, mx + d) = g(x), \quad \forall\ x \in \mathbb{R}, \tag{30}$$

where $x \in \mathbb{R}$ is a parameter and m is the slope of the line $\tilde{L} : y = mx + d$ and d is some number. Here $g(x)$ is a C^1 func. of x ($g(x)$ can also be a const., say $g(x) \equiv 0$). If the line $\tilde{L}$ is vertical, then the side condition has the form

$$u(d, y) = g(y), \quad \forall\ y \in \mathbb{R}.$$

First order linear PDE with const. coeff. and side condition.

The property we have is the following:

Theorem

If the line $\tilde{L}$ *is* **not a char. line** *of (4), then the PDE with the side condition (30) has a* **unique sol.**. *If the line* $\tilde{L}$ *is a* **char. line**, *the PDE with the side condition has* **either no sol. or infinitely many sol.**.

The key point is that if $\tilde{L}$ is **not** a char. line, then it **intersects each char. line** L **"transversally" at exactly one point.** By (30), it allows us to determine the integration const. func. $C(w) = C(bx - ay)$ **uniquely**. Note that $C(w)$ is const. along each char. line.

We shall not make the above theorem precise. Instead, we will just look at some examples to convince us the result.

First order linear PDE with const. coeff. and side condition.

Example

The eq. $2u_x(x,y) + 5u_y(x,y) = 0$ has char. lines of the form $5x - 2y = \lambda$ (same as $y = \frac{5}{2}x + \lambda$), $\lambda \in \mathbb{R}$. Its general sol. is given by

$$u(x,y) = C(5x - 2y), \tag{31}$$

for arbitrary C^1 func. $C(\cdot)$. Let L be the line $y = \frac{5}{2}x + 1$, which **is a char. line**. Consider the side condition problem:

$$\begin{cases} 2u_x(x,y) + 5u_y(x,y) = 0 \\ u\left(x, \frac{5}{2}x + 1\right) = g(x), \quad x \in (-\infty, \infty) \end{cases} \tag{32}$$

for some func. $g(x)$. By (31), we have

$$u\left(x, \frac{5}{2}x + 1\right) = C\left(5x - 2\left(\frac{5}{2}x + 1\right)\right) = C(-2) \text{ (this is a const. !!).}$$

First order linear PDE with const. coeff. and side condition.

Example

(**Continued**.) Therefore, if $g(x)$ is not a const. func., then (32) has **no sol.** at all. On the other hand, if $g(x)$ is a const. func., say $g(x) = 10$, then any $u(x, y)$ of the form $u(x, y) = C(5x - 2y)$ is a sol. as long as C satisfies $C(-2) = 10$. In such a case, we have **infinitely many sol.**. Finally, if we replace the side condition by

$$u(x, 2x + 7) = \sin x, \quad x \in (-\infty, \infty), \tag{33}$$

then the line $y = 2x + 7$ **is not a char. line** and by (31) we can solve the eq.

$$u(x, 2x + 7) = C(5x - 2(2x + 7)) = C(x - 14) = \sin x, \quad x \in (-\infty, \infty)$$

to get $C(\xi) = \sin(\xi + 14)$, $\xi \in (-\infty, \infty)$. Therefore, the side condition problem (33) has the **unique sol.** given by

$$u(x, y) = C(5x - 2y) = \sin(5x - 2y + 14), \quad (x, y) \in \mathbb{R}^2. \tag{34}$$

First order linear PDE with const. coeff. and side condition.

Example

You can check that the general sol. of the eq.

$$3u_x - 2u_y + u = x \tag{35}$$

is

$$u(x,y) = x - 3 + e^{\frac{y}{2}} C(2x+3y), \tag{36}$$

and if we put the side condition as

$$u\left(x, -\frac{2}{3}x + 2\right) = g(x), \quad x \in (-\infty, \infty), \tag{37}$$

for some $g(x)$, where the line $y = -\frac{2}{3}x + 2$ **is a char. line**, we have

$$u\left(x, -\frac{2}{3}x + 2\right) = x - 3 + C(6)\, e^{-x/3+1} = g(x), \quad x \in (-\infty, \infty).$$

First order linear PDE with const. coeff. and side condition.

Example

(**Continued**.) Therefore, unless $g(x)$ has the form $g(x) = x - 3 + ke^{-x/3+1}$ for some const. k, we have **no sol.** satisfying the side condition. On the other hand, if $g(x) = x - 3 + 100e^{-x/3+1}$, there are **infinitely many sol.** satisfying this side condition everywhere on $x \in (-\infty, \infty)$ as long as we choose the func. $C(\cdot)$ to satis. $C(6) = 100$.

Example

Solve the eq.

$$\begin{cases} u_x(x,y) - u_y(x,y) + 2u(x,y) = 1 \\ u(x,0) = x^2, \quad x \in (-\infty, \infty). \end{cases} \tag{38}$$

First order linear PDE with const. coeff. and side condition.

Solution: One can check that the general sol. of $u_x - u_y + 2u = 1$ is (it is easy to guess a particular sol., which is $1/2$)

$$u(x, y) = \frac{1}{2} + e^{2y} C(-x - y) \quad \left(\text{same as } \frac{1}{2} + e^{2y} C(x + y)\right)$$

for arbitrary func. $C(\cdot)$. The side condition is prescribed on the x-axis, which is **not a char. line**. We need to choose $C(\cdot)$ so that

$$u(x, 0) = \frac{1}{2} + C(x) = x^2,$$

Hence we have $C(x) = x^2 - 1/2$ and then

$$u(x, y) = \frac{1}{2} + e^{2y} C(x + y) = \frac{1}{2} + e^{2y} \left[(x + y)^2 - \frac{1}{2}\right], \quad (x, y) \in \mathbb{R}^2$$

is the unique sol. of (38). □

First order linear PDE with const. coeff. and side condition.

Example

Solve the eq.

$$\begin{cases} u_x(x,y) + 2u_y(x,y) - 4u(x,y) = e^{x+y} \\ u(x, 4x+2) = 0, \quad x \in (-\infty, \infty). \end{cases} \tag{39}$$

Solution: One can check that the general sol. of $u_x + 2u_y - 4u = e^{x+y}$ is (it is easy to guess a particular sol., which is $-e^{x+y}$)

$$u(x,y) = -e^{x+y} + e^{2y} C(2x - y).$$

By the side condition, we require (the line $y = 4x + 2$ is **not a char. line**)

$$0 = u(x, 4x+2) = -e^{5x+2} + e^{8x+4} C(-2x - 2),$$

i.e. we need to require $C(-2x-2) = e^{-3x-2}$.

First order linear PDE with const. coeff. and side condition.

To get $C(r)$, we let $r = -2x - 2$ and one can solve x in terms of r to get

$$-3x - 2 = -3\left(\frac{r+2}{-2}\right) - 2 = \frac{3}{2}r + 1.$$

Hence $C(r) = e^{\frac{3}{2}r+1}$ and the unique sol. is def. on $\mathbb{R}^2$, given by

$$u(x,y) = -e^{x+y} + e^{2y}e^{\frac{3}{2}(2x-y)+1} = -e^{x+y} + e^{3x+\frac{1}{2}y+1}, \quad (x,y) \in \mathbb{R}^2.$$

$\square$

First order linear PDE with const. coeff. and side condition.

In the next example, the side condition has the form $u\big|_C = g$, where C is a **curve** in $\mathbb{R}^2$, not a line.

Example

Solve the eq. with side condition

$$\begin{cases} u_x(x,y) + 2u(x,y) = y \\ u(x, e^x) = \sin x, \quad x \in (-\infty, \infty). \end{cases} \tag{40}$$

Remark: Strictly speaking, the eq. is an ODE, but we need the char. line to have the form $y = \lambda$ (since $b = 0$) to see that the dom. of $u(x,y)$ is $\mathbb{R}^2_+ = \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\}$.

Solution: The general sol. of the eq. $u_x + 2u = y$ is given by (in case $b = 0$, we use the formula $u(x,y) = e^{-\frac{c}{a}x} C(bx - ay)$, $a \neq 0$, for the general sol. of $au_x + bu_y + cu = 0$; see (20))

$$u(x,y) = \frac{y}{2} + e^{-2x} C(y), \quad \text{for arbitrary func. } C(\cdot).$$

First order linear PDE with const. coeff. and side condition.

Now we note that the side condition is specified on the **curve** $y = e^x, \ x \in (-\infty, \infty)$. **This curve intersects each char. line** $y = \lambda$ **at exactly one point only for** $\lambda > 0$**, but not for** $\lambda \leq 0$. **This may suggest that the sol. of (40) is def. only on the upper half-plane** $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. This is indeed so. By the condition

$$u(x, e^x) = \frac{e^x}{2} + e^{-2x} C(e^x) = \sin x, \quad x \in (-\infty, \infty),$$

we need to require

$$C(e^x) = e^{2x} \left(\sin x - \frac{e^x}{2} \right).$$

First order linear PDE with const. coeff. and side condition

If we let $\theta = e^x > 0$, we have

$$C(\theta) = \theta^2 \left(\sin(\log\theta) - \frac{\theta}{2} \right), \quad \theta \in (0, \infty).$$

Therefore, the unique sol. of (**40**) is def. only on $\mathbb{R}^2_+$, given by

$$u(x, y) = \frac{y}{2} + e^{-2x} C(y) = \frac{y}{2} + e^{-2x} y^2 \left(\sin(\log y) - \frac{y}{2} \right),$$

where $(x, y) \in \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. □

First order linear PDE with variable coeff..

