PDE Teaching for Spring, 2022

Revised on 2022-3-7

Remark 0.1 *This notes is based on "Lecture-notes-on-PDE-first-part-2022.tex".*

1 First order linear PDE with constant coefficients.

Assume $u(x,y)$ is a C^1 function of two variables defined on some open set $\Omega \subseteq \mathbb{R}^2$. If $u(x,y)$ satisfies an equation of the form

$$F(x,y,u,u_x,u_y) = 0, \quad \forall\ (x,y) \in \Omega, \tag{1}$$

or, more precisely

$$F(x,y,u(x,y),u_x(x,y),u_y(x,y)) = 0, \quad \forall\ (x,y) \in \Omega,$$

we say it satisfies a **first order** PDE on Ω. Here $F = F(x,y,z,p,q) : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a given function defined on $\Omega \times \mathbb{R}^3$.

On the other hand, if we are given an equation of the form

$$F(x,y,u,u_x,u_y) = 0, \quad (x,y) \in \Omega \subseteq \mathbb{R}^2, \tag{2}$$

and you want to **solve** the first-order PDE, that means you want to find a C^1 function $u(x,y)$ defined at least on some **small open set** $\tilde{\Omega} \subseteq \Omega$ such that (2) is satisfied for all $(x,y) \in \tilde{\Omega}$. In general, the solution $u(x,y)$ you find cannot be defined on the whole Ω.

Remark 1.1 *In case the domain $\Omega \subset \mathbb{R}^2$ in (2) is not specified, which is often the case, then we usually pick a "**natural domain**" for it and we want to find a C^1 solution $u(x,y)$ defined on some **open set** $\tilde{\Omega} \subseteq \Omega$.*

If the function $F(x,y,z,p,q)$ is **linear** in the three variables $z,\ p,\ q$, we say the PDE is **linear**, otherwise it is **nonlinear**. Therefore, a general first-order linear PDE for a function $u(x,y)$ of two variables has the form:

$$a(x,y)\,u_x(x,y) + b(x,y)\,u_y(x,y) + c(x,y)\,u(x,y) = f(x,y), \quad u = u(x,y),$$

where $a(x,y),\ b(x,y),\ c(x,y),\ f(x,y)$ are given functions, which are assumed to be continuous on a common domain $\Omega \subseteq \mathbb{R}^2$.

Solving a PDE is, in general, very difficult, even for a first-order linear PDE. In this elementary course, in most cases, we will focus on simple first and second order **linear PDE (with constant coefficients most of the time)**. For simplicity of discussions, we will focus on a function $u(x,y)$ with two variables, but some theory of solution method can be easily generalized to a function $u(x_1, \ldots, x_n)$ with n variables.

1.1 A simple first order linear PDE with constant coefficients.

Consider the equation

$$3u_x(x,y) + 4u_y(x,y) = 0, \quad u = u(x,y), \quad F(x,y,z,p,q) = 3p + 4q \tag{3}$$

and we want to solve it. Its natural domain for (x,y) is $\mathbb{R}^2$. Without further conditions, the PDE has infinitely many C^1 solutions, some of them are defined on whole $\mathbb{R}^2$, some of them are defined on open subset of $\mathbb{R}^2$ only. Since this equation is linear with constant coefficients, we can try to use a **linear change of variables** to convert it into an ODE and solve it. Let (w, z) be the new variables given by

$$\begin{pmatrix} w \\ z \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}, \quad J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4}$$

where A, ..., D are constants to be determined later on. Under the change of variables, the function $u(x, y)$ becomes a new function $U(w, z)$, i.e.

$$U(Ax + By, Cx + Dy) = u(x, y),$$

and by the chain rule (since we seek for a C^1 solution $u(x, y)$, it is differentiable and the chain rule holds) we have

$$u_x(x, y) = U_w(w, z) A + U_z(w, z) C, \qquad u_y(x, y) = U_w(w, z) B + U_z(w, z) D, \tag{5}$$

which, in terms of matrix notation, is equivalent to the following **gradient vector relation:**

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial U}{\partial w} \\ \frac{\partial U}{\partial z} \end{pmatrix}, \quad J = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6}$$

or the **first-order differential operator relation**:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial z} \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} w \\ z \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

For later use, we also need to know the **second-order operator relation**:

$$\begin{cases} \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(A\frac{\partial}{\partial w} + C\frac{\partial}{\partial z}\right)(\cdots) = A^2\frac{\partial^2}{\partial w^2} + 2AC\frac{\partial^2}{\partial w \partial z} + C^2\frac{\partial^2}{\partial z^2} \\ \frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \left(A\frac{\partial}{\partial w} + C\frac{\partial}{\partial z}\right)(\cdots) = AB\frac{\partial^2}{\partial w^2} + (AD + BC)\frac{\partial^2}{\partial w \partial z} + CD\frac{\partial^2}{\partial z^2} \\ \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(B\frac{\partial}{\partial w} + D\frac{\partial}{\partial z}\right)(\cdots) = B^2\frac{\partial^2}{\partial w^2} + 2BD\frac{\partial^2}{\partial w \partial z} + D^2\frac{\partial^2}{\partial z^2}, \end{cases} \tag{8}$$

i.e. we have

$$\begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} \frac{\partial^2}{\partial w^2} & \frac{\partial^2}{\partial w \partial z} \\ \frac{\partial^2}{\partial w \partial z} & \frac{\partial^2}{\partial z^2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial^2}{\partial w^2} & \frac{\partial^2}{\partial w \partial z} \\ \frac{\partial^2}{\partial w \partial z} & \frac{\partial^2}{\partial z^2} \end{pmatrix} J, \tag{9}$$

or equivalently, the **Hessain matrix relation**:

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial^2 U}{\partial w^2} & \frac{\partial^2 U}{\partial w \partial z} \\ \frac{\partial^2 U}{\partial w \partial z} & \frac{\partial^2 U}{\partial z^2} \end{pmatrix} J, \quad J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{10}$$

Back to the PDE, under the change of variables, the equation for $U(w, z)$ becomes

$$\begin{aligned} 0 &= 3u_x(x, y) + 4u_y(x, y) = 3\left[AU_w(w, z) + CU_z(w, z)\right] + 4\left[BU_w(w, z) + DU_z(w, z)\right] \\ &= \Big(\underbrace{3A + 4B}\Big) U_w(w, z) + \Big(\underbrace{3C + 4D}\Big) U_z(w, z). \end{aligned} \tag{11}$$

In terms of inner product in linear algebra, the above is the same as

$$0 = \langle v, \ \nabla u \rangle = \left\langle v, \ J^T \nabla U \right\rangle = \langle Jv, \ \nabla U \rangle, \quad v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{12}$$

If we choose $A = 4,\ B = -3$, we get $(3C + 4D)\, U_z(w, z) = 0$ and if we require $3C + 4D \neq 0$ (this is precisely the change of variables condition $\det J = AD - BC \neq 0$), we will get

$$U_z(w, z) = 0 \quad \textbf{(this is just an ODE with parameter } w\textbf{)}. \tag{13}$$

After integration, we get

$$U(w, z) = F(w), \tag{14}$$

where $F(w)$ is an arbitrary C^1 function defined on some open interval in $\mathbb{R}$. Another way to see (14) is to note that for each fixed w, the function $U(w, z)$ is independent of z due to $U_z(w, z) = 0$; hence $U(w, z)$ is a constant C with respect to z. But this constant may depend on w, so we have $U(w, z) = C(w)$, which is exactly (14). Therefore, we conclude that any C^1 solution of the PDE (3) **must have the form**:

$$u(x, y) = U(w, z) = F(w) = F(4x - 3y) \tag{15}$$

for some C^1 function $F(w)$ defined on some open interval $w \in I$. On the other hand, it is easy to check that for arbitrary C^1 function $F(w),\ w \in I$, the function

$$u(x, y) = F(4x - 3y) \tag{16}$$

is a solution of the PDE (3) defined on some open set $\Omega \subseteq \mathbb{R}^2$, where $(x, y) \in \Omega$ if and only if $4x - 3y \in I$.

From the above discussion, we can conclude the following:

Theorem 1.2 *The "**general solution**" of the PDE (3) is given by*

$$u(x, y) = U(w, z) = F(w) = F(4x - 3y) \tag{17}$$

for arbitrary C^1 function $F(w)$ defined on some open interval $w \in I$.

Example 1.3 *If we choose $F(w) = w^2,\ w \in (-\infty, \infty)$, we get the solution $u(x, y) = (4x - 3y)^2$ and it is defined on $\mathbb{R}^2$; if $F(w) = \sqrt{w},\ w \in (0, \infty)$, then $u(x, y) = \sqrt{4x - 3y}$ is defined only on the half-plane $4x - 3y > 0$, ..., etc.*

Definition 1.4 *Any solution $u(x, y)$ of equation (3) has **constant value** along each line of the form $4x - 3y = \lambda$, where λ is a constant parameter. We call $4x - 3y = \lambda$ a **characteristic line** of the equation (3). A characteristic line has $v = (3, 4)$ as its generating vector.*

Remark 1.5 ***(Important.)*** *The **geometric meaning** of the equation $3u_x(x, y) + 4u_y(x, y) = 0$ is the following: the gradient vector ∇u is everywhere **perpendicular to** the constant vector $(3, 4)$, therefore **perpendicular to** a characteristic line. One can obtain characteristic lines by solving the system of ODE*

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 4 \tag{18}$$

*to get $x(t) = 3t + c_1,\ y(t) = 4t + c_2$, where $c_1,\ c_2$ are integration constants. Therefore, if $u(x, y)$ is a solution of the PDE, then it is a **constant function** when restricted to each **solution curve** of the ODE (18). That is: each **level curve** of $u(x, y)$ is a **solution curve** of the ODE (18). Draw a picture for this ....*

1.2 First order linear PDE with constant coefficients; general discussion.

Let $a,\ b,\ c$ be constants with $a \neq 0$ and $b \neq 0$ and $f(x,y)$ be a given **continuous** function defined on domain $\Omega \subseteq \mathbb{R}^2$. Consider the PDE

$$au_x(x,y) + bu_y(x,y) + cu(x,y) = f(x,y), \quad (x,y) \in \Omega \subseteq \mathbb{R}^2 \tag{19}$$

and we want to find a C^1 solution (or to find the **general solution** if possible) $u(x,y)$ defined at least on some open subset $\tilde{\Omega}$ of Ω.

Remark 1.6 *(**Decomposing the general solution**.) Note that if $u(x,y)$ and $v(x,y)$ are two solutions of (19) defined on a common domain $\tilde{\Omega} \subseteq \Omega$, their difference $w(x,y) = u(x,y) - v(x,y)$ will satisfy the **homogeneous** equation on $\tilde{\Omega}$:*

$$aw_x(x,y) + bw_y(x,y) + cw(x,y) = 0. \tag{20}$$

*By this, the general solution of (19) can be decomposed as a **particular solution** of the nonhomogeneous equation (19) plus the **general solution** of the homogeneous equation (20).*

Motivated by the method in the example in Section 1.1, we consider the linear **change of variables**:

$$\begin{cases} w = Ax + By \\ z = Cx + Dy \end{cases} \quad \text{i.e.} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad AD - BC \neq 0 \tag{21}$$

and denote the function corresponding to $u(x,y)$ as $U(w,z)$ (i.e. $U(Ax+By, Cx+Dy) = u(x,y)$). By the chain rule we have

$$\begin{aligned} &au_x(x,y) + bu_y(x,y) + cu(x,y) \\ &= a\left[U_w(w,z)A + U_z(w,z)C\right] + b\left[U_w(w,z)B + U_z(w,z)D\right] + cU(w,z) \\ &= \big(\underbrace{aA + bB}\big) U_w(w,z) + \big(\underbrace{aC + bD}\big) U_z(w,z) + cU(w,z), \end{aligned} \tag{22}$$

which indicates that the new equation for $U(w,z)$ will become an **ODE** if and only if one of the following is satisfied (**but not both**):

$$\text{either} \quad \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad \text{or} \quad \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} C \\ D \end{pmatrix} = 0. \tag{23}$$

By (23), we can choose (A,B) so that $(a,b)\cdot(A,B) = 0$ and choose (C,D) so that $(a,b)\cdot(C,D) \neq 0$. Hence we can pick the change of variables as:

$$\begin{cases} w = Ax + By = bx - ay \\ z = Cx + Dy \ = y \end{cases} \quad \text{i.e.,} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} b & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{Jacobian} = b \neq 0 \tag{24}$$

and conclude the **linear ODE** for $U(w,z)$:

$$\underbrace{bU_z(w,z) + cU(w,z)} = F(w,z) \quad \left(\underbrace{= f\left(\frac{w+az}{b}, z\right)}\right), \tag{25}$$

where

$$u(x,y) = U(bx - ay, y) \quad \text{and} \quad U(w,z) = u\left(\frac{w+az}{b}, z\right). \tag{26}$$

Remark 1.7 *Since here $a \neq 0,\ b \neq 0,$ it is also OK if we use the change of variables $w = bx - ay,\ z = x$.*

The ODE is defined on a domain Σ in wz-plane (Σ is the image of Ω in xy-plane under the linear map in (24); it is in one-one correspondence with Ω). By ODE theory we can solve (25) to get its general solution:

$$\begin{aligned} U(w,z) &= e^{-\frac{c}{b}z}\left(\frac{1}{b}\int e^{\frac{c}{b}z} f\left(\frac{w+az}{b}, z\right) dz + C(w)\right) \\ &= \underbrace{\frac{e^{-\frac{c}{b}z}}{b}\cdot \int e^{\frac{c}{b}z} f\left(\frac{w+az}{b}, z\right) dz}\, + \underline{e^{-\frac{c}{b}z}C(w)} := I(w,z) + II(w,z), \quad w = bx - ay, \quad z = y, \end{aligned} \tag{27}$$

where $C(w)$ is an arbitrary C^1 function of w ($C(w)$ is "**integration constant**" for the integral $\int dz$).

