# PDE Teaching for Spring, 2022

**Revised on 2022-6-2**

**Remark 0.1** *This notes is based on "Lecture-notes-on-elliptic-equation-for-student-2021-5-10.tex".*

# 1 Laplace equation on $\mathbb{R}^n$ (for $n = 2,\ 3$) and the divergence theorem.

From now on, we shall focus on second order elliptic equation with constant coefficients. If $u(x, y)$ is a function of two variables, its canonical form is

$$u_{xx}(x, y) + u_{yy}(x, y) + (\text{lower order terms}) = f(x, y),$$

where $f(x, y)$ is a given continuous function defined on some domain $\Omega \subseteq \mathbb{R}^2$. To begin with, for simplicity, we shall look at the 2-dimensional **Laplace equation:**

$$\triangle u(x, y) := u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad (x, y) \in \Omega \subseteq \mathbb{R}^2. \tag{1}$$

The purpose is to find $C^2$ solutions $u(x, y) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$. If the domain $\Omega$ is not specified in advance, we want the solutions $u(x, y)$ to be defined on some open set $\Omega$ in $\mathbb{R}^2$, as large as possible. A $C^2$ solution $u$ of (1), defined on $\Omega$, is called a **harmonic function** on $\Omega$. In case $u = u(x, y, z)$ is a three-variable function, then the above becomes

$$\triangle u(x, y, z) := u_{xx}(x, y, z) + u_{yy}(x, y, z) + u_{zz}(x, y, z) = 0, \quad (x, y, z) \in \Omega \subseteq \mathbb{R}^3. \tag{2}$$

**Example 1.1** *The following functions*

$$u(x, y) = x, \quad y, \quad x^2 - y^2, \quad 2xy, \quad x^3 - 3xy^2, \quad y^3 - 3x^2 y, \quad e^x \cos y, \quad e^x \sin y \tag{3}$$

*are all harmonic functions defined on the whole plane $\mathbb{R}^2$. The functions*

$$u(x, y) = \frac{x}{x^2 + y^2}, \quad \frac{-y}{x^2 + y^2}$$

*are harmonic functions defined on $\mathbb{R}^2 \backslash \{(0, 0)\}$. In the above, the polynomials $x,\ y,\ x^2 - y^2,\ 2xy,\ x^3 - 3xy^2,\ y^3 - 3x^2 y$, ... etc. are called **harmonic polynomials** with degree 1, 2, 3, ... etc.*

**Definition 1.2** *Let $\Omega \subset \mathbb{R}^n$ be a domain (open and connected) and let $V(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ **vector field** on $\Omega$ given by*

$$V(x) = (V_1(x), \ \dots, V_n(x)), \quad V_i(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R} \text{ is a } C^1 \text{ function.}$$

*Its **divergence**, denoted as $(\operatorname{div} V)(x)$ (or just $\operatorname{div} V(x)$): $\Omega \subset \mathbb{R}^n \to \mathbb{R}$, is a **scalar function** defined as*

$$\operatorname{div} V(x) = \frac{\partial V_1}{\partial x_1}(x) + \cdots + \frac{\partial V_n}{\partial x_n}(x), \quad x = (x_1, \ \dots, x_n) \in \Omega.$$

*Note that $\operatorname{div} V(x)$ is a continuous function on $\Omega$.*

**Remark 1.3** *Note that we can also express $\operatorname{div} V(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ as*

$$\operatorname{div} V(x) = Tr(DV)(x), \quad x \in \Omega, \tag{4}$$

*where $DV$ is the derivative of the map $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$.*

We first note that for $C^2$ function $u(x,y):\Omega\subseteq\mathbb{R}^2\to\mathbb{R},\ \triangle u(x,y)$ can be decomposed as

$$\triangle u(x,y)=(\operatorname{div}(\nabla u))(x,y)=Tr(D(\nabla u))(x,y),\quad (x,y)\in\Omega. \tag{5}$$

Similar identity holds for $\triangle u(x),\ x\in\Omega\subseteq\mathbb{R}^3$ (or $x\in\Omega\subseteq\mathbb{R}^n$).