Now we look at first order linear PDE for $u = u(x, y)$ with variable coeff. The eq. has the form

$$\underbrace{a(x, y)\, u_x + b(x, y)\, u_y} + c(x, y)\, u = f(x, y), \quad u = u(x, y), \tag{41}$$

where the coeff. $a(\cdot),\ b(\cdot),\ c(\cdot),\ f(\cdot)$ are given C^1 func. def. on some open set $\Omega \subseteq \mathbb{R}^2$ with

$$a^2(x, y) + b^2(x, y) > 0 \quad \text{on} \quad \Omega. \tag{42}$$

To solve (41) means to find a C^1 func. $u(x, y)$ def. at least on some **open subset** $\tilde{\Omega}$ of Ω and satis. (41) on $\tilde{\Omega}$. The sol. method will involve some ODE theory.

First order linear PDE with variable coeff..

The **nondegenerate condition** (42) on Ω" is important. In case there is $(x_0, y_0) \in \Omega$ such that $a^2(x_0, y_0) + b^2(x_0, y_0) = 0$, then it is possible that the eq. (41) has **no sol.** def. on any open set containing (x_0, y_0). For example:

Example

Consider the eq.

$$xu_x(x, y) + yu_y(x, y) = e^{x+y}, \quad (x, y) \in \mathbb{R}^2, \tag{43}$$

where $x^2 + y^2 = 0$ at $(0, 0)$. There is no C^1 sol. $u(x, y)$ def. near some open set containing $(0, 0)$ due to

$$0 = 0u_x(0, 0) + 0u_y(0, 0) = e^{0+0} = 1.$$

Therefore, the **natural dom.** for the PDE (43) is $\mathbb{R}^2 \setminus \{(0, 0)\}$. There is no sol. def. on any open set containing $(0, 0)$.

First order linear PDE with variable coeff..

It is also possible that the eq. (41) has only **trivial sol.** (i.e. **const. sol.**) def. near $(x_0, y_0) \in \Omega$ if we have $a^2(x_0, y_0) + b^2(x_0, y_0) = 0$, which is of little interest. We have:

Example (Example A.)

Let $B_R(0) \subset \mathbb{R}^2$ be an open ball centered at the origin $0 = (0, 0)$ with radius $R > 0$. If $u(x, y)$, def. on $B_R(0) \subset \mathbb{R}^2$, is a C^1 sol. of the PDE

$$xu_x(x, y) + yu_y(x, y) = 0, \quad (x, y) \in B_R(0),$$

then it must be a **const. sol.** on $B_R(0)$.

Solution: I will leave the proof of the above example to you (in HW 2). □

First order linear PDE with variable coeff..

We first look at the special case of (41), i.e.

$$a(x,y)\,u_x + b(x,y)\,u_y = 0, \quad u = u(x,y), \quad (x,y) \in \Omega, \qquad (44)$$

where $a^2(\cdot) + b^2(\cdot) > 0$ on Ω. The PDE (44) gives rise to a **nonzero** C^1 **vector field** $V(x,y) = (a(x,y), b(x,y))$ on Ω and we can rewrite the eq. as

$$\nabla u(x,y) \cdot \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} = 0, \quad (x,y) \in \Omega,$$

which means that the gradient vector field $\nabla u(x,y)$ of $u(x,y)$ is **everywhere perpendicular** to the vector field $V(x,y)$ on Ω.

Also, by ODE theory, $V(x, y)$ will generate a family of **integral curves** $C : \alpha(t) = (x(t), y(t)) \in \Omega,\ \ t \in I$ (some open interval) on Ω, where $(x(t), y(t))$ satisfies

$$\frac{dx}{dt}(t) = a(x(t), y(t)), \quad \frac{dy}{dt}(t) = b(x(t), y(t)), \quad \forall\ t \in I. \qquad (45)$$

We have the following **geometric meaning** of the PDE (44) on Ω :

Lemma (Lemma A.)

Let $V(x, y) = (a(x, y), b(x, y))$ be a C^1 vector field on $\Omega \subseteq \mathbb{R}^2$ with $a^2(x, y) + b^2(x, y) > 0$ on Ω. Then $u(x, y)$ is a C^1 sol. of (44) on Ω **if and only if** *for* **any** *C^1 integral curve $C : \alpha(t) = (x(t), y(t)),\ t \in I$, of the ODE (45) lying on Ω, the func. $u(\alpha(t)),\ t \in I$, is a* **const.** *func. along the curve C.*

First order linear PDE with variable coeff..

Proof: $(\Longrightarrow)$: Assume $u(x,y)$ is a C^1 sol. of (44) on Ω and $\alpha(t) = (x(t), y(t))$ satis. the ODE (45) on I. Then we have

$$\frac{d}{dt}u(x(t), y(t)) = u_x(\alpha(t))\frac{dx}{dt}(t) + u_y(\alpha(t))\frac{dy}{dt}(t)$$
$$= a(\alpha(t))u_x(\alpha(t)) + b(\alpha(t))u_y(\alpha(t)) = 0, \quad \forall\ t \in I. \tag{46}$$

Hence, $u(\alpha(t))$, $t \in I$, is a **const.** func.

$(\Longleftarrow)$: By ODE theory, for any point $p = (x_0, y_0) \in \Omega$, **there is** a C^1 integral curve $C : \alpha(t) = (x(t), y(t)) \in \Omega$, $t \in (a,b)$, satis. $\alpha(0) = p$. Here (a,b) is a maximal time interval with $0 \in (a,b)$ and $\alpha(t)$ approaches $\partial\Omega$ as $t \to a$ and $t \to b$. By the assumption we have (46) for all $t \in (a,b)$ and at $t = 0$ it gives $a(p)u_x(p) + b(p)u_y(p) = 0$. Since $p = (x_0, y_0) \in \Omega$ is arbitrary, $u(x,y)$ satis. the eq. (44) on Ω. $\square$

First order linear PDE with variable coeff..

Definition (Definition A.)

(**Geometric definition**.) A curve C in the plane is called a **char. curve** of the PDE (44) (or the general PDE (41)) if at each point $(x, y) \in C$, the nonzero vector $V(x, y) = (a(x, y), b(x, y))$ is **tangent** to C at (x, y).

Remark: By a **suitable parametrization** (i.e., by solving the ODE (45) with initial condition at some $p \in C$), a **char. curve** C can be parametrized as $C : \alpha(t) = (x(t), y(t)) \in \Omega,\ t \in I$, where $x(t)$ and $y(t)$ satis. the ODE (45) on some interval I.

Example

Consider the eq. $au_x + bu_y + cu = f(x, y)$. Then, according to Definition A, each char. line $bx - ay = \lambda$ (passing through (x_0, y_0)) is a char. curve of the eq.. It can be parametrized as $\alpha(t) = (x(t), y(t)) = (at + x_0, bt + y_0)$, where $x(t)$ and $y(t)$ satis. the ODE:

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b. \tag{47}$$

First order linear PDE with variable coeff..

In Lemma A., the const. may be different on different integral curves. The lemma says that an **integral curve** $\alpha(t) = (x(t), y(t))$, $t \in I$, of the ODE (45) is a **char. curve of the PDE (44)** and is also a **level curve** (in parametric form) of the PDE sol. $u(x, y)$ on Ω.

Remark: Roughly speaking, if we can know "**all integral curves** of the ODE (45) on Ω", then one can find **all sol.** of PDE (44) on Ω. If the PDE has a **side condition,** then we can obtain a **unique** sol. (if the curve C in the side condition intersects each integral curve **"transversally"** at exactly one point). Two curves are called **"transversal"** if at the intersection point p, their tangent lines are not equal to each other.

Finding general sol. of the PDE (44).

Based on the above remark, we can follow the steps below to solve the PDE (44) on some open subset $\tilde{\Omega}$ of Ω :

1. Solve the ODE (45) on Ω to get all **integral curves** $\alpha(t) = (x(t), y(t))$, $t \in I$, on Ω.

2. Convert the family of sol. $(x(t), y(t))$ into the following **implicit form** (either **by deleting the time variable** t or **by the Inverse Function Theorem to get an algebraic relation between** x **and** y)

$$h(x, y) = C, \tag{48}$$

where $h(x, y)$ is a C^1 func. and C is a **free const.** (serving as a **free parameter**). **The func.** $h(x, y)$, **when restricted to each integral curve** $(x(t), y(t))$**, is a const. func.** Note that the free constant C in (48) may lie on certain range. For example, if the **integral curves** are a family of circles centered at the origin, then we have $x^2 + y^2 = C$, where $C \in (0, \infty)$.

Finding general sol. of the PDE (44).

In general, the func. $h(x,y)$ is def. only on some **open subset** $\tilde{\Omega}$ of Ω. As the const. C varies, the identity $h(x,y) = C$ describes all possible different char. curves (in algebra identity, not in parametric form) of the eq. $a(x,y)\,u_x + b(x,y)\,u_y = 0$ lying on $\tilde{\Omega}$.

Moreover, since the algebraic form in (48) is **not unique** in general, **the dom. of** $h(x,y)$ **may depend on how you choose your** $h(x,y)$. For example, if the identity you derived is $h(x,y) = y/x = C$, then the dom. is either $\mathbb{R}^2_{x^+}$ or $\mathbb{R}^2_{x^-}$, but if the identity you derived is $h(x,y) = x/y = C$ (which is equivalent to $y/x = C$), then the dom. is either $\mathbb{R}^2_{y^+}$ or $\mathbb{R}^2_{y^-}$.

Finding general sol. of the PDE (44).