We note that $U(w,z) = I(w,z) + II(w,z)$ is the sum of a particular solution of $bU_z + cU = F$ and the general solution of $bU_z + cU = 0$. **Moreover, we see that $I(w,z)$ is defined on Σ. However, the domain of $II(w,z)$ depends on how you choose $C(w)$.** Back to $u(x,y)$, we will have

$$\begin{aligned} & u(x,y) \\ &= \begin{cases} \text{a } \textbf{particular solution} \text{ of } au_x + bu_y + cu = f \ (\textbf{this is defined on } \Omega) \\ + \text{ the } \textbf{general solution} \text{ of } au_x + bu_y + cu = 0 \ (\text{this is given by } e^{-\frac{c}{b}y}C(bx-ay)) \end{cases} \end{aligned} \tag{28}$$

and we note that **the domain of $e^{-\frac{c}{b}y}C(bx-ay)$ depends on how you choose** $C(w)$.

Note that (28) describes all possible solutions of (19) and is called the **general solution** of the equation (19).

Remark 1.8 *(**Omit this in class.**) This is to explain why the integral function $\int e^{\frac{c}{b}z} f\left(\frac{w+az}{b}, z\right) dz$ is defined on the domain Σ in wz-plane. Denote the integrand as $H(w,z)$ and we look at $\int H(w,z)\,dz$. For each fixed $w_0 \in \mathbb{R}$ with $(w_0, z_0) \in \Sigma$ for some z_0, since Σ is an open set in wz-plane, the cross section $\{z \in \mathbb{R} : (w_0, z) \in \Sigma\}$ is an open set O in $\mathbb{R}$ (a countable union of disjoint open intervals) and we note that the indefinite integral*

$$\int H(w_0, z)\,dz, \quad z \in O$$

is a well-defined function on O (since $H(w,z)$ is a C^1 function of $(w,z) \in \Sigma$). One can use an example to demonstrate this. We look at the ODE

$$u_x(x,y) = e^{xy}. \tag{29}$$

The function e^{xy} is a C^1 function on $\mathbb{R}^2$ and by integration we get

$$u(x,y) = \int e^{xy}dx = \begin{cases} \frac{1}{y}e^{xy}, & y \in (-\infty, 0)\bigcup(0,\infty), \quad y \neq 0 \\ x, & y = 0, \end{cases} + C(y) \tag{30}$$

*where $C(y)$ is an arbitrary C^1 function of $y \in \mathbb{R}^2$. At this moment, it is not clear that we can find a C^1 **particular solution** of the ODE (29) **defined on all** $\mathbb{R}^2$. However, if we use the **definite integral***

$$u(x,y) = \int_0^x e^{\theta y}d\theta = \begin{cases} \frac{1}{y}\left(e^{xy} - 1\right), & y \neq 0 \\ x, & y = 0, \end{cases} \quad \textit{(denote this function as } v(x,y)\textit{)} \tag{31}$$

then we can obtain the following form of general solution

$$u(x,y) = v(x,y) + C(y), \quad (x,y) \in \mathbb{R}^2. \tag{32}$$

One can see that the function $v(x,y)$ ***is indeed a*** C^1 ***function on*** $\mathbb{R}^2$ *due to the following (write* e^{xy} *as* $1 + xy + \frac{x^2y^2}{2!} + \frac{x^3y^3}{3!} + \cdots$*):*

$$\begin{cases} (1).\ \lim_{y\to 0} \frac{e^{xy}-1}{y} = x, \\ (2).\ v_x(x,y) = e^{xy},\ y \neq 0; \qquad v_x(x,0) = 1; \qquad \lim_{y\to 0} v_x(x,y) = 1, \\ (3A).\ v_y(x,y) = \frac{-1}{y^2}\left(e^{xy}-1\right) + \frac{x}{y}e^{xy}, \quad y \neq 0, \\ (3B).\ v_y(x,0) = \lim_{y\to 0} \frac{v(x,y)-v(x,0)}{y} = \lim_{y\to 0} \frac{\frac{1}{y}(e^{xy}-1)-x}{y} = \frac{x^2}{2}, \\ (3C).\ \lim_{y\to 0} v_y(x,y) = \lim_{y\to 0}\left(\frac{-1}{y^2}\left(e^{xy}-1\right) + \frac{x}{y}e^{xy}\right) = \lim_{y\to 0}\left(\frac{xye^{xy}-e^{xy}+1}{y^2}\right) = \frac{x^2}{2}. \end{cases} \tag{33}$$

Remark 1.9 *One can also express the general solution of* $au_x + bu_y + cu = 0$ *as* $u(x,y) = e^{-\frac{c}{a}x}C(bx-ay)$. *Reason: multiply* $e^{-\frac{c}{b}y}C(bx-ay)$ *by* $e^{\lambda(bx-ay)}$, *where* $\lambda = -c/ab$.

Remark 1.10 *(A trick to absorb the term cu.) For the homogeneous equation* $au_x + bu_y + cu = 0$, $a \neq 0,\ b \neq 0$, *you can let* $v(x,y) = e^{\frac{c}{b}y}u(x,y)$, *then it will satisfy* $av_x + bv_y = 0$. *similarly, you can let* $w(x,y) = e^{\frac{c}{a}x}u(x,y)$, *then it will satisfy* $aw_x + bw_y = 0$. *By this, we see that the general solution of the homogeneous equation* $au_x + bu_y + cu = 0$ *is*

$$u(x,y) = e^{-\frac{c}{a}x}C(bx-ay) \quad \text{or} \quad u(x,y) = e^{-\frac{c}{b}y}C(bx-ay), \tag{34}$$

where $C(w)$ *is arbitrary* C^1 *functions of* w.

Definition 1.11 *Any line L in xy-space of the form*

$$bx - ay = \lambda, \quad \lambda \text{ is a constant} \tag{35}$$

is called a ***characteristic line*** *of the PDE (19). Characteristic line* $bx - ay = \lambda$ *correspond to the coordinate line* $w = \lambda$ *in the* wz*-space.* ***We use characteristic lines to do change of variables.***

Remark 1.12 *(**Important.**) The above says that the* ***characteristic lines*** *play an important role in solving the PDE (19). They are the* ***coordinate lines*** *for the new variable* (w,z). ***Without using the characteristic lines, we cannot convert the PDE into ODE.*** *Draw a picture for this ....*

1.2.1 ODE along a characteristic line.

Note that if $c = 0$ and $f(x,y) = 0$ in the PDE (19), i.e. $au_x(x,y) + bu_y(x,y) = 0$, where $a \neq 0$ and $b \neq 0$, then we know its general solution is given by $u(x,y) = F(bx-ay)$, where $F(w)$ is an arbitrary C^1 function of w. In this situation, we know $u(x,y)$ is **constant along each characteristic line** $bx - ay = \lambda$.

On the other hand, if $c \neq 0$ or $f(x,y)$ is not a zero function, then $u(x,y)$ is in general **no longer a constant function along each characteristic line**. However, if we know the value of $u(x,y)$ at **a point** $p_0 = (x_0,y_0) \in L$, where L is the characteristic line passing through p_0 with equation $bx - ay = \lambda_0,\ \lambda_0 = bx_0 - ay_0$, then the function $u(x,y)$ will satisfy an ODE on L (with respect to the **parameter** of the line L; you can use x variable or y variable) and by solving the ODE we can know all values of $u(x,y)$ on L. More precisely, one can parametrize the line L as (here we use y variable as the parameter; you also use use other parameters like x or t)

$$L = \left\{ (x,y) = \left(\frac{\lambda_0 + ay}{b}, y \right), \quad y \in (-\infty, \infty) \right\}$$

and let

$$Q(y) = u\left(\frac{\lambda_0 + ay}{b}, y \right), \quad Q(y_0) = u(x_0, y_0) \text{ is known.}$$

Then from the PDE (for simplicity of discussion, here we assume both $u(x,y)$ and $f(x,y)$ are defined on the whole plane $\mathbb{R}^2$)

$$au_x(x,y) + bu_y(x,y) + cu(x,y) = f(x,y), \quad \forall\ (x,y) \in \mathbb{R}^2, \tag{36}$$

we can derive

$$\begin{aligned} Q'(y) &= \frac{a}{b} u_x \left(\frac{\lambda_0 + ay}{b}, y \right) + u_y \left(\frac{\lambda_0 + ay}{b}, y \right) \\ &= \frac{1}{b} f\left(\frac{\lambda_0 + ay}{b}, y \right) - \frac{c}{b} u \left(\frac{\lambda_0 + ay}{b}, y \right) = \frac{1}{b} f\left(\frac{\lambda_0 + ay}{b}, y \right) - \frac{c}{b} Q(y), \quad \forall\ x \in (-\infty, \infty), \end{aligned}$$

and so we obtain **the ODE along** L (which is similar to (25)):

$$bQ'(y) + cQ(y) = f\left(\frac{\lambda_0 + ay}{b}, y \right), \quad y \in (-\infty, \infty), \quad Q(y_0) = u(x_0, y_0) \text{ is known.} \tag{37}$$

By ODE theory, one can know $Q(y)$ for all $y \in (-\infty, \infty)$. We can summarize the following: if $u(x,y)$ satisfies the PDE (36) on $\mathbb{R}^2$, then it satisfies the ODE (37) on any characteristic line L (or: if $u(x,y)$ satisfies the PDE (19) on open domain $\Omega \subseteq \mathbb{R}^2$, then it satisfies the ODE (37) on any characteristic line L lying inside Ω).

1.2.2 Some examples.

Example 1.13 *Find the general solution of the PDE*

$$3u_x - 2u_y + u = x, \quad u = u(x,y). \tag{38}$$

Solution:

To make the change of variables and the ODE look better, we rewrite the equation as $-3u_x + 2u_y - u = -x$ with $a = -3,\ b = 2,\ c = -1$. According to the method, we introduce the change of variables

$$w = bx - ay = 2x + 3y, \quad z = y, \tag{39}$$

and the function $u(x,y)$ becomes $U(w,z)$, where by (25) we have

$$-3u_x(x,y) + 2u_y(x,y) - u(x,y) = 2U_z(w,z) - U(w,z) = -x = -\frac{w - 3z}{2},$$

and the ODE (in the variable z) for $U(w,z)$ is

$$U_z(w,z) - \frac{1}{2} U(w,z) = -\frac{1}{4}(w - 3z).$$

It has the solution

$$\begin{aligned} U(w,z) &= e^{\frac{z}{2}} \left(\int e^{-\frac{z}{2}} \left(-\frac{1}{4}(w - 3z) \right) dz + C(w) \right) \\ &= e^{\frac{z}{2}} \left(-\frac{w}{4} \int e^{-\frac{z}{2}} dz + \frac{3}{4} \int z e^{-\frac{z}{2}} dz + C(w) \right), \end{aligned}$$

where by

$$\int e^{-\frac{z}{2}}dz = -2e^{-\frac{z}{2}}, \quad \int ze^{-\frac{z}{2}}dz = -2ze^{-\frac{z}{2}} - 4e^{-\frac{z}{2}}$$

we get

$$\begin{aligned} U(w,z) &= e^{\frac{z}{2}}\left(-\frac{w}{4}\left(-2e^{-\frac{z}{2}}\right) + \frac{3}{4}\left(-2ze^{-\frac{z}{2}} - 4e^{-\frac{z}{2}}\right) + C(w)\right) \\ &= \frac{w}{2} - \frac{3}{2}z - 3 + e^{\frac{z}{2}}C(w). \end{aligned}$$