The following **divergence theorem** (general version of the Fundamental Theorem of Calculus) will be needed often later on:

**Theorem 1.4** *(**Divergence theorem.**) Let $\Omega$ be a $C^1$ bounded domain in $\mathbb{R}^n$ (which means that its boundary $\partial\Omega$ is a $C^1$ $(n-1)$-dimensional surface in $\mathbb{R}^n$) and $\mathbf{W}:\bar{\Omega}\to\mathbb{R}^n$ is a vector field on $\bar{\Omega}$ with $\mathbf{W}\in C^1(\Omega)\bigcap C^0(\bar{\Omega})$. We have the identity*

$$\begin{aligned}&\int_\Omega \operatorname{div}\mathbf{W}dx \ \textit{(volume integral in } \mathbb{R}^n\textit{)}\\ &=\int_{\partial\Omega}\mathbf{W}\cdot\mathbf{N}d\sigma\ \textit{((}n-1\textit{)-dimensional surface integral in } \mathbb{R}^n\textit{)}\end{aligned} \tag{6}$$

*where $\mathbf{N}$ is the unit **outward** normal to $\partial\Omega$. Here $\operatorname{div}\mathbf{W}:\Omega\to\mathbb{R}$ is the **divergence** of the vector field $\mathbf{W}$ and $d\sigma$ is the "**surface measure**" on $\partial\Omega$.*

**Remark 1.5** *$\Omega$ is a $C^1$ bounded domain in $\mathbb{R}^n$ means that its boundary $\partial\Omega$ is locally a $C^1$ graph everywhere. For example, if $\Omega$ is a $C^1$ bounded domain in $\mathbb{R}^3$, then near any $p\in\partial\Omega$, the boundary $\partial\Omega$ can be expressed as a graph $z=f(x,y)$ for some $C^1$ function $f(x,y)$ defined on some open subset of $\mathbb{R}^2$. Therefore the boundary $\partial\Omega$ is a $C^1$ surface in $\mathbb{R}^3$.*

**Remark 1.6** *(**Important.**) Be careful that the divergence theorem **is valid only when** $\Omega\subset\mathbb{R}^n$ **is bounded**. For example, let $\Omega=\mathbb{R}^n_+=\{(x_1,\ \dots\ ,x_n)\in\mathbb{R}^n: x_n>0\}$ be the upper half-space of $\mathbb{R}^n$ and let $\mathbf{W}$ be the smooth vector field on $\bar{\Omega}$ given by*

$$\mathbf{W}(x)=(0,0,...,0,x_n),\quad x=(x_1,\ \dots\ ,x_n)\in\mathbb{R}^n_+\bigcup\partial\mathbb{R}^n_+.$$

*Then we have $\operatorname{div}\mathbf{W}\equiv1$ on $\mathbb{R}^n_+$ and $\mathbf{W}\equiv0$ on $\partial\mathbb{R}^n_+$. In such a case, the identity (6) clearly fails. In view of this, the **Green identities also fail on unbounded domains**.*

**Remark 1.7** *If $n=2$, then $d\sigma$ means $ds$, where $ds$ is the **arc length differential** and the above theorem is the same as the familiar **Green Theorem** for plane region enclosed by a **simple closed curve** $\Gamma$. More precisely, let $C\subset\mathbb{R}^2$ be a **counterclockwise** simple closed curve parametrized by $\alpha(t)=(x(t),y(t)),\ t\in[a,b]$, where $t$ is an arbitrary parameter (**not necessarily the arc length parameter**) and let $\Omega\subset\mathbb{R}^2$ be the open region enclosed by $C$ with $\partial\Omega=C$ ($\partial\Omega$ means the boundary of $\Omega$). Let $\mathbf{W}\in C^1(\Omega)\cap C^0(\bar{\Omega})$ be a vector field on $\bar{\Omega}$ given by*

$$\mathbf{W}(x,y)=(p(x,y),q(x,y)),\quad (x,y)\in\bar{\Omega}.$$

*Now we first have*

$$\int_\Omega\operatorname{div}\mathbf{W}dx\ \textit{(volume integral in } \mathbb{R}^2\textit{)}=\iint_\Omega\left(\frac{\partial p}{\partial x}(x,y)+\frac{\partial q}{\partial y}(x,y)\right)dxdy \tag{7}$$

*and the unit **outward** normal $\mathbf{N}$ to $(x(t),y(t))\in\partial\Omega,\ t\in[a,b]$, is given by*

$$\mathbf{N}(x(t),y(t))=\left(\frac{y'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}},\ \frac{-x'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}\right),\quad t\in[a,b]$$