3. Another way to obtain (48) is to **rewrite the system of ODE (45) as the single eq. (here we view y as a func. of x)**

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}, \quad \text{where } a(x,y) \neq 0 \text{ on some dom. } \tilde{\Omega} \subseteq \Omega, \tag{49}$$

or **as the single eq. (here we view x as a func. of y)**

$$\frac{dx}{dy} = \frac{a(x,y)}{b(x,y)}, \quad \text{where } b(x,y) \neq 0 \text{ on some dom. } \tilde{\Omega} \subseteq \Omega, \tag{50}$$

and solve it to get a relation between y and x and the free integration const. C (from the above process we can obtain an identity of the form (48)).

Finding general sol. of the PDE (44).

One very important property for $h(x, y)$ is that, since the vector field $V(x, y) = (a(x, y), b(x, y))$ is **everywhere nonzero** on Ω, by **restricting the dom.** $\tilde{\Omega}$ **to an even smaller open subset** if necessary, the func. $h(x, y) : \tilde{\Omega} \to \mathbb{R}$ **will satisfy either**

$$\frac{\partial h}{\partial x}(x, y) \neq 0 \quad \text{everywhere on} \quad \tilde{\Omega} \tag{51}$$

or

$$\frac{\partial h}{\partial y}(x, y) \neq 0 \quad \text{everywhere on} \quad \tilde{\Omega}. \tag{52}$$

Remark: See the explanation in my lecture notes.

Finding general sol. of the PDE (44).

Lemma (Lemma B.)

We have the following important results:

(1) . *The above* C^1 *func.* $h(x,y):\tilde{\Omega}\to\mathbb{R}$ *is a* **sol.** *of the PDE (44) on* $\tilde{\Omega}$.

(2) . *If* $\frac{\partial h}{\partial x}(x,y)\neq 0$ *on* $\tilde{\Omega}$ *(i.e. nonzero everywhere on* $\tilde{\Omega}$*), it implies* $b(x,y)\neq 0$ *on* $\tilde{\Omega}$.

(3) . *If* $\frac{\partial h}{\partial y}(x,y)\neq 0$ *on* $\tilde{\Omega}$*, it implies* $a(x,y)\neq 0$ *on* $\tilde{\Omega}$.

Proof: (1) . For each $\alpha(0)=(x(0),y(0))\in\tilde{\Omega}$, one can solve the ODE (45) to get a unique integral curve $\alpha(t)=(x(t),y(t))\in\tilde{\Omega}$ for $t\in I$ (some interval containing $t=0$).

Finding general sol. of the PDE (44).

Since we have

$$h(x(t), y(t)) = const., \quad \forall\ t \in I,$$

the chain rule implies

$$\begin{aligned} &\frac{\partial h}{\partial x}(\alpha(t))\frac{dx}{dt}(t) + \frac{\partial h}{\partial y}(\alpha(t))\frac{dy}{dt}(t) \\ &= a(\alpha(t))\frac{\partial h}{\partial x}(\alpha(t)) + b(\alpha(t))\frac{\partial h}{\partial y}(\alpha(t)) = 0, \quad \forall\ t \in I. \end{aligned}$$

In particular, at $t = 0$, we get

$$a(\alpha(0))\frac{\partial h}{\partial x}(\alpha(0)) + b(\alpha(0))\frac{\partial h}{\partial y}(\alpha(0)) = 0.$$

Finding general sol. of the PDE (44).

As $\alpha(0) \in \tilde{\Omega}$ is arbitrary, we conclude

$$a(x,y)\frac{\partial h}{\partial x}(x,y) + b(x,y)\frac{\partial h}{\partial y}(x,y) = 0, \quad \forall\ (x,y) \in \tilde{\Omega}. \tag{53}$$

Hence $h(x,y) : \tilde{\Omega} \to \mathbb{R}$ is a **sol.** of the PDE (44) on $\tilde{\Omega}$.

(2). If $\frac{\partial h}{\partial x}(x,y) \neq 0$ on $\tilde{\Omega}$, and $b(x_0,y_0) = 0$ at some $(x_0,y_0) \in \tilde{\Omega}$, then (53) will imply $a(x_0,y_0) = 0$, a contradiction due to our main assumption $a^2(x,y) + b^2(x,y) > 0$ on Ω. Similarly, if $\frac{\partial h}{\partial y}(x,y) \neq 0$ on $\tilde{\Omega}$, we must have $a(x,y) \neq 0$ on $\tilde{\Omega}$.

The proof of (3) is similar. □

Remark: At this moment, $h(x,y)$ is only **one sol.** of the PDE on $\tilde{\Omega}$. We will use $h(x,y)$ to do **ch. of var.** (similar to the previous case when $h(x,y) = bx - ay$ and we let $w = bx - ay,\ z = y$) to find the **general sol.** of the eq. (53).

Finding general sol. of the PDE (44).

Example

Consider the eq.

$$xu_x + yu_y = 0, \quad u = u(x,y), \quad (x,y) \in \Omega = \mathbb{R}^2 \setminus \{(0,0)\}. \tag{54}$$

If we solve the system of ODE

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y, \tag{55}$$

we get

$$x(t) = x_0 e^t, \quad y(t) = y_0 e^t, \quad t \in (-\infty, \infty), \quad (x_0, y_0) \neq (0,0).$$

Example

(**Continued**.) We can use one of the following **three methods** to find $h(x,y)$:

(1) . Cancel the parameter t to get a relation between x and y. We can look at the ratio x/y (if $y_0 \neq 0$) or y/x (if $x_0 \neq 0$) to get

$$h(x,y) = \frac{x}{y} = \frac{x_0 e^t}{y_0 e^t} = \frac{x_0}{y_0} := C \quad (\text{missing } x\text{-axis, i.e. } y = 0),$$

or

$$h(x,y) = \frac{y}{x} = \frac{y_0 e^t}{x_0 e^t} = \frac{y_0}{x_0} := C \quad (\text{missing } y\text{-axis, i.e. } x = 0),$$

along each integral curve, where C is an arbitrary const.. Note that the func. $h(x,y) = x/y$ (or y/x) cannot be def. on the whole Ω due to the denominator. One can check that $h(x,y) = x/y$ (or y/x) is a C^1 sol. of the PDE (54) def. on $\Omega \setminus \{x\text{-axis}\}$ $(\Omega \setminus \{y\text{-axis}\})$.

Finding general sol. of the PDE (44).

Example

(**Continued**.) (2) . Rewrite the system of ODE (55) as (we view y as a func. of x)

$$\frac{dy}{dx} = \frac{y}{x} \text{ (\textbf{separable ODE})},$$

and obtain

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx, \quad \log|y| = \log|x| + K, \quad y = \pm e^K x,$$

which gives the identity $h(x, y) = \frac{x}{y} = C$ (or $\frac{y}{x} = C$).

(3) . From the eq. $x(t) = x_0 e^t$, we get $t = \log\left(\frac{x}{x_0}\right)$ (if $x_0 \neq 0$) and then

$$y = y(t) = y_0 e^t = y_0 e^{\log\left(\frac{x}{x_0}\right)} = \frac{y_0}{x_0} x, \text{ i.e. } h(x, y) = \frac{y}{x} = \frac{y_0}{x_0} := C.$$

Using $h(x,y)$ to do ch. of var. to solve (44).

To obtain **general sol.** of the PDE $a(x,y)\,u_x + b(x,y)\,u_y = 0$, similar to the case of const. coeff. (see (12)), we can use the func. $h(x,y)$ to do the ch. of var., where we know that $h(x,y)$, coming from (48), is a sol. of the PDE on $\tilde{\Omega}$, and satis. either (51) or (52) if we make $\tilde{\Omega}$ smaller. If $h(x,y)$ satisfies (51), then we can do the ch. of var.

$$w = h(x,y), \quad z = y, \quad (x,y) \in \tilde{\Omega}, \tag{56}$$

which satis. the nonzero Jacobian condition,
i.e. $J(x,y) = \frac{\partial h}{\partial x}(x,y) \neq 0$ on $\tilde{\Omega}$. Moreover, by Lemma B., we also have $b(x,y) \neq 0$ on $\tilde{\Omega}$.

Remark: Since $J(x,y) \neq 0$ on $\tilde{\Omega}$, by the **Inverse Function Theorem** in Advanced Calculus and restricting $\tilde{\Omega}$ to an even smaller open subset if necessary, (56) is a **ch. of var.** on $\tilde{\Omega}$.

Using $h(x,y)$ to do ch. of var. to solve (44).

Now the func. $u(x,y)$ becomes $U(w,z)$ (i.e. $U(h(x,y),y) = u(x,y)$) and by the chain rule, we get

$$u_x(x,y) = U_w(w,z)\frac{\partial h}{\partial x}, \quad u_y(x,y) = U_w(w,z)\frac{\partial h}{\partial y} + U_z(w,z)$$

Therefore, in terms of the new variables (w,z), the PDE for $U(w,z)$ becomes:

$$\begin{aligned} 0 &= a(x,y)\,U_w(w,z)\frac{\partial h}{\partial x} + b(x,y)\left[U_w(w,z)\frac{\partial h}{\partial y} + U_z(w,z)\right] \\ &= \left[a(x,y)\frac{\partial h}{\partial x}(x,y) + b(x,y)\frac{\partial h}{\partial y}(x,y)\right]U_w(w,z) + b(x,y)\,U_z(w,z) \\ &= \underbrace{b(x,y)\,U_z(w,z)}, \quad (x,y) \in \tilde{\Omega}. \end{aligned} \tag{57}$$

Using $h(x,y)$ to do ch. of var. to solve (44).