As a result, the general solution of the original equation is

$$u(x,y) = \frac{2x+3y}{2} - \frac{3}{2}y - 3 + e^{\frac{y}{2}}C(2x+3y) = x - 3 + e^{\frac{y}{2}}C(2x+3y),$$

where $C(\cdot)$ is an arbitrary C^1 function. □

Remark 1.14 *One can see that*

$$u(x,y) = \begin{cases} x-3 \text{ (a particular solution of (38))} \\ +e^{\frac{y}{2}}C(2x+3y) \text{ (the general solution of } 3u_x - 2u_y + u = 0\text{).} \end{cases}$$

Example 1.15 *Find the general solution of the PDE*

$$u_x + u_y + u = e^{x+2y}, \quad u = u(x,y). \tag{40}$$

<u>**Solution**</u>:

We do the change of variables

$$w = bx - ay = x - y, \quad z = y,$$

and get the **linear ODE** for $U(w,z)$:

$$U_z(w,z) + U(w,z) = e^{x+2y} = e^{(w+z)+2z} = e^{w+3z},$$

which we can solve it to get

$$U(w,z) = e^{-z}\left(\int e^z \cdot e^{w+3z}dz + C(w)\right) = e^{-z}\left(\frac{1}{4}e^{w+4z} + C(w)\right) = \frac{1}{4}e^{w+3z} + e^{-z}C(w).$$

Therefore, the general solution of (40) is given by

$$u(x,t) = U(x-y,y) = \frac{1}{4}e^{(x-y)+3y} + e^{-y}C(x-y) = \frac{1}{4}e^{x+2y} + e^{-y}C(x-y),$$

where the particular solution $e^{x+2y}/4$ for the equation is defined on $\mathbb{R}^2$ and the domain of $e^{-y}C(x-y)$ depends on your choice of $C(w)$. □

1.3 The physical meaning of the equation $cu_x + u_t = 0$.

Let $c > 0$ be a constant. The equation $cu_x + u_t = 0$, $u = u(x,t)$, is called a **simple transport equation**. Here $x \in \mathbb{R}$ represents space (one-dimensional) coordinate and $t \in \mathbb{R}$ represents time. Assume we have a fluid (water, say), moving to the right with **constant speed** c, along a horizontal **thin pipe** which we view it as a one-dimensional space with coordinate $x \in \mathbb{R}$. Let $u(x,t)$ be the

concentration of some substance suspended in the water. The amount of the substance in the interval $[0, x_0]$ (assume $x_0 > 0$) at time t_0 is given by (here we view the space dimension as 1)

$$M = \int_0^{x_0} u(x, t_0)\, dx.$$

A small time $h > 0$ later, the same substance (previously lies in the interval $[0, x_0]$) has moved to the right by distance $c \cdot h$. Hence we have the identity

$$M = \int_0^{x_0} u(x, t_0)\, dx = \int_{ch}^{x_0 + ch} u(x, t_0 + h)\, dx.$$

Note that the above identity is valid **for all** $x_0 > 0$ **and all** $h > 0$. Differentiation with respect to x_0 gives

$$u(x_0, t_0) = u(x_0 + ch, t_0 + h), \quad \forall\, x_0 > 0, \quad \forall\, h > 0,$$

which says that u is constant in $h \in (0, \infty)$ along the ray $(x_0 + ch, t_0 + h)$ for all $h > 0$) and differentiation with respect to h and letting $h = 0$ gives

$$cu_x(x_0, t_0) + u_t(x_0, t_0) = 0, \quad \forall\, x_0 > 0, \quad t_0 \in \mathbb{R}. \tag{41}$$

The same analysis on the interval $[x_0, 0]$, $x_0 < 0$, also give us the same equation. Hence we conclude

$$cu_x(x_0, t_0) + u_{t_0}(x_0, t_0) = 0, \quad \forall\, x_0 \in \mathbb{R}, \quad t_0 \in \mathbb{R}. \tag{42}$$

Note that for fixed (x_0, t_0), $u(x, t)$ is constant along the line $x - ct = x_0 - ct_0$ (it has a simple physical interpretation due to the identity $u(x_0, t_0) = u(x_0 + ch, t_0 + h)$ for any $h \in \mathbb{R}$). The general solution of the equation is given by $u(x, t) = F(x - ct)$ for arbitrary C^1 function $F(w)$.

1.4 First order linear PDE with constant coefficients plus side condition.

Consider equation (19). We know that it has infinitely many solutions if we do not impose any condition on the solutions. The purpose of the extra "**side condition**" is to ensure that the solution is unique. A general side condition has the form $u|_C = g$, where C is a given **curve** in the plane $\mathbb{R}^2$, **which intersects each characteristic line** L **"transversally" at exactly one point**, and g is a given function on C. In most cases, we only consider the case when $C = \tilde{L}$ is a **line** in $\mathbb{R}^2$. Now the side condition has the form

$$u(x, mx + d) = g(x), \quad \forall\, x \in \mathbb{R}, \tag{43}$$

where m is the slope of the line $\tilde{L}: y = mx + d$ and d is some number. Here $g(x)$ is a C^1 function of x ($g(x)$ can also be a constant, say $g(x) \equiv 0$). If the line $\tilde{L}$ is vertical, then the side condition has the form

$$u(d, y) = g(y), \quad \forall\, y \in \mathbb{R}. \tag{44}$$

The general fact we know is the following:

Theorem 1.16 ***(Roughly speaking.)*** *If the line* $\tilde{L}$ *is* ***not a characteristic line*** *of (19), then the PDE with the side condition has a* ***unique*** *solution. If the line* $\tilde{L}$ *is a* ***characteristic line****, the PDE with the side condition has* ***either no solution or infinitely many solutions****.*

Remark 1.17 ***(Important observation.) The key point is that if*** $\tilde{L}$ ***is not a characteristic line, then it intersects each characteristic line*** L ***at exactly one point. This allows us to determine the integration function*** $C(\cdot)$ ***uniquely.*** *Note that* $C(\cdot)$ *is constant along each characteristic line* L*.*

We shall not make the above theorem precise. Instead, we will just look at some examples to convince us the result.

Example 1.18 *The equation* $2u_x(x,y)+5u_y(x,y)=0$ *has characteristic lines of the form* $5x-2y=\lambda$ *(same as* $y=\frac{5}{2}x+\lambda$*),* $\lambda\in\mathbb{R}$*. Its general solution is given by*

$$u(x,y)=C(5x-2y), \tag{45}$$

for arbitrary C^1 *function* $C(\cdot)$*. Let* L *be the line* $y=\frac{5}{2}x+1$*, which* ***is a characteristic line****. Consider the side condition problem*

$$\begin{cases} 2u_x(x,y)+5u_y(x,y)=0 \\ u\left(x,\frac{5}{2}x+1\right)=g(x), \quad x\in(-\infty,\infty) \end{cases} \tag{46}$$

for some function $g(x)$*. By (45), we have*

$$u\left(x,\frac{5}{2}x+1\right)=C\left(5x-2\left(\frac{5}{2}x+1\right)\right)=C(-2) \text{ (this is a constant !!).} \tag{47}$$

Therefore, if $g(x)$ *is not a constant function, then (46) has* ***no*** *solution at all. On the other hand, if* $g(x)$ *is a constant function, say* $g(x)=10$*, then any* $u(x,y)$ *of the form* $u(x,y)=C(5x-2y)$ *is a solution as long as* C *satisfies* $C(-2)=10$*. In such a case, we have* ***infinitely many*** *solutions. Finally, if we replace the side condition by*

$$u(x,2x+7)=\sin x, \quad x\in(-\infty,\infty), \tag{48}$$

then the line $y=2x+7$ ***is not a characteristic line*** *and by (45) we can solve the equation*

$$u(x,2x+7)=C(5x-2(2x+7))=C(x-14)=\sin x, \quad x\in(-\infty,\infty)$$

to get $C(\xi)=\sin(\xi+14)$*,* $\xi\in(-\infty,\infty)$*. Therefore, the side condition problem (48) has the* ***unique*** *solution given by*

$$u(x,y)=C(5x-2y)=\sin(5x-2y+14), \quad (x,y)\in\mathbb{R}^2. \tag{49}$$

Example 1.19 *In Example 1.13, the general solution of the equation*

$$3u_x-2u_y+u=x \tag{50}$$

is

$$u(x,y)=x-3+e^{\frac{y}{2}}C(2x+3y), \tag{51}$$

and if we put the side condition as

$$u\left(x,-\frac{2}{3}x+2\right)=g(x), \quad x\in(-\infty,\infty), \tag{52}$$

for some $g(x)$*, then the line* $y=-\frac{2}{3}x+2$ ***is a characteristic line****, and we have*

$$u\left(x,-\frac{2}{3}x+2\right)=x-3+C(6)e^{-\frac{1}{3}x+1}, \quad x\in(-\infty,\infty)$$

Therefore, unless $g(x)$ *has the form* $g(x)=x-3+ke^{-\frac{1}{3}x+1}$ *for some constant* k*, we have* ***no*** *solution satisfying the side condition. On the other hand, if* $g(x)$ *is given by, say,* $g(x)=x-3+100e^{-\frac{1}{3}x+1}$*, there are* ***infinitely many*** *solutions satisfying this side condition everywhere on* $x\in(-\infty,\infty)$ *as long as we choose the function* $C(\cdot)$ *to satisfy* $C(6)=100$*.*

Example 1.20 *Consider the same equation as in (50), but now with the condition*

$$u(x, 4x-2) = g(x), \quad x \in (-\infty, \infty) \tag{53}$$

for some $g(x)$*. The line* $y = 4x - 2$ ***is not a characteristic line****, and by (51) we need to require*

$$u(x, 4x-2) = x - 3 + e^{2x-1} C(14x - 6) = g(x), \quad \forall\, x \in (-\infty, \infty),$$

i.e.

$$C(14x-6) = \frac{g(x) - x + 3}{e^{2x-1}}, \quad x \in (-\infty, \infty),$$

which implies (let $\xi = 14x - 6,\ x = \frac{1}{14}(\xi + 6)$*) the* ***unique choice*** *of the integration constant function* $C(\cdot)$*, namely*

$$C(\xi) = \frac{g\left(\frac{1}{14}(\xi+6)\right) - \frac{1}{14}(\xi+6) + 3}{e^{\frac{2}{14}(\xi+6)-1}}, \quad \xi \in (-\infty, \infty).$$

Hence for the side condition (53) with arbitrary function $g(x)$*, the unique solution* $u(x,y)$ *is*

$$u(x,y) = x - 3 + e^{\frac{y}{2}} \left(\frac{g\left(\frac{1}{14}(2x+3y+6)\right) - \frac{1}{14}(2x+3y+6) + 3}{e^{\frac{2}{14}(2x+3y+6)-1}} \right).$$

It satisfies $u(x, 4x-2) = g(x)$ *for all* $x \in (-\infty, \infty)$*.*

Example 1.21 *Solve the equation*

$$\begin{cases} u_x(x,y) - u_y(x,y) + 2u(x,y) = 1 \\ u(x,0) = x^2, \quad x \in (-\infty, \infty). \end{cases} \tag{54}$$

Solution:

By (27), one can check (or "guess") that the general solution of $u_x - u_y + 2u = 1$ is

$$u(x,y) = \frac{1}{2} + e^{2y} C(-x-y) \quad \left(\text{same as } \frac{1}{2} + e^{2y} C(x+y)\right)$$

for arbitrary function $C(\cdot)$. The side condition is prescribed on the x-axis, which is **not a characteristic line**. We need to choose $C(\cdot)$ so that

$$u(x,0) = \frac{1}{2} + C(x) = x^2.$$

Hence we have $C(x) = x^2 - 1/2$ and then

$$u(x,y) = \frac{1}{2} + e^{2y} C(x+y) = \frac{1}{2} + e^{2y} \left[(x+y)^2 - \frac{1}{2} \right], \quad (x,y) \in \mathbb{R}^2 \tag{55}$$

is the unique solution of (54). □

Example 1.22 *Solve the equation*

$$\begin{cases} u_x(x,y) + 2u_y(x,y) - 4u(x,y) = e^{x+y} \\ u(x, 4x+2) = 0, \quad x \in (-\infty, \infty). \end{cases} \tag{56}$$

Solution:

By (27), one can check (or "guess") that the general solution of $u_x + 2u_y - 4u = e^{x+y}$ is

$$u(x,y) = -e^{x+y} + e^{2y} C(2x - y).$$

By the side condition, we require (the line $y = 4x + 2$ is **not a characteristic line**)

$$0 = u(x, 4x+2) = -e^{5x+2} + e^{8x+4} C(-2x - 2),$$

i.e. we need to require $C(-2x-2) = e^{-3x-2}$. To get $C(r)$, we let $r = -2x - 2$ and one can solve x in terms of r to get

$$-3x - 2 = -3\left(\frac{r+2}{-2}\right) - 2 = \frac{3}{2}r + 1.$$

Hence $C(r) = e^{\frac{3}{2}r+1}$ and the unique solution is

$$u(x,y) = -e^{x+y} + e^{2y} e^{\frac{3}{2}(2x-y)+1} = -e^{x+y} + e^{3x+\frac{1}{2}y+1}, \quad (x,y) \in \mathbb{R}^2. \tag{57}$$

□

In the next example, the side condition has the form $u|_C = g$, where C is a **curve** in $\mathbb{R}^2$, not a line.

Example 1.23 *Solve the equation*

$$\begin{cases} u_x(x,y) - u_y(x,y) + u(x,y) = 0 \\ u(x, x^3) = e^{-x}(x + x^3), \quad x \in (-\infty, \infty). \end{cases} \tag{58}$$

Solution:

Here u is specified on the **curve** (unlike the previous examples, it is not a line) $y = x^3$. **This curve intersects each characteristic line at exactly one point, which is good (otherwise, the solution may not exist).** The general solution of $u_x - u_y + u = 0$ is given by $u(x,y) = e^y C(-x-y)$ (same as $u(x,y) = e^y C(x+y)$) for arbitrary function $C(\cdot)$. Then we need to solve

$$u\left(x, x^3\right) = e^{x^3} C\left(x + x^3\right) = e^{-x}\left(x + x^3\right)$$

and get

$$C\left(x + x^3\right) = \frac{e^{-x}\left(x + x^3\right)}{e^{x^3}} = e^{-\left(x+x^3\right)}\left(x + x^3\right). \tag{59}$$

Hence $C(r) = e^{-r} r$ and the unique solution is

$$u(x,y) = e^y C(x+y) = e^y (x+y) e^{-(x+y)} = (x+y) e^{-x}, \quad (x,y) \in \mathbb{R}^2. \tag{60}$$

□

Remark 1.24 *In the above example, the curve $y = x^3$ intersects each characteristic line $x + y = \lambda,\ \lambda \in (-\infty, \infty)$, at* ***exactly one point****. With this, we have the unique solution $u(x,y)$ satisfying the side condition. If this is not the case, the solution* ***may not exist or may not be unique****. If we replace the side condition by*

$$u\left(x, x^2\right) = e^{-x}\left(x + x^3\right), \quad x \in (-\infty, \infty),$$

then we note that the curve $y = x^2$ intersects each characteristic line $x + y = \lambda$, $\lambda \in (-\infty, \infty)$, at either two points, or one point, or no intersection at all. Now the condition (59) becomes

$$C\left(x + x^2\right) = e^{-\left(x+x^2\right)}\left(x + x^3\right), \quad x \in (-\infty, \infty). \tag{61}$$

One can check that it is ***"impossible"*** *to find a function $C(\cdot)$ satisfying $C(x + x^2) = e^{-\left(x+x^2\right)}(x + x^3)$ for all $x \in (-\infty, \infty)$. Exercise: Show that it is* ***impossible*** *to find a C^1 function $C(\theta)$ defined on $\theta \in [-1/4, \infty)$ satisfying $C(x + x^2) = e^{-\left(x+x^2\right)}(x + x^3)$ for all $x \in (-\infty, \infty)$. Hint: look at $x = 1$ and $x = -2$.*

Remark 1.25 *In general, if we want to solve an equation of the form $C(h(x)) = g(x)$, then if $h'(x) > 0$ (or < 0) for all $x \in (-\infty, \infty)$, one can solve $x = h^{-1}(r)$ and get the function $C(r) = g(h^{-1}(r))$. It satisfies $C(h(x)) = g(h^{-1}(h(x))) = g(x)$.*

2 First order linear PDE with variable coefficients.

In this section, we look at first order linear PDE for a 2-variable function $u = u(x, y)$ with variable coefficients. The equation has the form

$$\underbrace{a(x,y)\,u_x + b(x,y)\,u_y}+c(x,y)\,u = f(x,y), \quad u = u(x,y), \quad (x,y) \in \Omega \subseteq \mathbb{R}^2, \tag{62}$$

where the coefficients $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$ are given C^1 functions defined on some common open set $\Omega \subseteq \mathbb{R}^2$, with

$$a^2(x,y) + b^2(x,y) > 0 \quad \text{on} \quad \Omega. \tag{63}$$

To solve (62) means to find a C^1 function $u(x,y)$ defined on some **open subset** $\tilde{\Omega}$ of Ω which satisfies (62) on $\tilde{\Omega}$. The solution method will involve some ODE theory.

Remark 2.1 *In case (62) does not specify its domain $\Omega \subseteq \mathbb{R}^2$ for the equation, we usually take $\Omega \subseteq \mathbb{R}^2$ to be the* ***largest natural domain****, which means that $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$ are all defined on Ω with $a^2(x,y) + b^2(x,y) > 0$ on Ω.*

Remark 2.2 *The reason of assuming $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$ to be C^1 functions on Ω (not just C^0 functions on Ω) is to ensure that the system of ODE (69) below has a* ***unique solution*** *passing through each point $p \in \Omega$. This is because the vector field $V(x,y) = (a(x,y), b(x,y))$ is a C^1 vector field.*

Our purpose is to find a "**general solution**", which can encompass as many solutions as possible. Here, roughly speaking, "**general**" means that we have a **solution formula** and it contains an arbitrary **integration function** $C(w)$ as in formula (27).

In this elementary course, we will be satisfied if we can find some open subset $\tilde{\Omega}$ of Ω and you can find a general solution formula for the PDE (62) on $\tilde{\Omega}$.

The following example says that the **nondegenerate condition** "$a^2(x,y) + b^2(x,y) > 0$ on Ω" is important. In case there is some point $(x_0, y_0) \in \Omega$ such that $a^2(x_0, y_0) + b^2(x_0, y_0) = 0$ (i.e. the PDE **degenerates** at (x_0, y_0)), then it is possible that the equation (62) has **no** solution defined near $(x_0, y_0) \in \Omega$. We have:

Example 2.3 *Consider the equation given by*

$$xu_x(x,y) + yu_y(x,y) = e^{x+y}, \quad (x,y) \in \Omega = \mathbb{R}^2 \setminus \{(0,0)\}, \quad x^2 + y^2 > 0 \text{ on } \Omega. \tag{64}$$

We note that if we take $\Omega = \mathbb{R}^2$, then the condition (63) is not satisfied at $(0,0)$ (i.e. the PDE has a ***singularity*** *(i.e.* ***degenerate****) at $(0,0)$). One can see that* ***"any"*** C^1 ***solution*** $u(x,y)$ ***cannot be defined on any open set containing the origin*** $(0,0)$. *Otherwise, we will obtain the absurd identity*

$$0 = 0u_x(0,0) + 0u_y(0,0) = e^{0+0} = 1.$$

Therefore, the ***natural domain*** *for the PDE (64) is $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$.*

It is also possible that the equation (62) has only **trivial solution** (i.e. **constant solution**) defined near $(x_0, y_0) \in \Omega$ if we have $a^2(x_0,y_0) + b^2(x_0,y_0) = 0$, which is of no interest at all. We have:

Lemma 2.4 *Let $B_R(0) \subset \mathbb{R}^2$ be an open ball centered at the origin $0 = (0,0)$ with radius $R > 0$. Show that if $u(x,y)$, defined on $B_R(0) \subset \mathbb{R}^2$, is a C^1 solution of the PDE*

$$xu_x(x,y) + yu_y(x,y) = 0, \quad (x,y) \in B_R(0), \tag{65}$$

then it must be a ***constant solution*** *on $B_R(0)$.*

Proof. I will leave its proof as an homework problem for you. See HW 3. □

Remark 2.5 ***The phenomena in Example 2.3 and Lemma 2.4 will not happen if*** $a^2(x_0,y_0) + b^2(x_0,y_0) > 0$.

Example 2.6 *For $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$, there exists a* ***non-constant*** *solution defined on the whole Ω. You can check that (see HW3) the function*

$$u(x,y) = \begin{cases} e^{-y^2/x^2}, & x \neq 0, \quad y \in \mathbb{R} \\ 0, & x = 0, \quad y \neq 0 \in \mathbb{R}, \end{cases} \tag{66}$$

lies in the space $C^1(\Omega)$ and satisfies equation (65) on $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$. However, note that the limit $\lim_{(x,y)\to(0,0)} u(x,y)$ ***does not*** *exist. Another C^1 solution is $u(x,y) = y/x$, which is defined only on the open subset $\tilde{\Omega}$ of Ω, where $\tilde{\Omega} = \Omega \setminus \{y\text{-axis}\}$. Below, we will show you how to find the* ***general solution*** *of (65) on some open subset $\tilde{\Omega}$ of Ω.*

We first look at the easier case

$$a(x,y)\,u_x + b(x,y)\,u_y = 0, \quad u = u(x,y), \tag{67}$$

where $a(x,y)$, $b(x,y)$ are given C^1 functions on $\Omega \subseteq \mathbb{R}^2$ with $a^2 + b^2 > 0$ on Ω and here we also assume that $u(x,y)$ is a C^1 solution **defined on** Ω. One can rewrite the equation as

$$a(x,y)\,u_x + b(x,y)\,u_y = \nabla u(x,y) \cdot \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} = 0, \quad (x,y) \in \Omega. \tag{68}$$

If we view the vector $V(x,y) = (a(x,y), b(x,y))$ as a C^1 vector field on Ω (the vector field $V(x,y)$ has **no equilibrium point** on Ω due to $a^2(x,y) + b^2(x,y) > 0$ on Ω), then (68) has a **geometric meaning**. It says that if a C^1 curve $C : \alpha(t) = (x(t), y(t)) \in \Omega$, $t \in I$ (some open interval), satisfies the system of ODE (such a C^1 curve is called a **solution curve** of the vector field $V(x,y)$ on Ω)

$$\begin{cases} \dfrac{dx}{dt} = a(x,y), \\ \dfrac{dy}{dt} = b(x,y), \quad t \in I, \end{cases} \tag{69}$$

then by the chain rule we have

$$\begin{aligned}&\frac{d}{dt}u(x(t),y(t)) = u_x(x(t),y(t))\frac{dx}{dt} + u_y(x(t),y(t))\frac{dy}{dt}\\&= a(x(t),y(t))\,u_x(x(t),y(t)) + b(x(t),y(t))\,u_y(x(t),y(t)) = 0, \quad \forall\, t \in I.\end{aligned}$$

Therefore, we can conclude:

Lemma 2.7 *Let $V(x,y) = (a(x,y), b(x,y))$ be a C^1 vector field on $\Omega \subseteq \mathbb{R}^2$ with $a^2(x,y) + b^2(x,y) > 0$ on Ω. Then $u(x,y)$ is a C^1 solution of the homogeneous equation*

$$a(x,y)\,u_x + b(x,y)\,u_y = 0, \quad u = u(x,y), \quad (x,y) \in \Omega \tag{70}$$

*on Ω **if and only if** for **any** C^1 solution curve $C : \alpha(t) = (x(t), y(t))$, $t \in I$, of the ODE (69) lying on Ω, the function $u(\alpha(t))$, $t \in I$, is a **constant** function along the curve C.*

Proof. The direction $\Longrightarrow$ is clear. For the direction $\Longleftarrow$, we note that at any point $p = (x_0, y_0) \in \Omega$ (open set), there is a C^1 solution curve $C : \alpha(t) = (x(t), y(t))$, $t \in (-\varepsilon, \varepsilon)$, $\alpha(0) = p$, lying on Ω and passing through it. By the assumption we have

$$\frac{d}{dt}u(x(t),y(t)) = a(x(t),y(t))\,u_x(x(t),y(t)) + b(x(t),y(t))\,u_y(x(t),y(t)) = 0, \quad \forall\, t \in I,$$

which, at $t = 0$, implies $a(p)\,u_x(p) + b(p)\,u_y(p) = 0$. Since $p \in \Omega$ can be arbitrary, $u(x,y)$ satisfies the equation (70) on Ω. □

Definition 2.8 *A curve C in the plane is called a **characteristic curve** of the PDE (67) (or the more general PDE (62)) if at each point $(x,y) \in C$, the vector $V(x,y) = (a(x,y), b(x,y))$ is **tangent** to C at (x,y). By a **suitable parametrization** (i.e., by solving the ODE (69)), a **characteristic curve** C can be parametrized as $C : \alpha(t) = (x(t), y(t)) \in \Omega$, $t \in I$, where $x(t)$ and $y(t)$ satisfy the system of ODE (69) on some interval I.*

Remark 2.9 *In Lemma 2.7, the constant may be different on different solution curves. The lemma says that a **solution curve** $\alpha(t) = (x(t), y(t))$, $t \in I$, is a **level curve** (in parametric form) of the function $u(x,y)$ lying inside Ω. Thus we have*

$$\textbf{solution curve of ODE } \textit{(69)} = \textbf{characteristic curve} = \textbf{level curve of } u. \tag{71}$$

Remark 2.10 ***(Important.)** Roughly speaking, if we can know "**all solution curves** of the ODE (69) inside Ω", then one can find solutions for PDE (67) on Ω. If the PDE has a **side condition,** then we can obtain an unique solution (if the curve C in the side condition intersects each solution curve transversally at exactly one point).*

2.1 Finding general solution of the PDE (67).

Same as before, the characteristic curves play an important role in solving the PDE (67) (or the more general PDE (62)). We solve the ODE (69) first and obtain a family of solutions $(x(t), y(t))$ with arbitrary initial data $(x(0), y(0)) \in \Omega$ (each solution is defined on some maximal time interval I with $0 \in I$). By ODE theory, solution curves with different initial data **will not intersect at all**. Therefore, the domain Ω can be viewed as the disjoint union of **all** solution curves in Ω. We then try to convert (**by deleting the time variable** t or **by the Inverse Function Theorem** or by **rewriting the system as** $dy/dx = b(x,y)/a(x,y)$ **and solving it to get a relation between** x **and** y) this family of ODE solutions $(x(t), y(t))$ into the **implicit form**

$$h(x,y) = d \tag{72}$$

where $h(x,y)$ is a C^1 function and d is an **integration constant** (serving as a parameter). The above function $h(x,y)$, when restricted to each solution curve $(x(t), y(t))$, is a constant function.