Note that in the above we have used the identity (53). Since $b(x,y) \neq 0$ on $\tilde{\Omega}$, we conclude $U_z(w,z) \equiv 0$ on its dom. in wz-space. The general sol. of the **ODE** (57) is given by

$$U(w,z) = F(w) = F(h(x,y)), \quad w = h(x,y), \quad (x,y) \in \tilde{\Omega} \tag{58}$$

for arbitrary C^1 func. $F(w)$.

On the other hand, if $h(x,y)$ also satisfies (52), then we can also do the ch. of var.

$$w = h(x,y), \quad z = x, \quad (x,y) \in \tilde{\Omega}, \tag{59}$$

which satis. the nonzero Jacobian condition,
i.e. $J(x,y) = -\frac{\partial h}{\partial y}(x,y) \neq 0$ on $\tilde{\Omega}$. Moreover, by Lemma B., we also have $a(x,y) \neq 0$ on $\tilde{\Omega}$.

Using $h(x, y)$ to do ch. of var. to solve (44).

Now the PDE for $U(w, z)$ becomes:

$$
\begin{aligned}
0 &= a(x, y)\left[U_w(w, z)\frac{\partial h}{\partial x} + U_z(w, z)\right] + b(x, y)\, U_w(w, z)\frac{\partial h}{\partial y} \\
&= \left[a(x, y)\frac{\partial h}{\partial x}(x, y) + b(x, y)\frac{\partial h}{\partial y}(x, y)\right] U_w(w, z) + a(x, y)\, U_z(w, z) \\
&= \underbrace{a(x, y)\, U_z(w, z)}, \quad (x, y) \in \tilde{\Omega}. \qquad (60)
\end{aligned}
$$

Since $a(x, y) \neq 0$ on $\tilde{\Omega}$, we conclude $U_z(w, z) \equiv 0$ on its dom. in wz-space. The general sol. of the **ODE** (60) is also given by (58) for arbitrary C^1 func. $F(w)$.

Remark: In case the eq. is $au_x + bu_y = 0$, where a, b are nonzero const., then $h(x, y) = bx - ay$ and the ch. of var. (56) is reduced to the previous ch. of var. (12).

Finding general sol. of the PDE (44).

We can now conclude the following result:

Theorem (Theorem A.)

(**General sol. of the PDE (44).**) *Let $a(\cdot)$, $b(\cdot)$ be two C^1 func. on a dom. $\Omega \subseteq \mathbb{R}^2$ with $a^2(\cdot) + b^2(\cdot) > 0$ on Ω. Consider the PDE $a(x, y)\, u_x + b(x, y)\, u_y = 0$ on Ω. There exists some* **open subset** *$\tilde{\Omega}$ of Ω such that on $\tilde{\Omega}$ any C^1 sol. $u(x, y)$ of the PDE has the form*

$$u(x, y) = F(h(x, y)), \quad (x, y) \in \tilde{\Omega}, \tag{61}$$

where $F(\cdot)$ is an arbitrary C^1 func. def. on $\mathbb{R}$ and the func. $h(x, y)$, which is def. on $\tilde{\Omega}$, comes from solving the ODE (45) on Ω. If $F(\cdot)$ is not def. on $\mathbb{R}$, the dom. of $u(x, y)$ may be smaller than $\tilde{\Omega}$ (depending on your choice of $F(\cdot)$).

Finding general sol. of the PDE (44).

Remark: In particular, if we choose $F(\cdot)$ to be a **constant** function, then $u(x,y)$ is a **constant** function. Any constant func. is clearly a sol. of the PDE $a(x,y)\,u_x + b(x,y)\,u_y = 0$ on Ω.

Remark: Due to our **ch. of var. method** of solving the PDE, it is possible that there are different **disjoint** open subsets $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ of Ω such that the func. $h_1(x,y)$ on $\tilde{\Omega}_1$ and $h_2(x,y)$ on $\tilde{\Omega}_2$ are **different** and the integration const. func. F and G on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are **independent** to each other. See Example A. below.

Finding general sol. of the PDE (44).

Let us go back to the eq. $xu_x + yu_y = 0$ again:

Example (Example A.)

Find the general sol. of the eq.

$$xu_x + yu_y = 0, \quad u = u(x, y), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{62}$$

Solution:

We already know that the implicit form of the integral curves is given by $h(x, y) = \frac{x}{y} = C$, $y \neq 0$, and it is a sol. on $\mathbb{R}^2_{y+} \bigcup \mathbb{R}^2_{y-}$. We do the ch. of var.

$$\begin{cases} w = \frac{x}{y} \\ z = y \neq 0 \end{cases} \quad \text{same as} \quad \begin{cases} x = wz, \\ y = z \neq 0. \end{cases} \tag{63}$$

Finding general sol. of the PDE (44).

Then the Jacobian for the ch. of var. is

$$J(x,y) = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y} \neq 0, \quad \forall \ (x,y) \in \mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}.$$

One can check that. the ch. of var. (63) is a **bijection map** between the open set $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$ in xy-space and the open set $\mathbb{R}^2_{z^+} \bigcup \mathbb{R}^2_{z^-}$ in wz-space ($\mathbb{R}^2_{y^+}$ maps to $\mathbb{R}^2_{z^+}$ and $\mathbb{R}^2_{y^-}$ maps to $\mathbb{R}^2_{z^-}$). The eq. $xu_x + yu_y = 0$ on $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$ is **equivalent to** the eq. on $\mathbb{R}^2_{z^+} \bigcup \mathbb{R}^2_{z^-}$:

$$0 = xu_x + yu_y = x\left(U_w(w,z)\frac{1}{y}\right) + y\left(U_w(w,z)\left(-\frac{x}{y^2}\right) + U_z(w,z)\right)$$
$$= yU_z(w,z), \quad y \neq 0.$$

Finding general sol. of the PDE (44).

Therefore, the general sol. for $u(x, y)$ on $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$ is given by:

$$u(x, y) = \begin{cases} F\left(\frac{x}{y}\right), & (x, y) \in \mathbb{R}^2_{y^+}, \\ G\left(\frac{x}{y}\right), & (x, y) \in \mathbb{R}^2_{y^-}, \end{cases} \tag{64}$$

where $F(\cdot)$ and $G(\cdot)$ are two **arbitrary** C^1 func. def. on open intervals I and J of $\mathbb{R}$. Note that each integral curve of the ODE on $\mathbb{R}^2_{y^+}$ (given by $(x(t), y(t)) = (x_0 e^t, y_0 e^t)$, $t \in (-\infty, \infty)$, $y_0 > 0$) will not traverse into $\mathbb{R}^2_{y^-}$ for all $t \in (-\infty, \infty)$. They all stay on $\mathbb{R}^2_{y^+}$. Similarly, each integral curve of the ODE on $\mathbb{R}^2_{y^-}$ will not traverse into $\mathbb{R}^2_{y^+}$.

Finding general sol. of the PDE (44).

In view of the above observation, the two integration const. func. $F(\cdot)$ and $G(\cdot)$ in (64) can be **independent** to each other. In case $I = J = \mathbb{R}$, then $u(x, y)$ is defined on $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$. □

Remark: If we choose the ch. of var. as $w = \frac{y}{x}$, $x \neq 0$, $z = x$, then then the Jacobian is $J(x, y) = -\frac{1}{x}$ and we obtain

$$0 = xu_x + yu_y = xU_z(w, z), \quad x \neq 0,$$

and the general sol. is

$$u(x, y) = \begin{cases} F\left(\frac{y}{x}\right), & (x, y) \in \mathbb{R}^2_{x^+}, \\ G\left(\frac{y}{x}\right), & (x, y) \in \mathbb{R}^2_{x^-}, \end{cases}$$

where $F(\cdot)$ and $G(\cdot)$ can be **independent** to each other.

Finding general sol. of the PDE (44).

Remark: However, it is **possible** to choose some **special** F and G in (64) so that $u(x,y)$ can be def. **across the** x-**axis** and is def. on the whole dom. $\mathbb{R}^2\setminus\{(0,0)\}$. One trivial sol. is $u(x,y)$ is a **const. sol.**, i.e. $u(x,y)\equiv C$ for all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$. Another nontrivial sol. is the func. (we choose $F(w)=G(w)=e^{-w^2}$ in (64), $w\in\mathbb{R}$, $\lim_{w\to\pm\infty}e^{-w^2}=0$)

$$u(x,y)=\begin{cases} e^{-\left(\frac{x}{y}\right)^2}, & y\neq 0, \quad x\in\mathbb{R}, \\ 0, & y=0, \quad x\neq 0\in\mathbb{R}. \end{cases} \tag{65}$$

Note that the above $u(x,y)$ is defined on $\mathbb{R}^2\setminus\{(0,0)\}$. We leave it to you (in HW2) to check that the func. $u(x,y)$ in (65) satis. $u\in C^1\left(\mathbb{R}^2\setminus\{(0,0)\}\right)$ with

$$xu_x+yu_y=0 \text{ on } \mathbb{R}^2\setminus\{(0,0)\}, \quad u_x(x_0,0)=u_y(x_0,0), \quad \forall\, x_0\neq 0.$$

Note that the 2-dimensional limit $\lim_{(x,y)\to(0,0)}u(x,y)$ does not exist. **We cannot make** $u(x,y)$ **to be continuous at** $(0,0)$.

Finding general sol. of the PDE (44).