In general, the function $h(x,y)$ is defined only on some open subset $\tilde{\Omega}$ of Ω. However, it is also possible for $h(x,y)$ to be defined on all Ω. Moreover, since the implicit form in (72) is **not unique** in general, **the domain of $h(x,y)$ may depend on how you choose your $h(x,y)$**. For example, if the equation is $h(x,y) = y/x = d$, then the domain is either $\mathbb{R}_{x^+}$ or $\mathbb{R}_{x^-}$, but if one derive the equivalent equation $h(x,y) = x/y = d$, then the domain is either $\mathbb{R}_{y^+}$ or $\mathbb{R}_{y^-}$. As the constant d varies, the identity $h(x,y) = d$ describes different characteristic curves (in implicit form, not in parametric form) of the equation $a(x,y) u_x + b(x,y) u_y = 0$ lying inside $\tilde{\Omega}$.

Finally, we note that, since the vector field $V(x,y) = (a(x,y), b(x,y))$ is everywhere **nonzero** on Ω, by **restricting the domain $\tilde{\Omega}$ to a smaller open subset** if necessary, the function $h(x,y) : \tilde{\Omega} \to \mathbb{R}$ will not be a constant function on $\tilde{\Omega}$, and we must have either

$$\frac{\partial h}{\partial x}(x,y) \neq 0 \quad \text{everywhere on} \quad \tilde{\Omega} \tag{73}$$

or

$$\frac{\partial h}{\partial y}(x,y) \neq 0 \quad \text{everywhere on} \quad \tilde{\Omega}. \tag{74}$$

The next important result is the following:

Lemma 2.11 *The above C^1 function $h(x,y) : \tilde{\Omega} \to \mathbb{R}$ is a **solution** of the PDE (67) on $\tilde{\Omega}$. Moreover, if $\frac{\partial h}{\partial x}(x,y) \neq 0$ everywhere on $\tilde{\Omega}$, we must have $b(x,y) \neq 0$ everywhere on $\tilde{\Omega}$; and if $\frac{\partial h}{\partial y}(x,y) \neq 0$ everywhere on $\tilde{\Omega}$, we must have $a(x,y) \neq 0$ everywhere on $\tilde{\Omega}$.*

Proof. For each $(x(0), y(0)) \in \tilde{\Omega}$, one can solve the ODE (69) to get an unique solution curve $(x(t), y(t)) \in \tilde{\Omega}$ for $t \in I$ (some interval containing $t = 0$). Since we have

$$h(x(t), y(t)) = d, \quad \forall\, t \in I, \tag{75}$$

chain rule implies

$$\begin{aligned} &\frac{\partial h}{\partial x}(x(t), y(t)) \frac{dx}{dt}(t) + \frac{\partial h}{\partial y}(x(t), y(t)) \frac{dy}{dt}(t) \\ &= a(x(t), y(t)) \frac{\partial h}{\partial x}(x(t), y(t)) + b(x(t), y(t)) \frac{\partial h}{\partial y}(x(t), y(t)) = 0, \quad \forall\, t \in I. \end{aligned}$$

In particular, at $t = 0$, we get

$$a(x(0), y(0)) \frac{\partial h}{\partial x}(x(0), y(0)) + b(x(0), y(0)) \frac{\partial h}{\partial y}(x(0), y(0)) = 0. \tag{76}$$

As $(x(0), y(0)) \in \tilde{\Omega}$ can be arbitrary, we conclude

$$a(x,y) \frac{\partial h}{\partial x}(x,y) + b(x,y) \frac{\partial h}{\partial y}(x,y) = 0, \quad \forall\ (x,y) \in \tilde{\Omega}. \tag{77}$$

Hence $h(x,y) : \tilde{\Omega} \to \mathbb{R}$ is a **solution** of the PDE (67) on $\tilde{\Omega}$.

Finally, if $\frac{\partial h}{\partial x}(x,y) \neq 0$ on $\tilde{\Omega}$, and $b(x_0, y_0) = 0$ at some $(x_0, y_0) \in \tilde{\Omega}$, then (77) gives

$$0 = a(x_0, y_0) \frac{\partial h}{\partial x}(x_0, y_0) + b(x_0, y_0) \frac{\partial h}{\partial y}(x_0, y_0) = a(x_0, y_0) \frac{\partial h}{\partial x}(x_0, y_0),$$

which will imply $a(x_0, y_0) = 0$, a contradiction due to our main assumption $a^2 + b^2 > 0$ on Ω. Similarly, if $\frac{\partial h}{\partial y}(x,y) \neq 0$ on $\tilde{\Omega}$, we must have $a(x,y) \neq 0$ on $\tilde{\Omega}$. The proof is done. □

Remark 2.12 ***However, at this moment, the function*** $h(x,y)$ ***is not yet a "general solution" of the PDE on*** $\tilde{\Omega}$***.***

Example 2.13 *Consider the equation*

$$xu_x + yu_y = 0, \quad u = u(x,y), \quad (x,y) \in \Omega = \mathbb{R}^2 \backslash \{(0,0)\}. \tag{78}$$

If we solve the system of ODE (69), we get $x(t) = x_0 e^t$ *and* $y(t) = y_0 e^t$, $t \in (-\infty, \infty)$, *where* $(x_0, y_0) \neq (0,0)$. *We can use one of the following three ways to find* $h(x,y)$:

1. Cancel the parameter t to get a relation between x and y. An obvious way is to look at the expression x/y (if $y_0 \neq 0$) or y/x (if $x_0 \neq 0$). Therefore, we get the identity

$$\frac{x}{y} = \frac{x_0 e^t}{y_0 e^t} = \frac{x_0}{y_0} := d \quad \text{(missing } x\text{-axis)}, \quad d \text{ is a constant}$$

   or the identity

$$\frac{y}{x} = \frac{y_0 e^t}{x_0 e^t} = \frac{y_0}{x_0} := d \quad \text{(missing } y\text{-axis)}, \quad d \text{ is a constant}$$

   along each solution curve. Note that the function $h(x,y) = x/y$ (or y/x) cannot be defined on the whole Ω due to the denominator. One can check that $u(x,y) = x/y$ (or y/x) is a C^1 solution of the PDE (78) defined on $\Omega \backslash \{x\text{-axis}\}$ ($\Omega \backslash \{y\text{-axis}\}$).

2. Rewrite the system of ODE (69) as

$$\frac{dy}{dx} = \frac{y}{x} \text{ (separable equation)}, \quad \int \frac{1}{y} dy = \int \frac{1}{x} dx$$

   and obtain the identity

$$\frac{x}{y} = d \quad (\text{or } \frac{y}{x} = d).$$

3. From the equation $x(t) = x_0 e^t$, we get $t = \log\left(\frac{x}{x_0}\right)$ (if $x_0 \neq 0$) and then

$$y = y(t) = y_0 e^t = y_0 e^{\log\left(\frac{x}{x_0}\right)} = y_0 \frac{x}{x_0}, \quad \text{i.e. } \frac{y}{x} = \frac{y_0}{x_0} := d.$$

2.1.1 Using $h(x,y)$ to do change of variables.

To obtain **general solution** of the PDE $a(x,y)u_x + b(x,y)u_y = 0$, similar to the case of constant coefficients (see (24)), we can use the function $h(x,y)$ to do the change of variables, where we know that $h(x,y)$, coming from (72), is a solution of the PDE $a(x,y)u_x + b(x,y)u_y = 0$ on $\tilde{\Omega}$, and satisfies either (73) or (74) if we make $\tilde{\Omega}$ smaller.

If $h(x,y)$ satisfies (73), then we can do the change of variables

$$\begin{cases} w = h(x,y), & (x,y) \in \tilde{\Omega}, \\ z = y, \end{cases} \tag{79}$$

which has the Jacobian nonzero condition, i.e.

$$J(x,y) = \begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} = \begin{vmatrix} \frac{\partial h}{\partial x}(x,y) & \frac{\partial h}{\partial y}(x,y) \\ 0 & 1 \end{vmatrix} = \frac{\partial h}{\partial x}(x,y) \neq 0 \quad \text{on} \quad \tilde{\Omega}. \tag{80}$$

Moreover, by Lemma 2.11, we also have $b(x,y) \neq 0$ on $\tilde{\Omega}$. Now the function $u(x,y)$ becomes $U(w,z)$ (i.e. $U(h(x,y),y) = u(x,y)$ or $U(w,z) = u(x(w,z),z)$) and by the chain rule, we get

$$\begin{cases} u_x(x,y) = U_w(w,z)\frac{\partial w}{\partial x} + U_z(w,z)\frac{\partial z}{\partial x} = U_w(w,z)\frac{\partial h}{\partial x} \\ u_y(x,y) = U_w(w,z)\frac{\partial w}{\partial y} + U_z(w,z)\frac{\partial z}{\partial y} = U_w(w,z)\frac{\partial h}{\partial y} + U_z(w,z) \end{cases}$$

Therefore, in terms of the new variables (w,z), the PDE for $U(w,z)$ becomes:

$$\begin{aligned} 0 &= a(x,y)\,u_x(x,y) + b(x,y)\,u_y(x,y) \\ &= a(x,y)\,U_w(w,z)\frac{\partial h}{\partial x} + b(x,y)\left[U_w(w,z)\frac{\partial h}{\partial y} + U_z(w,z)\right] \\ &= \left[a(x,y)\frac{\partial h}{\partial x}(x,y) + b(x,y)\frac{\partial h}{\partial y}(x,y)\right]U_w(w,z) + b(x,y)\,U_z(w,z) = b(x,y)\,U_z(w,z), \end{aligned} \tag{81}$$

where in the above we have used the identity (77). Since $b(x,y) \neq 0$ on $\tilde{\Omega}$, we conclude $U_z(w,z) \equiv 0$ on its domain in wz-space. The general solution of the **ODE** (81) is given by

$$U(w,z) = F(w) = F(h(x,y)), \quad w = h(x,y), \quad (x,y) \in \tilde{\Omega} \tag{82}$$

for arbitrary C^1 function $F(w)$.