Example

Find the general sol. of the eq.

$$xu_x - yu_y = 0, \quad u = u(x, y), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{66}$$

Solution:

The system of ODE for the eq. is

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y, \quad x(t) = x_0 e^t, \quad y(t) = y_0 e^{-t}$$

and we obtain the eq. $h(x, y) = xy = k$ ($k = c_1 c_2 \in (-\infty, \infty)$ is a const.) on integral curves. Now we do the ch. of var.

$$\begin{cases} w = xy \\ z = y \end{cases} \quad \text{same as} \quad \begin{cases} x = \frac{w}{z}, \quad z \neq 0 \\ y = z. \end{cases}$$

Finding general sol. of the PDE (44).

Its Jacobian is

$$J(x,y) = \begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y \neq 0.$$

One can check that. the ch. of var. (63) is a **bijection map** between the open set $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$ in xy-space and the open set $\mathbb{R}^2_{z^+} \bigcup \mathbb{R}^2_{z^-}$ in wz-space ($\mathbb{R}^2_{y^+}$ maps to $\mathbb{R}^2_{z^+}$ and $\mathbb{R}^2_{y^-}$ maps to $\mathbb{R}^2_{z^-}$). The eq. $xu_x - yu_y = 0$ on $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$ is **equivalent to** the eq. on $\mathbb{R}^2_{z^+} \bigcup \mathbb{R}^2_{z^-}$:

$$\begin{aligned} 0 &= xu_x - yu_y = x\left(U_w(w,z)\,y\right) - y\left(U_w(w,z)\,x + U_z(w,z)\right) \\ &= -yU_z(w,z), \quad y \neq 0. \end{aligned}$$

Finding general sol. of the PDE (44).

Therefore the general sol. on $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$ is given by

$$u(x,y) = \begin{cases} F(xy), & (x,y) \in \mathbb{R}^2_{y^+}, \\ G(xy), & (x,y) \in \mathbb{R}^2_{y^-}, \end{cases} \tag{67}$$

where $F(\cdot)$ and $G(\cdot)$ are two **arbitrary** C^1 func. def. on open intervals I and J of $\mathbb{R}$. Again, each integral curve of the ODE on $\mathbb{R}^2_{y^+}$ (given by $(x(t), y(t)) = (x_0 e^t, y_0 e^{-t})$,
$t \in (-\infty, \infty)$, $x_0 \in (-\infty, \infty)$, $y_0 > 0$) will not traverse into $\mathbb{R}^2_{y^-}$ for all $t \in (-\infty, \infty)$, and vice versa. In view of this, the two integration const. func. $F(\cdot)$ and $G(\cdot)$ in (67) can be **independent** to each other. In case $I = J = \mathbb{R}$, then $u(x,y)$ is defined on $\mathbb{R}^2_{y^+} \bigcup \mathbb{R}^2_{y^-}$. □

Finding general sol. of the PDE (44).

Remark: If we choose the ch. of var. as $w = xy,\ z = x$, then the Jacobian is $J(x,y) = -x$ and we obtain

$$0 = xu_x - yu_y = x\left(U_w y + U_z\right) - y\left(U_w x\right) = xU_z.$$

Now the general sol. is

$$u(x,y) = \begin{cases} F(xy), & (x,y) \in \mathbb{R}^2_{x^+}, \\ G(xy), & (x,y) \in \mathbb{R}^2_{x^-}. \end{cases}$$

Remark: The func. $u(x,y) = xy$ is a sol. of $xu_x - yu_y = 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$. In fact, it also defined on the whole $\mathbb{R}^2$. That is, there exists a **non-const. sol.** def. near the singularity $(0,0)$ of the PDE $xu_x - yu_y = 0$. This is different from the eq. $xu_x + yu_y = 0$. See Example A..

Finding general sol. of the PDE (44).

Example

Find the general solution of the eq.

$$yu_x - xu_y = 0, \quad u = u(x, y), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}. \qquad (68)$$

Solution:

The system of ODE for the eq. is (let $\mathbf{x}(t) = (x(t), y(t))$)

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x; \quad \mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}(t),$$

which gives $x''(t) + x(t) = 0$ and $y''(t) + y(t) = 0$. The general sol. for $(x(t), y(t))$ is given by

$$\begin{cases} x(t) = A\cos t + B\sin t, & x(0) = A, \\ y(t) = B\cos t - A\sin t, & y(0) = B, \quad t \in (-\infty, \infty), \end{cases}$$

Finding general sol. of the PDE (44).

where $A,\ B$ are two arbitrary const. with $(A,B) \neq (0,0)$. The implicit form of the family of sol. $(x(t), y(t))$ is given by the eq.

$$h(x,y) = x^2 + y^2 = A^2 + B^2 = C > 0, \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \tag{69}$$

for arbitrary number $C > 0$. Each integral curve is a circle with radius $\sqrt{C} > 0$, centered at the origin.
Another way to find (69) is to rewrite the ODE system as (we view y as a function of x)

$$\frac{dy}{dx} = -\frac{x}{y} \ \text{(separable eq., } \int y dy = -\int x dx\text{)} \tag{70}$$

and solve it to get the implicit sol. for $y(x)$, i.e.

$$x^2 + y^2 = C > 0, \quad C \text{ is a constant,}$$

which is the same as (69) (the explicit solution for (70) is given by $y(x) = \sqrt{C - x^2}$).

Finding general sol. of the PDE (44).

Now we do the ch. of var.

$$\begin{cases} w = x^2 + y^2, \\ z = y. \end{cases}$$

The Jacobian $J(x, y)$ is

$$J(x, y) = \begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x.$$

Thus the ch. of var. is good **only on** $\mathbb{R}^2_{x^+}$ or **only on** $\mathbb{R}^2_{x^-}$, but **not on** $\mathbb{R}^2_{x^+} \bigcup \mathbb{R}^2_{x^-}$ (note that both $(x, y) = (1, 1) \in \mathbb{R}^2_{x^+}$ and $(x, y) = (-1, 1) \in \mathbb{R}^2_{x^-}$ are mapped to the same point $(w, z) = (2, 1)$).

Finding general sol. of the PDE (44).

On $\mathbb{R}^2_{x^+}$, the inverse func. relation $(x, y) \longleftrightarrow (w, z)$ is **globally valid** with $x = \sqrt{w - z^2}$, $y = z$, if $(w, z) \in \Sigma$, where Σ is the open dom. in wz-space given by:

$$\Sigma = \left\{(w, z) \in \mathbb{R}^2 : w > z^2\right\}, \quad \mathbb{R}^2_{x^+} \longleftrightarrow \Sigma \text{ is a bijection,}$$

and the eq. $yu_x - xu_y = 0$ is **equivalent to** the eq. $-xU_z = 0$. Similarly, on $\mathbb{R}^2_{x^-}$, the inverse func. relation $(x, y) \longleftrightarrow (w, z)$ is **globally valid** between $\mathbb{R}^2_{x^-}$ and Σ with $x = -\sqrt{w - z^2}$, $y = z$.

Finding general sol. of the PDE (44).

Therefore on $\mathbb{R}^2_{x^+}$ the general sol. of the PDE is

$$u(x, y) = F\left(x^2 + y^2\right), \quad (x, y) \in \mathbb{R}^2_{x^+},$$

where $F(\cdot)$ is an arbitrary C^1 func. def. on some open interval $I \subset (0, \infty)$. Similarly, on $\mathbb{R}^2_{x^-}$ the general sol. of the PDE is

$$u(x, y) = G\left(x^2 + y^2\right), \quad (x, y) \in \mathbb{R}^2_{x^-},$$

where $G(w)$ is an arbitrary C^1 func. def. on open interval $J \subset (0, \infty)$. If we focus on $\mathbb{R}^2_{x^+}$ and $\mathbb{R}^2_{x^-}$ separately, $F(\cdot)$ and $G(\cdot)$ in the above can be **independent** to each other. In case $I = J = \mathbb{R}$, then $u(x, y)$ is defined on $\mathbb{R}^2_{x^+} \bigcup \mathbb{R}^2_{x^-}$. $\square$

Finding general sol. of the PDE (44).

Unlike the previous two examples, here the **integral curve on $\mathbb{R}^2_{x^+}$ will traverse into $\mathbb{R}^2_{x^-}$**. Hence, if we want to make $u(x,y)$ **consistent on both sides of y-axis (continuously across both sides of y-axis),** we must choose

$$F(w) = G(w) \quad \text{on some common interval } I \subseteq (0,\infty).$$

Hence any sol. $u(x,y)$ of the eq. on $\mathbb{R}^2 \backslash \{(0,0)\}$ must have the form

$$u(x,y) = F\left(x^2+y^2\right), \quad \text{where} \quad (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\} \tag{71}$$

for arbitrary C^1 func. $F(\cdot)$ def. on some open interval $I \subset (0,\infty)$. In case $F(w)$ is def. on $(0,\infty)$, then $u(x,y)$ is def. on $\mathbb{R}^2 \backslash \{(0,0)\}$. We call (71) the **general sol.** of the eq. (68) on $\mathbb{R}^2 \backslash \{(0,0)\}$. □

Finding general sol. of the PDE (44).

Remark: One can also choose the ch. of var. as $w = x^2 + y^2$, $z = x$, and do similar analysis on $\mathbb{R}^2_{y^+}$ and $\mathbb{R}^2_{y^-}$ respectively.

Remark: The func. $u(x, y) = \log(x^2 + y^2)$ is a sol. of $yu_x - xu_y = 0$ def. on the whole dom. $\mathbb{R}^2 \setminus \{(0, 0)\}$ (but not on $\mathbb{R}^2$).

Remark: The func. $u(x, y) = x^2 + y^2$ is a sol. of $yu_x - xu_y = 0$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$. However, it is also def. on the whole $\mathbb{R}^2$. That is, there exists a **non-constant solution** def. near the singularity $(0, 0)$ of the PDE $yu_x - xu_y = 0$. This is different from the eq. $xu_x + yu_y = 0$. See Example A..

Finding general sol. of the PDE (44).

Example

Find the general sol. of the eq.

$$u_x + yu_y = 0, \quad u = u(x, y), \quad (x, y) \in \mathbb{R}^2. \tag{72}$$

Note that the eq. has no singularity on $\mathbb{R}^2$.

Solution:

The system of ODE for the eq. is

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = y, \quad x(t) = t + x_0, \quad y(t) = y_0 e^t$$

or

$$\frac{dy}{dx} = \frac{y}{1}, \quad \int \frac{1}{y} dy = \int dx.$$

Finding general sol. of the PDE (44).

Each gives the implicit form for integral curves:

$$h(x,y) = e^{-x}y = C, \quad C \text{ is an arbitrary constant.}$$

One can see that $h(x,y) = e^{-x}y$ is a sol. of the eq. $u_x + yu_y = 0$ on $\mathbb{R}^2$. By $\frac{\partial h}{\partial x}(x,y) = -e^{-x}y$, $\frac{\partial h}{\partial y}(x,y) = e^{-x}$, we see that $\frac{\partial h}{\partial y}(x,y) \neq 0$ everywhere on $\mathbb{R}^2$. Hence we do the ch. of var.

$$\begin{cases} w = e^{-x}y, \\ z = x, \end{cases} \quad \text{with Jacobian } J(x,y) = -\frac{\partial h}{\partial y}(x,y) = -e^{-x} \neq 0 \text{ on } \mathbb{R}^2.$$

(if we choose $z = y$, then the Jacobian is $-e^{-x}y$, which is bad at $y = 0$). Thus this ch. of var. is a **bijection** from $\mathbb{R}^2$ (xy-space) to $\mathbb{R}^2$ (wz-space) with

$$\begin{cases} w = e^{-x}y, \\ z = x, \end{cases} \longleftrightarrow \begin{cases} x = z, \\ y = e^{z}w. \end{cases}$$

Finding general sol. of the PDE (44).

Now we have

$$\begin{aligned} 0 &= u_x + y u_y \\ &= \left(-U_w e^{-x} y + U_z\right) + y U_w e^{-x} = U_z, \quad U = U(w, z) \end{aligned}$$

and so the general sol. is

$$u(x, y) = U(w, z) = F(w) = F\left(e^{-x} y\right),$$

where $F(\cdot)$ is an arbitrary C^1 func. and if $F(\cdot)$ is def. on $(-\infty, \infty)$, then $u(x, y)$ is def. on $\mathbb{R}^2$. The sol. is const. along each char. curve $e^{-x} y = C$. □

Finding general sol. of the PDE (41).

We now focus on the general form for a first order linear PDE (41) with variable coeff., i.e.

$$a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = f(x,y)\,, \quad (x,y)\in\Omega, \tag{73}$$

where $a^2(x,y)+b^2(x,y)>0$ on Ω. The method is **the same as** that in solving the PDE (44). We can use the system of ODE (45) or the single ODE (49) to find the func. $h(x,y)$ which satis.

$$h(x,y) = C, \quad (x,y)\in\tilde{\Omega} \tag{74}$$

along each char. curve in $\tilde{\Omega}$.

Finding general sol. of the PDE (41).

After that we can use the ch. of var. (we **assume** $\frac{\partial h}{\partial x}(x,y) \neq 0$ on $\tilde{\Omega}$, which implies $b(x,y) \neq 0$ on $\tilde{\Omega}$)

$$\begin{cases} w = h(x,y), \\ z = y, \end{cases} \quad \text{with Jacobian } J(x,y) = \frac{\partial h}{\partial x}(x,y) \neq 0 \text{ on } \tilde{\Omega}$$

to **convert the PDE (73) into an ODE for** $U(w,z)$.
To see the details. we let

$$A(w,z),\ B(w,z),\ C(w,z),\ U(w,z),\ F(w,z)$$

be the func. corresponding
to $a(x,y),\ b(x,y),\ c(x,y),\ u(x,y),\ f(x,y)$.

Finding general sol. of the PDE (41).

By

$$\begin{aligned} & a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u \\ & = a(x,y)\,U_w h_x + b(x,y)\left(U_w h_y + U_z\right) + C(w,z)\,U \\ & = \Big(\underbrace{a(x,y)\,h_x + b(x,y)\,h_y}\Big)\,U_w + B(w,z)\,U_z + C(w,z)\,U \end{aligned}$$

and the identity $a(x,y)\,h_x + b(x,y)\,h_y = 0$, we see that the PDE (73) for the function $U(w,z)$ becomes an ODE of the form

$$B(w,z)\,U_z(w,z) + C(w,z)\,U(w,z) = F(w,z)\,, \qquad (75)$$

where $B(w,z) \neq 0$ on its domain (due to $b(x,y) \neq 0$ on $\tilde{\Omega}$). **Note that (75) is a first order linear ODE (in the variable z) containing a parameter w, which can be solved.**

Remark: For PDE (44), there are no $c(x,y)\,u$ and $f(x,y)$ terms and the ODE (**75**) becomes the trivial eq. $B(w,z)\,U_z(w,z) = 0$ (same as $U_z(w,z) = 0$). See (57).

Finding general sol. of the PDE (41).

Example

Find the general sol. of the eq.

$$u_x + yu_y + u = e^{x+y}, \quad u = u(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{76}$$

Note that the eq. has no singularity on $\mathbb{R}^2$.

Solution:

Similar to solving the eq. $u_x + yu_y = 0$, we
let $w = h(x,y) = e^{-x}y$, $z = x$, which is a **global** ch. of var. with Jacobian $J(x,y) = -e^{-x} \neq 0$ on $\mathbb{R}^2$. We have

$$\begin{aligned} u_x + yu_y + u &= \left(-U_w e^{-x} y + U_z\right) + y\left(U_w e^{-x}\right) + U \\ &= U_z + U = e^{z+e^z w}, \quad (w,z) \in \mathbb{R}^2 \end{aligned} \tag{77}$$

which is an ODE for $U(w,z)$ in the variable z with a parameter w.

Finding general sol. of the PDE (41).

We solve $U(w,z)$ to get

$$U(w,z) = e^{-z}\left[\int e^{2z}\cdot e^{e^z w}dz + F(w)\right]. \tag{78}$$

To find the integral, we let $s = e^z$ and get $ds = e^z dz$ (i.e. $dz = \frac{1}{s}ds$)

$$\underbrace{\int e^{2z}\cdot e^{e^z w}dz} = \int s^2\cdot e^{sw}\frac{1}{s}ds = \int se^{sw}ds = \int sd\left(\frac{1}{w}e^{sw}\right)$$

$$= \frac{s}{w}e^{sw} - \frac{1}{w}\int e^{sw}ds = \left(\frac{s}{w}-\frac{1}{w^2}\right)e^{sw} = \left(\frac{e^z}{w}-\frac{1}{w^2}\right)e^{e^z w}, \quad s = e^z$$

and so

$$U(w,z) = e^{-z}\left[\underbrace{\left(\frac{e^z}{w}-\frac{1}{w^2}\right)e^{e^z w}} + F(w)\right], \quad \text{where} \quad w = e^{-x}y, \quad z = x.$$

Finding general sol. of the PDE (41).

We obtain the general sol.

$$u(x,y) = e^{-x}\left[\left(\frac{e^x}{e^{-x}y} - \frac{1}{(e^{-x}y)^2}\right)e^y + F\left(e^{-x}y\right)\right]$$
$$= \underbrace{e^{x+y}\left(\frac{1}{y} - \frac{1}{y^2}\right)}\ \text{(particular sol.)} + e^{-x}F\left(e^{-x}y\right), \quad y \neq 0 \qquad (79)$$

where $F(\cdot)$ is an arbitrary C^1 func. and if $F(\cdot)$ is def. on $\mathbb{R}$, then $u(x,y)$ is def. on $\mathbb{R}^2 \backslash \{y=0\}$. □

Finding general sol. of the PDE (41).

Remark (important observation.): For PDE (76), it has no singularity at all; the func. $h(x,y) = e^{-x}y$ is defined on $\mathbb{R}^2$, and the ch. of var. $w = e^{-x}y,\ z = x$, is also good from $\mathbb{R}^2$ (xy-space) to $\mathbb{R}^2$ (wz-space) globally. In view of this, you may wonder why we have a particular sol. **undefined at $y = 0$**. This is due to our **indefinite integral sol. formula** (78) of the ODE (77). If we use the **definite integral sol. formula:**

$$U(w,z) = e^{-z}\left[\int_0^z e^{2\theta}\cdot e^{e^{\theta}w}d\theta + F(w)\right] = e^{-z}\int_0^z e^{2\theta}\cdot e^{e^{\theta}w}d\theta + e^{-z}F(w),$$

we will get ($w = e^{-x}y,\ z = x$)

$$u(x,y) = \underbrace{e^{-x}\int_0^x e^{2\theta}\cdot e^{e^{\theta}e^{-x}y}d\theta} + e^{-x}F\left(e^{-x}y\right),\quad (x,y)\in\mathbb{R}^2, \tag{80}$$

where the func. $\underbrace{\cdot}$ in (80) is a **smooth func. defined on** $\mathbb{R}^2$.