If $h(x,y)$ satisfies (74), then we can do the change of variables

$$\begin{cases} w = h(x,y), \quad (x,y) \in \tilde{\Omega}, \\ z = x, \end{cases} \tag{83}$$

and has the Jacobian nonzero condition, i.e.

$$J(x,y) = \begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} = \begin{vmatrix} \frac{\partial h}{\partial x}(x,y) & \frac{\partial h}{\partial y}(x,y) \\ 1 & 0 \end{vmatrix} = -\frac{\partial h}{\partial y}(x,y) \neq 0 \quad \text{on} \quad \tilde{\Omega}. \tag{84}$$

Moreover, by Lemma 2.11, we also have $a(x,y) \neq 0$ on $\tilde{\Omega}$. In terms of the new variables (w,z), the PDE for $U(w,z)$ becomes:

$$\begin{aligned} 0 &= a(x,y)\,u_x(x,y) + b(x,y)\,u_y(x,y) \\ &= a(x,y)\left[U_w(w,z)\frac{\partial h}{\partial x} + U_z(w,z)\right] + b(x,y)\,U_w(w,z)\frac{\partial h}{\partial y} \\ &= \left[a(x,y)\frac{\partial h}{\partial x}(x,y) + b(x,y)\frac{\partial h}{\partial y}(x,y)\right]U_w(w,z) + a(x,y)\,U_z(w,z) = a(x,y)\,U_z(w,z), \end{aligned} \tag{85}$$

Since $a(x,y) \neq 0$ on $\tilde{\Omega}$, we conclude $U_z(w,z) \equiv 0$ on its domain in wz-space. The general solution of the **ODE** (81) is also given by (82) for arbitrary C^1 function $F(w)$.

We can now conclude the following result:

Theorem 2.14 ***(General solution of the PDE (67).)*** *Let $a(x,y),\ b(x,y)$ be two C^1 functions on a domain $\Omega \subseteq \mathbb{R}^2$ with $a^2 + b^2 > 0$ on Ω. Consider the PDE $a(x,y)\,u_x + b(x,y)\,u_y = 0$ on Ω. There exists some* ***open subset*** *$\tilde{\Omega}$ of Ω such that on $\tilde{\Omega}$ any C^1 solution $u(x,y)$ of the PDE has the form*

$$u(x,y) = F(h(x,y)), \quad (x,y) \in \tilde{\Omega}, \tag{86}$$

where $F(\cdot)$ is an arbitrary C^1 function and the function $h(x,y)$, which is defined on $\tilde{\Omega}$, comes from solving the ODE (69) on Ω. Note that, in general, the open set $\tilde{\Omega} \subseteq \Omega$ is smaller than Ω.

Remark 2.15 *In the above theorem, the domain of the function* $F(\cdot)$ *will affect the domain of* $u(x,y)$*. However, if we choose* $F(\cdot)$ *a* C^1 *function defined on* $\mathbb{R}$*, then* $u(x,y) = F(h(x,y))$ *will be defined on* $\tilde{\Omega}$*.*

Remark 2.16 *It is possible that there are different disjoint open sets* $\tilde{\Omega}_1$*,* $\tilde{\Omega}_2$ *of* Ω *such that the functions* $h_1(x,y)$ *on* $\tilde{\Omega}_1$ *and* $h_2(x,y)$ *on* $\tilde{\Omega}_2$ *are* ***different*** *(and the integration constant functions* F *and* G *on* $\tilde{\Omega}_1$ *and* $\tilde{\Omega}_2$ *can be independent to each other).*

Since the PDE (67) is **nondegenerate** on Ω (due to the condition (63)), we also have the following **local result** similar to Theorem 2.14:

Theorem 2.17 *(**General solution of the PDE (67) near a point** $(x_0,y_0)\in\Omega$.) Let* $a(x,y)$*,* $b(x,y)$ *be two* C^1 *functions on a domain* $\Omega\subseteq\mathbb{R}^2$ *with* $a^2+b^2>0$ *on* Ω*. Consider the PDE* $a(x,y)u_x + b(x,y)u_y = 0$ *on* Ω*. For any point* $(x_0,y_0)\in\Omega$ *there exists some* ***small open set*** $\tilde{\Omega}\subseteq\Omega$ *containing* (x_0,y_0) *such that on* $\tilde{\Omega}$ *any* C^1 *solution* $u(x,y)$ *of the PDE has the form*

$$u(x,y) = F(h(x,y)), \quad (x,y)\in\tilde{\Omega}, \tag{87}$$

where $F(\cdot)$ *is an arbitrary* C^1 *function and the function* $h(x,y)$*, which is defined on* $\tilde{\Omega}$*, comes from solving the ODE (69) on* $\tilde{\Omega}$*.*

Let us go back to the equation $xu_x + yu_y = 0$ again:

Example 2.18 *Find the general solution of the equation*

$$xu_x + yu_y = 0, \quad u = u(x,y), \quad (x,y)\in\Omega = \mathbb{R}^2\backslash\{(0,0)\}. \tag{88}$$

<u>**Solution**</u>:

We already know that $h(x,y) = \frac{x}{y}$ is a solution on $\Omega\backslash\{x\text{-axis}\}$. We focus on the open half-plane $\mathbb{R}^2_{y+} = \{(x,y)\in\mathbb{R}^2 : y>0\}$ (or on $\mathbb{R}^2_{y-} = \{(x,y)\in\mathbb{R}^2 : y<0\}$) and on it we do the change of variables

$$\begin{cases} w = \frac{x}{y} \in (-\infty,\infty) \\ z = y \in (0,\infty) \end{cases} \quad \text{same as} \quad \begin{cases} x = wz, \quad z\neq 0, \\ y = z, \end{cases}$$

which has Jacobian

$$\begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y} \neq 0 \text{ on } \mathbb{R}^2_{y+}.$$

We note that the above change of variables is a **bijection map** between the open set $(-\infty,\infty)\times(0,\infty)$ in xy-space and the open set $(-\infty,\infty)\times(0,\infty)$ in wz-space. By chain rule the equation on $\mathbb{R}^2_{y+}$ becomes

$$\begin{aligned} 0 = xu_x + yu_y &= x\left(U_w(w,z)\frac{\partial w}{\partial x} + U_z(w,z)\frac{\partial z}{\partial x}\right) + y\left(U_w(w,z)\frac{\partial w}{\partial y} + U_z(w,z)\frac{\partial z}{\partial y}\right) \\ &= x\left(U_w(w,z)\frac{1}{y}\right) + y\left(U_w(w,z)\left(-\frac{x}{y^2}\right) + U_z(w,z)\right) = yU_z(w,z), \quad y>0, \end{aligned}$$

which gives the general solution $U(w,z) = F(w)$ and so we get $u(x,y) = F\left(\frac{x}{y}\right)$ for $(x,y)\in\mathbb{R}^2_{y+}$ and similarly on $\mathbb{R}^2_{y-}$ the general solution is given by $u(x,y) = G\left(\frac{x}{y}\right)$ for $(x,y)\in\mathbb{R}^2_{y-}$, where $F(\cdot)$ and $G(\cdot)$ are two **arbitrary** C^1 functions defined on $(-\infty,\infty)$ (in order for $u(x,y)$ to have maximal domain of definition).

If we allow a solution $u(x,y)$ to be **undefined** on x-axis ($y=0$), then the above two integration constant functions $F(\cdot)$ and $G(\cdot)$ can be **independent to each other**. **Note that each solution curve on** $\mathbb{R}^2_{y+}$ (given by $(x(t),y(t))=(x_0e^t,y_0e^t)$, $t\in(-\infty,\infty)$, $y_0>0$) **will not traverse into** $\mathbb{R}^2_{y-}$**, and vice versa.** Therefore, **there is no discontinuity/inconsistency problem here even if** $F(\cdot)$ **and** $G(\cdot)$ **are different.**

However, it is **possible** to choose some **special** F and G so that $u(x,y)$ can be defined **across the** x**-axis** (more precisely, defined on the whole $\Omega=\mathbb{R}^2\setminus\{(0,0)\}$). One trivial solution is $u(x,y)$ is a constant function; another less trivial one is

$$u(x,y)=\begin{cases} e^{-\left(\frac{x}{y}\right)^2}, & y\neq 0, \quad x\in(-\infty,\infty) \\ 0, \quad y=0, & x\neq 0\in(-\infty,\infty). \end{cases} \tag{89}$$

We leave it to you to check that the function $u(x,y)$ in (89) satisfies $u\in C^1(\Omega)$, where $\Omega=\mathbb{R}^2\setminus\{(0,0)\}$ and $xu_x+yu_y=0$ on Ω. Also, note that the 2-dimensional limit $\lim_{(x,y)\to(0,0)}u(x,y)$ does not exist. □

Exercise 2.19 *(This is a HW problem.) Find the general solution of the equation*

$$yu_x+xu_y=0, \quad u=u(x,y), \quad (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}. \tag{90}$$

Example 2.20 *Find the general solution of the equation*

$$yu_x-xu_y=0, \quad u=u(x,y), \quad (x,y)\in\Omega=\mathbb{R}^2\setminus\{(0,0)\}. \tag{91}$$

<u>**Solution**</u>:

The ODE for the equation is

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-x, \quad \frac{dy}{dx}=-\frac{x}{y} \tag{92}$$

which gives $x''(t)+x(t)=0$ and $y''(t)+y(t)=0$. The general solution $(x(t),y(t))$ of (92) is given by

$$x(t)=A\cos t+B\sin t, \quad y(t)=B\cos t-A\sin t, \quad t\in(-\infty,\infty), \quad x(0)=A, \quad y(0)=B,$$

where A, B are two arbitrary numbers with $(A,B)\in\Omega$. The implicit form of the characteristic curve $(x(t),y(t))$ is given by the equation

$$h(x,y)=x^2+y^2=A^2+B^2=d>0 \tag{93}$$

for arbitrary number $d>0$ (when $d=0$, the characteristic curve $(x(t),y(t))\equiv(0,0)$ is the **equilibrium solution** of the ODE). Each characteristic curve is a circle with radius $\sqrt{d}>0$, centered at the origin.

The function $h(x,y)=x^2+y^2$ is defined on Ω and is constant along each characteristic curve lying in Ω. Hence $h(x,y)=x^2+y^2:\Omega\to\mathbb{R}$ is a solution of (91). Moreover, it is defined on the whole Ω.

Another way to find (93) is to rewrite the ODE system as (now we view y as a function of x)

$$\frac{dy}{dx}=-\frac{x}{y} \quad \text{(this is a separable equation; we get } \int y\,dy=-\int x\,dx) \tag{94}$$

and solve it to get the implicit solution for $y(x)$, i.e.

$$x^2+y^2=C>0, \quad C \text{ is a constant,} \tag{95}$$

which is the same as (93) (the explicit solution for (94) is given by $y(x) = \sqrt{C - x^2}$).

Now we do the nonlinear change of variables

$$\begin{cases} w = x^2 + y^2, \\ z = y. \end{cases}$$

The Jacobian of the change of variables is

$$\begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x.$$

Thus the change of variables is good either on the half-plane $\mathbb{R}^2_{x+}$ or on the half-plane $\mathbb{R}^2_{x-}$ (note that both $(x, y) = (1, 1)$ and $(x, y) = (-1, 1)$ are mapped to $(w, z) = (2, 1)$). On $\mathbb{R}^2_{x+}$, the inverse function relation $(x, y) \longleftrightarrow (w, z)$ is **globally valid** with $x = \sqrt{w - z^2},\ y = z$. The open domain in wz-space corresponding to $\mathbb{R}^2_{x+}$ is

$$\Sigma = \left\{(w, z) \in \mathbb{R}^2 : w > z^2\right\}, \quad \Sigma \longleftrightarrow \mathbb{R}^2_{x+} \text{ is a bijection,}$$

and the equation $yu_x - xu_y = 0$ is **equivalent to** the equation $xU_z = 0$. Similarly, on $\mathbb{R}^2_{x-}$, the inverse function relation $(x, y) \longleftrightarrow (w, z)$ is **globally valid** with $x = -\sqrt{w - z^2},\ y = z$.

The general solution of the above ODE is $U(w, z) = F(w)$ for arbitrary C^1 function $F(w)$. Thus on $\mathbb{R}^2_{x+}$ the general solution of the PDE is

$$u(x, y) = F\left(x^2 + y^2\right), \quad x > 0, \tag{96}$$

where $F(w)$ is an arbitrary C^1 function defined on open interval $I \subset (0, \infty)$. Similarly, on $\mathbb{R}^2_{x-}$ the general solution of the PDE is

$$u(x, y) = G\left(x^2 + y^2\right), \quad x < 0, \tag{97}$$

where $G(w)$ is an arbitrary C^1 function defined on open interval $J \subset (0, \infty)$.