Finding general sol. of the PDE (41).

We can check that

$$
\begin{aligned}
& u_x + yu_y + u \\
= & \left\{ \begin{array}{l}
-e^{-x} \int_0^x e^{2\theta} \cdot e^{e^{\theta} e^{-x} y} d\theta + e^{-x} e^{2x} \cdot e^{y} \\
+e^{-x} \int_0^x e^{2\theta} \cdot e^{e^{\theta} e^{-x} y} \left(-e^{\theta} e^{-x} y \right) d\theta \\
+ye^{-x} \int_0^x e^{2\theta} \cdot e^{e^{\theta} e^{-x} y} \left(e^{\theta} e^{-x} \right) d\theta \\
+e^{-x} \int_0^x e^{2\theta} \cdot e^{e^{\theta} e^{-x} y} d\theta
\end{array} \right. \\
= & e^{-x} e^{2x} \cdot e^{y} = e^{x+y}, \quad \forall \ (x, y) \in \mathbb{R}^2.
\end{aligned}
$$

Finding general sol. of the PDE (41).

Another simple example is to compare the following:

(1) . The indefinite integral

$$\int e^{xy}\,dx = \frac{1}{y}e^{xy} + C\left(y\right)$$

will produce a singularity at $y = 0$ since for fixed x we have $\lim_{y\to 0} \frac{1}{y}e^{xy} = \pm\infty$. If we use Taylor series expansion for e^{xy}, we have

$$\begin{aligned}\frac{1}{y}e^{xy} &= \frac{1}{y}\left[1 + xy + \frac{\left(xy\right)^2}{2!} + \frac{\left(xy\right)^3}{3!} + \cdots\right] \\ &= \frac{1}{y}\ (\text{singular term}) + x + \frac{x^2 y}{2!} + \frac{x^3 y^2}{3!} + \cdots.\end{aligned}$$

Finding general sol. of the PDE (41).

(2) . On the other hand, if we look at the definite integral

$$\int_0^x e^{\theta y}\, d\theta = \frac{1}{y}e^{xy} - \frac{1}{y} = x\left(1 + \frac{xy}{2!} + \frac{x^2y^2}{3!} + + \frac{x^3y^3}{4!}\cdots\right), \qquad (81)$$

we see that the func. $\frac{1}{y}e^{xy} - \frac{1}{y}$ is actually a smooth func. def. on $\mathbb{R}^2$. According to (81), its value at $(x, 0)$ is defined as x for any $x \in \mathbb{R}$. One can also see this from the **L'Hospital rule**

$$\lim_{y\to 0} \frac{e^{xy} - 1}{y} = \lim_{y\to 0} \frac{d}{dy}\left(e^{xy} - 1\right) = x, \quad \forall\, x \in \mathbb{R}.$$

or from the limit

$$\lim_{y\to 0} \int_0^x e^{\theta y}\, d\theta = \int_0^x \lim_{y\to 0} e^{\theta y}\, d\theta = \int_0^x 1 d\theta = x, \quad \forall\, x \in \mathbb{R}.$$

ODE along a char. curve for eq. (41).

Remember that we have discussed the ODE along a char. line $bx - ay = \lambda$ for the eq.

$$au_x + bu_y + cu = f(x, y), \quad (x, y) \in \Omega,$$

where $a,\ b,\ c$ are const. (see eq. (25)) with $a \neq 0$ and $b \neq 0$. Now we can do the same for the eq.

$$a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = f(x, y), \quad (x, y) \in \Omega \tag{82}$$

and discuss the ODE along a char. curve C (parametrized by the integral curve $(x(t), y(t))$ of the ODE (45)).

ODE along a char. curve for eq. (41).

We have:

Lemma (Lemma C.)

Let $u(x,y)$ be a C^1 sol. of (82) on some open set Ω with $(x_0,y_0)\in\Omega$. Let $C:\alpha(t)=(x(t),y(t))\in\Omega,\ t\in I$ (with $0\in I$), be the unique integral curve of (82) satisfying $(x(0),y(0))=(x_0,y_0)\in C$, and lies in Ω. Then the func. $U(t)=u(x(t),y(t)),\ t\in I$, satis. the ODE

$$U'(t)+C(t)\,U(t)=F(t),\quad t\in I,\quad U(0)=u(x_0,y_0), \tag{83}$$

where $C(t)=c(x(t),y(t))$ and $F(t)=f(x(t),y(t))$. In particular, if we know the value of $U(0)$ (i.e. the value of $u(x_0,y_0)$), then we can know the value of $U(t)$ on the **whole interval** *I (by solving the ODE (83)).*

ODE along a char. curve. for eq. (41).

Proof: This is straightforward. We have

$$U'(t) = \frac{d}{dt} u(x(t), y(t)) = u_x(x(t), y(t)) x'(t) + u_y(x(t), y(t)) y'(t)$$
$$= a(x(t), y(t)) u_x(x(t), y(t)) + b(x(t), y(t)) u_y(x(t), y(t))$$
$$= -c(x(t), y(t)) u(x(t), y(t)) + f(x(t), y(t)), \quad t \in I. \tag{84}$$

Hence $U(t)$ satisfies the ODE (83) on I and if we know $U(0)$, then $U(t)$ can be expressed as

$$U(t) = e^{-\int_0^t C(\theta) d\theta} \left[\int_0^t \left(e^{\int_0^s C(\theta) d\theta} F(s) \right) ds + U(0) \right], \quad t \in I.$$

The result follows. □

Remark: In the special case when the equation is $a(x, y) u_x + b(x, y) u_y = 0$, (84) becomes $U'(t) = 0$, i.e. $u(x(t), y(t))$ is a **const. func.** along the char. curve C.

First order linear PDE with variable coeff. and side condition.

Due to Lemma C., a **"well-posed"** side condition on a curve γ is that **it intersects each char. curve C "transversally" at exactly one point** (so that **we have "exactly one" initial condition $U(0)$ for the ODE in Lemma C. on each char. curve**). Otherwise, the problem may have no sol. or infinitely many sol. This **"transversal condition"** is necessary for $u(x,y)$ to be differentiable across a char. curve. See Example B. below.

Example

Find the sol. of the problem with side condition

$$\begin{cases} yu_x - xu_y = 0, \quad u = u(x,y), \quad (x,y) \in \mathbb{R}^2_{x^+}, \\ u(s,s^2) = s^3, \quad s \in (0,\infty). \end{cases} \tag{85}$$

First order linear PDE with variable coeff. and side condition.

Solution:

We already know that the general sol. on $\mathbb{R}^2_{x^+}$ has the form $u(x,y) = F(x^2+y^2)$ for arbitrary C^1 func. $F(\cdot)$. Since the curve for the side condition is $\gamma = (s, s^2) \in \mathbb{R}^2_{x^+}$, $s \in (0,\infty)$, and it intersects each char. curve $x^2+y^2 = d > 0$ on $\mathbb{R}^2_{x^+}$ **transversally** at exactly one point, one should be able to find a **unique** $F(\cdot)$ satisfying the condition. For $(x,y) = (s, s^2)$, we need to solve the eq.

$$F(x^2+y^2) = F(s^2+s^4) = s^3, \quad s \in (0,\infty).$$

Let $r = s^2+s^4 > 0$. We want to solve s in terms of r, which is not easy. However, it is easy to solve $p := s^2 > 0$ in terms of s. We have $p^2+p-r = 0$, which gives

$$p = \frac{-1 \pm \sqrt{1+4r}}{2} > 0 \quad \text{(the minus sign does not make sense).}$$

First order linear PDE with variable coeff. and side condition.

Thus we have

$$F(r) = s^3 = p^{3/2} = \left(\frac{-1+\sqrt{1+4r}}{2} \right)^{3/2}$$

and the unique sol. of (85) is given by

$$u(x,y) = F\left(x^2+y^2\right) = \left(\frac{-1+\sqrt{1+4\left(x^2+y^2\right)}}{2} \right)^{3/2}, \quad (x,y) \in \mathbb{R}^2_{x^+}.$$

The sol. is defined on $\mathbb{R}^2_{x^+}$ and lies in the space $C^\infty\left(\mathbb{R}^2_{x^+}\right)$. To see this, we can decompose $u(x,y)$ as

$$u(x,y) = H(g(x,y)), \quad \text{where} \quad H(\theta) = \theta^{3/2} \in C^\infty(0,\infty)$$

and

$$g(x,y) = \frac{-1+\sqrt{1+4\left(x^2+y^2\right)}}{2} \in C^\infty\left(\mathbb{R}^2_{x^+}\right).$$

First order linear PDE with variable coeff. and side condition.

Since the composition of two smooth func. is a smooth func., we have

$$u(x,y) = \left(\frac{-1+\sqrt{1+4(x^2+y^2)}}{2} \right)^{3/2} \in C^\infty\left(\mathbb{R}^2_{x^+}\right),$$

and

$$u\left(s,s^2\right) = \left(\frac{-1+\sqrt{1+4(s^2+s^4)}}{2} \right)^{3/2}$$
$$= \left(\frac{-1+\left(2s^2+1\right)}{2} \right)^{3/2} = s^3, \quad \forall\, s \in (0,\infty).$$

□

First order linear PDE with variable coeff. and side condition.

Example

Find the sol. of the problem with side condition

$$\begin{cases} yu_x - xu_y + u = 0, \quad u = u(x,y), \quad (x,y) \in \mathbb{R}^2_{x^+}, \\ u(x,0) = h(x), \quad x \in (0,\infty). \end{cases} \tag{86}$$

Solution:

The curve for the side condition is $\gamma = (s,0) \in \mathbb{R}^2_{x^+}$, $s \in (0,\infty)$, and it intersects each char. curve $x^2 + y^2 = d > 0$ on $\mathbb{R}^2_{x^+}$ **transversally** at exactly one point. Therefore, one should be able to find a **unique** sol. for the problem (86). We find the general sol. of the PDE first. We already know that $h(x,y) = x^2 + y^2$ is a sol. of $yu_x - xu_y = 0$ on $\mathbb{R}^2_{x^+}$. Hence we do the ch. of var.

$$\begin{cases} w = x^2 + y^2, \\ z = y \end{cases} \qquad \text{Jacobian} = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x \neq 0 \ \text{ on } \ \mathbb{R}^2_{x^+}.$$

First order linear PDE with variable coeff. and side condition.

Now the new eq. for $U(w,z)$ becomes

$$y(U_w w_x) - x(U_w w_y + U_z z_y) + U = -xU_z + U = 0,$$

where $x = \sqrt{w - z^2}$, $w > z^2$, and the linear ODE for $U(w,z)$ is

$$U_z - \frac{1}{\sqrt{w - z^2}} U = 0$$

with integrating factor

$$e^{-\int \frac{1}{\sqrt{w-z^2}} dz} = e^{-\sin^{-1}\left(\frac{z}{\sqrt{w}}\right)},$$

where $w - z^2 = x^2 > 0$ on $\mathbb{R}^2_{x^+}$.

Hence we get

$$U(w,z) = e^{\sin^{-1}\left(\frac{z}{\sqrt{w}}\right)} \cdot F(w)$$

and the general sol. for $u(x,y)$ is

$$u(x,y) = e^{\sin^{-1}\left(y/\sqrt{x^2+y^2}\right)} \cdot F\left(x^2+y^2\right), \quad (x,y) \in \mathbb{R}^2_{x^+}, \tag{87}$$

where $F(\cdot)$ is an arbitrary C^1 func.. Hence we have $u(x,0) = F\left(x^2\right)$ and the unique sol. of the problem (86) is (we choose $F(w) = h\left(\sqrt{w}\right), \; w \in (0,\infty)$)

$$\begin{cases} u(x,y) = e^{\sin^{-1}\left(y/\sqrt{x^2+y^2}\right)} \cdot h\left(\sqrt{x^2+y^2}\right), & (x,y) \in \mathbb{R}^2_{x^+}, \\ u(x,0) = h(x). \end{cases}$$

The proof is done. □

First order linear PDE with variable coeff. and side condition.

Remark: One can also write the solution as

$$u(x,y) = e^{\tan^{-1}(y/x)} \cdot h\left(\sqrt{x^2+y^2}\right), \quad (x,y) \in \mathbb{R}^2_{x^+},$$

which looks better, due to

$$\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \tan^{-1}\left(\frac{y}{x}\right), \quad (x,y) \in \mathbb{R}^2_{x^+}.$$

Hint: use a right triangle to see the above identity.

First order linear PDE with variable coeff. and side condition.

What happens if the side condition is prescribed on a char. curve?
We look at the problem:

Example

Find the sol. of the problem with side condition

$$\begin{cases} yu_x - xu_y + u = 0, \quad u = u(x, y), \quad (x, y) \in \mathbb{R}^2_{x^+}, \\ u(\cos s, \sin s) = h(s), \quad s \in (-\pi/2, \pi/2), \end{cases} \tag{88}$$

where $h(s)$ is a given func. on $(-\pi/2, \pi/2)$. Note that the curve $\gamma = (\cos s, \sin s)$, $s \in (-\pi/2, \pi/2)$, for the side condition is a char. curve.

Solution:

By (87), we know the general sol. of the PDE is

$$u(x, y) = e^{\sin^{-1}\left(y/\sqrt{x^2+y^2}\right)} \cdot F\left(x^2 + y^2\right), \quad (x, y) \in \mathbb{R}^2_{x^+},$$

where $F(\cdot)$ is an arbitrary C^1 func.

First order linear PDE with variable coeff. and side condition.

Therefore, we have

$$u(\cos s, \sin s) = e^{\sin^{-1}(\sin s)} \cdot F(1) = e^{s} \cdot F(1), \quad s \in (-\pi/2, \pi/2).$$

By the above, if the given func. $h(s)$ has the form $h(s) = \lambda e^{s}$ for some const. λ, then any func. $F(\cdot)$ satisfying $F(1) = \lambda$ will give a sol. for the problem (88). Hence we have **infinitely many** solutions.
On the other hand, if $h(s)$ does not have the form $h(s) = \lambda e^{s}$, then (88) has **no sol.** at all. □

We can conclude the following: **For the first order linear PDE (41) with variable coeff., if the side condition is prescribed on a char. curve, then it may have infinitely many solutions or no solution at all.**

First order linear PDE with variable coeff. and side condition.

What happens if the side condition is prescribed on a curve which does not intersect some char. curve transversally? We look at the problem:

Example (Example B.)

Find the sol. of the problem with side condition

$$\begin{cases} u_x + u_y + u = 1, \quad u = u(x,y), \quad (x,y) \in \mathbb{R}^2, \\ u(s, \tan^{-1} s) = h(s), \quad s \in (-\infty, \infty), \end{cases} \tag{89}$$

where $h(s)$ is a given differentiable func. on $(-\infty, \infty)$.

First order linear PDE with variable coeff. and side condition.

Solution:

The curve for the side condition is $\gamma = \left(s, \tan^{-1} s\right), \ (-\infty, \infty)$, and it intersects each char. line $L_\lambda : x - y = \lambda, \ \lambda \neq 0$, on $\mathbb{R}^2$ **transversally** at exactly one point. However, for $\lambda = 0, \ \gamma$ is **tangent** to the line $L_0 : x - y = 0$ at $(0, 0)$, so it is **not transversal** at $(0, 0)$.
The general sol. of the PDE is

$$u(x, y) = 1 + e^{-x} C(x - y) \ \left(\text{or } 1 + e^{-y} C(x - y)\right)$$

for arbitrary C^1 func. $C(w)$. To satisfy the side condition we need to require

$$u\left(x, \tan^{-1} x\right) = 1 + e^{-x} C\left(x - \tan^{-1} x\right) = h(x), \quad x \in (-\infty, \infty),$$

which is the same as

$$C\left(x - \tan^{-1} x\right) = e^{x}\left(h(x) - 1\right), \quad x \in (-\infty, \infty), \tag{90}$$

First order linear PDE with variable coeff. and side condition.

where now $C(w)$ is def. on $(-\infty, \infty)$ and we note that

$$\frac{d}{dx}\left(x - \tan^{-1} x\right) = 1 - \frac{1}{1+x^2}$$
$$= \begin{cases} > 0, & \text{if } x \neq 0 \text{ (transversal)}, \\ = 0, & \text{if } x = 0 \text{ (not transversal)}. \end{cases}$$

If $C(w)$ is differentiable at $w = 0$, then by (90) we would have

$$\left.\frac{d}{dx}\right|_{x=0} C\left(x - \tan^{-1} x\right)$$
$$= C'(0) \left.\frac{d}{dx}\right|_{x=0} \left(x - \tan^{-1} x\right) = C'(0)(1-1) = 0.$$

First order linear PDE with variable coeff. and side condition.

Therefore, if the func. $h(x)$ satis.

$$\left.\frac{d}{dx}\right|_{x=0}\left[e^{x}\left(h(x)-1\right)\right]\neq 0, \tag{91}$$

we will get a **contradiction** due to the identity (90). We can conclude the following: Assume $h(s)$ is a given differentiable func. on $(-\infty,\infty)$ and it satis. (91). Then the sol. of problem (89) is given by

$$u(x,y)=1+e^{-x}C(x-y), \tag{92}$$

for some unique differentiable C^1 **func.** $C(w)$ **defined only on** $(-\infty,0)\bigcup(0,\infty)$. **To find** $C(w)$ **we need to solve** x **in terms of** w **from the identity**

$$w=x-\tan^{-1}x,\quad \frac{dw}{dx}=\begin{cases}>0, & \text{if } x\,(-\infty,0)\bigcup(0,\infty),\\ =0, & \text{if } x=0.\end{cases}$$

First order linear PDE with variable coeff. and side condition.

On $(0,\infty)$, $C(w)$ is uniquely given by

$$C(w) = e^{x(w)}\left(h(x(w)) - 1\right), \quad w \in (0,\infty), \tag{93}$$

where $x(w)$ is the **inverse differentiable func.** of $w = x - \tan^{-1} x$ on $x \in (0,\infty)$. Similarly, on $(-\infty, 0)$, $C(w)$ is uniquely given by

$$C(w) = e^{x(w)}\left(h(x(w)) - 1\right), \quad w \in (-\infty, 0), \tag{94}$$

where $x(w)$ is the inverse differentiable func. of $w = x - \tan^{-1} x$ on $x \in (-\infty, 0)$. $x(w)$ is **not** a differentiable func. of w near $w = 0$ due to $w'(0) = 0$.

The solution $u(x,y)$ given by (92) (with $C(w)$ given by (93) and (94)) is a C^1 sol. of problem (89) on $\mathbb{R}^2_{x>y} \bigcup \mathbb{R}^2_{x<y}$. It is not differentiable on the char. line $L_0 : x - y = 0$. □

Remark: Draw a picture of the graph of the func. $w = x - \tan^{-1} x$ on $x \in (0,\infty) \bigcup (-\infty, 0)$.

This is the end of first order linear PDE, 2025-3-10