Unlike the previous example, where **each solution curve on $\mathbb{R}^2_{y+}$ will not traverse into** $\mathbb{R}^2_{y-}$ (so $F(\cdot)$ on $\mathbb{R}^2_{y+}$ and $G(\cdot)$ on $\mathbb{R}^2_{y-}$ can be independent), here **solution curve on $\mathbb{R}^2_{x+}$ will traverse into** $\mathbb{R}^2_{x-}$. Hence we must choose $F(\cdot) = G(\cdot)$ on $(0, \infty)$ to make $u(x, y)$ consistent on **both sides of y-axis and have larger domain (across both sides of y-axis).** Hence any solution $u(x, y)$ of the equation on $\Omega = \mathbb{R}^2 \backslash \{(0, 0)\}$ must have the form

$$u(x, y) = F\left(x^2 + y^2\right), \quad \text{where} \quad (x, y) \in \Omega = \mathbb{R}^2 \backslash \{(0, 0)\} \tag{98}$$

for arbitrary C^1 function $F(\cdot)$ defined on $(0, \infty)$. In case $F(w)$ has smaller domain $(a, b) \subset (0, \infty)$, then $u(x, y)$ has smaller domain

$$\left\{(x, y) \in \mathbb{R}^2 : a < x^2 + y^2 < b\right\} \subset \Omega.$$

We call (98) the **general solution** of the equation. Note that $u(x, y)$ is constant along each characteristic curve (circle) $x^2 + y^2 = C > 0$. □

Remark 2.21 *The function $u(x, y) = x^2 + y^2$ is a solution of $yu_x - xu_y = 0$ on $\Omega = \mathbb{R}^2 \backslash \{(0, 0)\}$. In fact, it also defined on the whole $\mathbb{R}^2$. That is, there exists a* ***non-constant solution*** *defined near the singularity $(0, 0)$ of the PDE $yu_x - xu_y = 0$.*

Example 2.22 *Find the general solution of the equation*

$$xu_x - yu_y = 0, \quad u = u(x, y), \quad (x, y) \in \Omega = \mathbb{R}^2 \backslash \{(0, 0)\}. \tag{99}$$

Solution:

The system of ODE is

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y, \quad x(t) = c_1 e^t, \quad y(t) = c_2 e^{-t}$$

and we obtain the equation $h(x,y) = xy = k$ ($k = c_1 c_2 \in (-\infty, \infty)$ is a constant) on characteristic curves. Now we do the change of variables

$$\begin{cases} w = xy \\ z = y \end{cases} \quad \text{same as} \quad \begin{cases} x = \frac{w}{z}, \quad z \neq 0 \\ y = z. \end{cases}$$

Its Jacobian is

$$\begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = \frac{\partial w}{\partial x} = y.$$

Thus the change of variables is good either on $\mathbb{R}^2_{y+} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ or on $\mathbb{R}^2_{y-} = \{(x,y) \in \mathbb{R}^2 : y < 0\}$. C $\mathbb{R}^2_{y+}$ the change of variables is a bijection map between $\mathbb{R}^2_{y+}$ and the open set $\Sigma = \{(w,z) \in \mathbb{R}^2 : z > 0\}$ in wz-space and the equation $xu_x - yu_y = 0$ on $\mathbb{R}^2_{y+}$ is **equivalent to** the equation on Σ :

$$0 = xu_x - yu_y = x(U_w w_x + U_z z_x) - y(U_w w_y + U_z z_y) = -yU_z, \quad y > 0.$$

Therefore the general solution on $\mathbb{R}^2_{y+}$ is given by

$$u(x,y) = F(w) = F(xy), \quad (x,y) \in \mathbb{R}^2_{y+},$$

where $F(\cdot)$ is an arbitrary C^1 function defined on $(-\infty, \infty)$ and $u(x,y)$ is defined on $\mathbb{R}^2_{y+}$. Similarly, on $\mathbb{R}^2_{y-}$ the general solution is given by

$$u(x,y) = G(xy), \quad (x,y) \in \mathbb{R}^2_{y-},$$

where $G(\cdot)$ is an arbitrary C^1 function defined on $(-\infty, \infty)$. Unlike the example $yu_x - xu_y = 0$, now each characteristic curve **does not** traverse from $\mathbb{R}^2_{y+}$ to $\mathbb{R}^2_{y-}$, and vice versa. If $u(x,y)$ is not defined on the x-axis ($y = 0$), then the function F and the function G in the above can be arbitrary and **independent** to each other. Compare with Example 2.18. □

Remark 2.23 *If we choose $z = x$, the analysis is exactly the same except that the two open sets $\mathbb{R}^2_{y+}$ and $\mathbb{R}^2_{y-}$ become $\mathbb{R}^2_{x+}$ and $\mathbb{R}^2_{x-}$.*

Remark 2.24 (***Interesting.***) *The function $u(x,y) = xy$ is a solution of $xu_x - yu_y = 0$ on $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$. In fact, it also defined on the whole $\mathbb{R}^2$. That is, there exists a* ***non-constant solution*** *defined near the singularity $(0,0)$ of the PDE $xu_x - yu_y = 0$. More generally, you can check that for two arbitrary C^1 functions $F(\theta)$, $G(\theta)$ defined on $\mathbb{R}$ with $F(0) = G(0)$ and $F'(0) = G'(0)$ the function*

$$u(x,y) = \begin{cases} F(xy) & x \in \mathbb{R}, \quad y \in [0, \infty) \\ G(xy), & x \in \mathbb{R}, \quad y \in (-\infty, 0] \end{cases}$$

is a C^1 function on $\mathbb{R}^2$ and it satisfies the equation $xu_x - yu_y = 0$ on $\mathbb{R}^2$. This fact says that it is ***not enough*** *to use just one arbitrary function to describe the general solution of the equation $xu_x - yu_y = 0$ on $\mathbb{R}^2$.*

Example 2.25 *Find the general solution of the equation*

$$u_x + yu_y = 0, \quad u = u(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{100}$$

Note that the equation has no singularity on $\mathbb{R}^2$.

Solution:

The ODE has the form

$$\frac{dx}{dt}=1,\quad \frac{dy}{dt}=y,\quad x(t)=c_1+t,\quad y(t)=c_2e^t$$

or

$$\frac{dy}{dx}=\frac{y}{1},\quad \int\frac{1}{y}dy=\int dx.$$

Each gives the implicit solution equation

$$h(x,y)=e^{-x}y=k,\quad k \text{ is an arbitrary constant.}$$

One can see that $h(x,y)=e^{-x}y$ is a solution of the equation $u_x+yu_y=0$ on $\mathbb{R}^2$. By $\frac{\partial h}{\partial x}(x,y)=-e^{-x}y$, $\frac{\partial h}{\partial y}(x,y)=e^{-x}$, we see that $\frac{\partial h}{\partial y}(x,y)\neq 0$ everywhere on $\mathbb{R}^2$. Hence we do the change of variables

$$\begin{cases} w=e^{-x}y, \\ z=x, \end{cases}\quad \text{with Jacobian } J(x,y)=-\frac{\partial h}{\partial y}(x,y)=-e^{-x}\neq 0 \text{ on } \mathbb{R}^2.$$

(if we choose $z=y$, then the Jacobian is $-e^{-x}y$, which is bad at $y=0$). Thus this change of variables is a **bijection** from $\mathbb{R}^2$ to $\mathbb{R}^2$, with

$$\begin{cases} w=e^{-x}y, \\ z=x, \end{cases}\quad \longleftrightarrow \quad \begin{cases} x=z, \\ y=e^zw. \end{cases}$$

Now we have

$$\begin{aligned} 0=u_x+yu_y&=(U_ww_x+U_zz_x)+y(U_ww_y+U_zz_y)\\ &=\left(-U_we^{-x}y+U_z\right)+yU_we^{-x}=U_z,\quad U=U(w,z) \end{aligned}$$

and so the general solution is

$$u(x,y)=U(w,z)=F(w)=F\left(e^{-x}y\right),$$

where $F(\cdot)$ is an arbitrary C^1 function and if $F(\cdot)$ is defined on $(-\infty,\infty)$, then $u(x,y)$ is defined on $\mathbb{R}^2$. The solution is constant along each characteristic curve $e^{-x}y=k$. □

2.2 Finding general solution of the PDE (62).

We now focus on the general form for first order linear PDE with variable coefficients:

$$a(x,y)u_x+b(x,y)u_y+c(x,y)u=f(x,y),\quad u=u(x,y),\quad (x,y)\in\Omega. \tag{101}$$

The method is **the same as** that in solving the PDE (67). We can use the system of characteristic ODE (69) to find the function $h(x,y)$, which satisfies

$$h(x,y)=d,\quad (x,y)\in\tilde{\Omega} \tag{102}$$

along each characteristic curve in $\tilde{\Omega}$. After that we can use the change of variables (we assume $\frac{\partial h}{\partial x}(x,y)\neq 0$ on $\tilde{\Omega}$, which implies $b(x,y)\neq 0$ on $\tilde{\Omega}$)

$$\begin{cases} w=h(x,y), \\ z=y \quad (\text{or } z=x), \end{cases}\quad \text{where } \begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix}=\frac{\partial h}{\partial x}(x,y)\neq 0 \quad \text{on} \quad \tilde{\Omega}$$

to **convert the PDE (101) into an ODE for** $U(w,z)$. To see this, let $A(w,z)$, $B(w,z)$, $C(w,z)$, $U(w,z)$ $F(w,z)$ be the functions corresponding to $a(x,y)$, $b(x,y)$, $c(x,y)$, $u(x,y)$, $f(x,y)$. Then by

$$\begin{aligned}
&a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u \\
&= a(x,y)\,U_w h_x + b(x,y)\left(U_w h_y + U_z\right) + C(w,z)\,U \\
&= \Big(\underbrace{a(x,y)\,h_x + b(x,y)\,h_y}\Big)U_w + B(w,z)\,U_z + C(w,z)\,U = B(w,z)\,U_z + C(w,z)\,U,
\end{aligned}$$

we see that the PDE (101) for the function $U(w,z)$ becomes an ODE of the form

$$B(w,z)\,U_z(w,z) + C(w,z)\,U(w,z) = F(w,z), \tag{103}$$

which can be rewritten as

$$U_z(w,z) + \frac{C(w,z)}{B(w,z)}\cdot U(w,z) = \frac{F(w,z)}{B(w,z)}, \tag{104}$$

where $B(w,z)\neq 0$ on its domain. **Note that (104) is a first order linear ODE (in the variable** z**) containing a parameter** w**, which can be solved.**

Example 2.26 *Find the general solution of the equation*

$$u_x + yu_y + u = e^{x+y}, \quad u = u(x,y), \quad (x,y)\in\mathbb{R}^2. \tag{105}$$

Note that the equation has no singularity on $\mathbb{R}^2$.

Solution:

Similar to Example 2.25, we let $w = e^{-x}y$, $z = x$, which is a **global** change of variables with Jacobian matrix $J(x,y) = -e^{-x}\neq 0$ on $\mathbb{R}^2$. We have

$$\begin{aligned}
&u_x + yu_y + u = \left(-U_w e^{-x}y + U_z\right) + y\left(U_w e^{-x}\right) + U \\
&= U_z + U = e^{z+e^z w}, \quad (w,z)\in\mathbb{R}^2
\end{aligned} \tag{106}$$

which is an ODE for $U(w,z)$ in the variable z with a parameter w, and get

$$U(w,z) = e^{-z}\left[\int e^{2z}\cdot e^{e^z w}dz + F(w)\right]. \tag{107}$$

To find the integral, we let $s = e^z$ and get $ds = e^z dz$ (i.e. $dz = \frac{1}{s}ds$)

$$\begin{aligned}
&\int e^{2z}\cdot e^{e^z w}dz = \int s^2\cdot e^{sw}\frac{1}{s}ds = \int se^{sw}ds = \int s\,d\left(\frac{1}{w}e^{sw}\right) \\
&= \frac{s}{w}e^{sw} - \frac{1}{w}\int e^{sw}ds = \frac{s}{w}e^{sw} - \frac{1}{w^2}e^{sw} = \left(\frac{e^z}{w} - \frac{1}{w^2}\right)e^{e^z w}, \quad s = e^z
\end{aligned}$$

and so

$$U(w,z) = e^{-z}\left[\left(\frac{e^z}{w} - \frac{1}{w^2}\right)e^{e^z w} + F(w)\right], \quad \text{where} \quad w = e^{-x}y, \quad z = x.$$

We obtain the general solution

$$\begin{aligned}
u(x,y) &= e^{-x}\left[\left(\frac{e^x}{e^{-x}y} - \frac{1}{\left(e^{-x}y\right)^2}\right)e^y + F\left(e^{-x}y\right)\right] \\
&= e^{x+y}\left(\frac{1}{y} - \frac{1}{y^2}\right) + e^{-x}F\left(e^{-x}y\right), \quad y\neq 0
\end{aligned} \tag{108}$$

where $F(\cdot)$ is an arbitrary C^1 function and if $F(\cdot)$ is defined on $\mathbb{R}$, then $u(x,y)$ is defined on $\mathbb{R}^2\backslash\{y=0\}$. □

Remark 2.27 ***(Omit this in class.)*** *Since the ODE $U_z + U = e^{z+e^z w}$ is defined on $\mathbb{R}^2$ (the function $e^{z+e^z w}$ is defined on $\mathbb{R}^2$), one should be able to find a solution $u(x,y)$ of the PDE (105) defined on $\mathbb{R}^2$. We can use* ***definite integral*** *to get (see Remark 1.8 also)*

$$e^z U(w,z) = \int_0^z e^{2\theta} \cdot e^{e^\theta w} dz + F(w) = \underbrace{\int_0^z e^{2\theta + e^\theta w} d\theta} + F(w), \tag{109}$$

where by the indefinite integral formula

$$\int e^{2\theta} e^{e^\theta w} dz = \left(\frac{e^\theta}{w} - \frac{1}{w^2} \right) e^{e^\theta w},$$

we get

$$\underbrace{\int_0^z e^{2\theta + e^\theta w} d\theta} = \begin{cases} \left(\frac{e^z}{w} - \frac{1}{w^2}\right) e^{e^z w} - \left(\frac{1}{w} - \frac{1}{w^2}\right) e^w, & w \neq 0 \\ \frac{1}{2} e^{2z} - \frac{1}{2}, & w = 0. \end{cases} := V(w,z) \tag{110}$$

By (109) and (110), we conclude

$$U(w,z) = e^{-z} V(w,z) + e^{-z} F(w), \quad w = e^{-x} y, \quad z = x. \tag{111}$$

We claim that the $V(w,z) \in C^1(\mathbb{R}^2)$. To see this, it suffices to check the continuity and derivatives continuity across $w = 0$. By L'Hospital rule, we first have

$$\begin{aligned} &\lim_{w\to 0} \left[\left(\frac{e^z}{w} - \frac{1}{w^2} \right) e^{e^z w} - \left(\frac{1}{w} - \frac{1}{w^2} \right) e^w \right] = \lim_{w\to 0} \frac{(e^z w - 1) e^{e^z w} - (w-1) e^w}{w^2} \quad \left(\frac{0}{0} \text{ form}\right) \\ &= \lim_{w\to 0} \frac{e^z e^{e^z w} + (e^z w - 1) e^{e^z w} e^z - e^w - (w-1) e^w}{2w} = \lim_{w\to 0} \frac{(e^z w) e^{e^z w} e^z - w e^w}{2w} \\ &= \lim_{w\to 0} \frac{(e^z) e^{e^z w} e^z - e^w}{2} = \frac{1}{2} e^{2z} - \frac{1}{2}. \end{aligned} \tag{112}$$

Hence $V(w,z)$ is ***continuous across*** *$w = 0$. We leave the check of the rest as an exercise.* ***From (111), we can obtain the general solution formula for $u(x,y)$, which is slightly different from (108),*** *given by*

$$u(x,y) = e^{-x} F\left(e^{-x} y\right) + e^{-x} V\left(e^{-x} y, x\right), \quad \text{with} \quad e^{-x} V\left(e^{-x} y, x\right) \in C^1\left(\mathbb{R}^2\right), \tag{113}$$

where

$$e^{-x} V\left(e^{-x} y, x\right) = \begin{cases} e^{-x} \left[\left(\frac{e^x}{e^{-x} y} - \frac{1}{(e^{-x} y)^2} \right) e^y - \underbrace{\left(\frac{1}{e^{-x} y} - \frac{1}{(e^{-x} y)^2} \right) e^{e^{-x} y}} \right] \text{ (call it } \Gamma), \quad y \neq 0 \\ \frac{1}{2} e^x - \frac{1}{2} e^{-x}, \quad y = 0. \end{cases} \tag{114}$$

By applying L'Hospital rule twice, one can check the continuity of the function $\{\cdots$ in (114) across $y = 0$:

$$\begin{aligned} &\lim_{y\to 0} \Gamma \quad \left(\frac{0}{0} \text{ form}\right) = \lim_{y\to 0} \left(\frac{e^{-x} y e^y - e^{-x} e^y - e^{-2x} y e^{e^{-x} y} + e^{-x} e^{e^{-x} y}}{(e^{-x} y)^2} \right) \\ &= \lim_{y\to 0} \left(\frac{e^x e^y + e^x y e^y - e^x e^y - e^{e^{-x} y} - y e^{e^{-x} y} e^{-x} + e^{e^{-x} y}}{2y} \right) \\ &= \lim_{y\to 0} \left(\frac{e^x e^y + e^x e^y + e^x y e^y - e^x e^y - e^{e^{-x} y} e^{-x} - e^{e^{-x} y} e^{-x} - y e^{e^{-x} y} e^{-x} e^{-x} + e^{e^{-x} y} e^{-x}}{2} \right) \\ &= \frac{e^x - e^{-x}}{2}. \end{aligned}$$

One can continue the process to check that $e^{-x} V\left(e^{-x} y, x\right) \in C^1\left(\mathbb{R}^2\right)$.

2.2.1 ODE along a characteristic curve.

Recall the important fact that every solution of the PDE $a(x,y)\,u_x + b(x,y)\,u_y = 0$ is **constant** along each characteristic curve lying in its domain. That means if we know the value of $u(x,y)$ at **one** point $p \in C$ (C is a characteristic curve), then we know the values of $u(x,y)$ at every point of C. This is also true for the general **nonhomogeneous** equation

$$a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = f(x,y), \quad u = u(x,y). \tag{115}$$

More precisely, we have:

Lemma 2.28 *Let $u(x,y)$ be a C^1 solution of (115) on some open set Ω with $(x_0,y_0) \in \Omega$. Let C : $\alpha(t) = (x(t), y(t)) \in \Omega,\ t \in I$ (some open interval containing $t=0$), be the unique characteristic curve of (115) satisfying $(x(0), y(0)) = (x_0, y_0) \in C$, and lies in Ω. Then the function $U(t) = u(x(t), y(t)),\ t \in I$, satisfies the ODE*

$$U'(t) + C(t)\,U(t) = F(t), \quad t \in I, \quad U(0) = u(x_0, y_0), \tag{116}$$

*where $C(t) = c(x(t), y(t))$ and $F(t) = f(x(t), y(t))$. In particular, if we know the value of $U(0)$ (i.e. if we know the value of $u(x_0,y_0)$), then we can know the value of $U(t)$ on the **whole interval** I (by solving the ODE (116)).*

Proof. This is straightforward. We have

$$\begin{aligned} U'(t) &= \frac{d}{dt}u(x(t), y(t)) = u_x(x(t), y(t))\,x'(t) + u_y(x(t), y(t))\,y'(t) \\ &= a(x(t), y(t))\,u_x(x(t), y(t)) + b(x(t), y(t))\,u_y(x(t), y(t)) \\ &= -c(x(t), y(t))\,u(x(t), y(t)) + f(x(t), y(t)) = -C(t)\,U(t) + F(t), \quad t \in I. \end{aligned} \tag{117}$$

Hence $U(t)$ satisfies the ODE (116) on I and if we know $U(0)$, $U(t)$ can be expressed as

$$U(t) = e^{-\int_0^t C(\theta)d\theta}\left[\int_0^t \left(e^{\int_0^s C(\theta)d\theta} F(s)\right) ds + U(0)\right], \quad t \in I.$$

The result follows. □

Remark 2.29 *In the special case when the equation is $a(x,y)\,u_x + b(x,y)\,u_y = 0$, (117) becomes $U'(t) = 0$, i.e. $u(x(t), y(t))$ is a constant function along the characteristic curve C.*

2.3 First order linear PDE with variable coefficients and side condition.

Due to Lemma 2.28, a **"well-posed"** side condition on a curve γ is that it intersects each characteristic curve C at **exactly one point** (so that **we have exactly one initial condition $U(0)$ for the ODE in Lemma 2.28 on each characteristic curve**). Otherwise, the problem may have no solution or infinitely many solutions.

Example 2.30 *Find the solution to the problem*

$$\begin{cases} yu_x - xu_y = 0, \quad u = u(x,y), \quad (x,y) \in \mathbb{R}^2_{x+}, \\ u(s, s^2) = s^3, \quad s \in (0, \infty). \end{cases}$$

Solution:

We already know that the general solution on $\mathbb{R}^2_{x+}$ has the form $u(x,y) = F(x^2+y^2)$ for arbitrary C^1 function $F(\cdot)$. Since the curve $\gamma = (s, s^2)$, $s \in (0,\infty)$, intersects each characteristic curve $x^2+y^2 = d > 0$, lying on $\mathbb{R}^2_{x+}$, at exactly one point, one should be able to find a **unique** $F(\cdot)$ satisfying the condition. We need to solve the equation

$$F\left(s^2+s^4\right) = s^3, \quad s \in (0,\infty).$$

Let $r = s^2+s^4 > 0$, and let $p = s^2 > 0$. We have $p^2+p-r = 0$ and so

$$p = \frac{-1 \pm \sqrt{1+4r}}{2} > 0 \quad \text{(the minus sign does not make sense).}$$

Thus we have

$$F(r) = s^3 = p^{3/2} = \left(\frac{-1+\sqrt{1+4r}}{2}\right)^{3/2}$$

and the solution is given by

$$u(x,y) = F\left(x^2+y^2\right) = \left(\frac{-1+\sqrt{1+4\left(x^2+y^2\right)}}{2}\right)^{3/2}, \quad (x,y) \in \mathbb{R}^2_{x+} \tag{118}$$

The solution is defined on $\mathbb{R}^2_{x+}$ and lies in the space $C^\infty\left(\mathbb{R}^2_{x+}\right)$. To see this, note that we have

$$g(x,y) = \frac{-1+\sqrt{1+4\left(x^2+y^2\right)}}{2} \in C^\infty\left(\mathbb{R}^2_{x+}\right), \quad g(x,y) : \mathbb{R}^2_{x+} \to (0,\infty)$$

and the function $h(\theta) = \theta^{3/2} \in C^\infty(0,\infty)$. Therefore, chain rule implies

$$u(x,y) = h(g(x,y)) \in C^\infty\left(\mathbb{R}^2_{x+}\right), \tag{119}$$

and we have

$$u\left(s,s^2\right) = \left(\frac{-1+\sqrt{1+4\left(s^2+s^4\right)}}{2}\right)^{3/2} = \left(\frac{-1+\left(2s^2+1\right)}{2}\right)^{3/2} = s^3, \quad \forall\, s \in (0,\infty).$$

□

Example 2.31 *Find the solution to the problem*

$$\begin{cases} yu_x - xu_y + u = 0, \quad u = u(x,y), \quad (x,y) \in \mathbb{R}^2_{x+}, \\ u(x,0) = h(x), \quad x \in (0,\infty). \end{cases}$$

Solution:

We already know that $h(x,y) = x^2+y^2$ is a solution of $yu_x - xu_y = 0$ on $\mathbb{R}^2_{x+}$. According to the standard method, we do the change of variables

$$\begin{cases} w = x^2+y^2, \\ z = y \end{cases} \qquad \text{Jacobian} = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x \neq 0 \ \text{ on } \ \mathbb{R}^2_{x+}$$

and the new equation for $U(w,z)$ becomes

$$y\left(U_w w_x\right) - x\left(U_w w_y + U_z z_y\right) + U = -xU_z + U = 0, \quad x = \sqrt{w-z^2}, \quad w > z^2$$

and the linear ODE for $U(w,z)$ is

$$U_z - \frac{1}{\sqrt{w-z^2}} U = 0,$$

where we see that the integrating factor of the above ODE is

$$\exp\left(-\int \frac{1}{\sqrt{w-z^2}} dz\right) = \exp\left(-\sin^{-1}\left(\frac{z}{\sqrt{w}}\right)\right).$$

Hence we get

$$U(w,z) = e^{\sin^{-1}\left(z/\sqrt{w}\right)} \cdot F(w), \quad F(w) \text{ is integration constant,}$$

which implies the general solution for $u(x,y)$:

$$u(x,y) = e^{\sin^{-1}\left(y/\sqrt{x^2+y^2}\right)} \cdot F\left(x^2+y^2\right), \quad (x,y) \in \mathbb{R}^2_{x+},$$

where $F(\cdot)$ is an arbitrary C^1 function. Thus the solution satisfying the side condition is

$$u(x,y) = e^{\sin^{-1}\left(y/\sqrt{x^2+y^2}\right)} \cdot h\left(\sqrt{x^2+y^2}\right), \quad (x,y) \in \mathbb{R}^2_{x+}.$$

The proof is done. □

Remark 2.32 *One can also write the solution as*

$$u(x,y) = u(x,y) = e^{\tan^{-1}(y/x)} \cdot h\left(\sqrt{x^2+y^2}\right), \quad (x,y) \in \mathbb{R}^2_{x+}$$

due to

$$\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \tan^{-1}\left(\frac{y}{x}\right), \quad (x,y) \in \mathbb{R}^2_{x+}.$$

This is the end of first order linear PDE, 2022-3-7