*and we also have*

$$d\sigma = ds = \sqrt{(x'(t))^2 + (y'(t))^2}dt, \quad t \in [a,b].$$

*Hence we conclude*

$$\begin{aligned}
&\int_{\partial\Omega} \mathbf{W}\cdot\mathbf{N}d\sigma \\
&= \int_a^b \left[ (p(x(t),y(t)), q(x(t),y(t))) \cdot \left( \frac{y'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}, \frac{-x'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}} \right) \right] \sqrt{(x'(t))^2 + (y'(t))^2}dt \\
&= \int_a^b \left[ (p(x(t),y(t)) y'(t) - q(x(t),y(t))) x'(t) \right] dt = \int_C -qdx + pdy \ \text{(this is line integral).} \qquad (8)
\end{aligned}$$

*By (7) and (8), we conclude the identity*

$$\iint_\Omega \left( \frac{\partial p}{\partial x}(x,y) + \frac{\partial q}{\partial y}(x,y) \right) dxdy = \int_C -qdx + pdy, \tag{9}$$

***which is exactly the familiar Green Theorem (in a slightly different way)****. Note that the value of the line integral* $\int_C -qdx + pdy$ *is* ***independent of*** *parametrization.*

**Remark 1.8** *If* $n = 3$*, then the surface measure* $d\sigma$ *means*

$$d\sigma = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy \tag{10}$$

*if the surface is represented by* ***the graph of a function*** $z = f(x,y)$*. Therefore, we are doing* ***surface integrals*** *in* $\mathbb{R}^3$*. Also, if the surface in* $\mathbb{R}^3$ *is given by the* ***parametrization form***

$$X(u,v) : (u,v) \in U \subset \mathbb{R}^2 \to (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3,$$

*then*

$$d\sigma = |X_u \times X_v| \, dudv, \quad \text{where } \times \text{ is the cross product in } \mathbb{R}^3. \tag{11}$$

*In case the surface in* $\mathbb{R}^3$ *is given by the equation*

$$\varphi(x,y,z) = 0$$

*for some smooth function* $\varphi(x,y,z) : O \subset \mathbb{R}^3 \to \mathbb{R}$*, then (assuming the surface can be expressed as* $z = f(x,y)$ *for* $(x,y) \in U \subset \mathbb{R}^2$*) we have*

$$X(x,y) = (x, y, f(x,y)) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$$

*and by chain rule we have*

$$d\sigma = |X_x \times X_y| \, dxdy = \sqrt{1 + f_x^2 + f_y^2}dxdy = \frac{\sqrt{\varphi_x^2 + \varphi_y^2 + \varphi_z^2}}{|\varphi_z|}dxdy. \tag{12}$$

*The first identity in (12) is clear, to see the second identity, note that by* $\varphi(x,y,f(x,y)) = 0$ *for all* $(x,y) \in U \subset \mathbb{R}^2$*, we have*

$$f_x = -\frac{\varphi_x}{\varphi_z}, \quad f_y = -\frac{\varphi_y}{\varphi_z} \tag{13}$$

*and so*

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \left(\frac{\varphi_x}{\varphi_z}\right)^2 + \left(\frac{\varphi_y}{\varphi_z}\right)^2} = \frac{\sqrt{\varphi_x^2 + \varphi_y^2 + \varphi_z^2}}{|\varphi_z|}. \tag{14}$$

**Proof. (Omit this in class.) (Special case only.)** We will give a proof of the theorem only for the case $n=2$ and assuming that $\Omega$ takes the simple form:

$$\Omega = \{(x,y) : \ a \le x \le b, \quad 0 \le y \le f(x), \quad f(a) = f(b) = 0\} \tag{15}$$

where $f(x)$ is a $C^1$ function defined on $[a,b]$. Although $\Omega$ may not be smooth at $(a, f(a))$ and $(b, f(b))$, divergence theorem still holds for such $\Omega$ as shown below. Its boundary $\partial\Omega$ has two parts: the graph $y = f(x)$ and the segment $(x,0)$, $a \le x \le b$. Call them $\partial_1\Omega$ and $\partial_2\Omega$ respectively. We have

$$\begin{cases} \mathbf{N} \text{ at } (x, f(x)) = \dfrac{(-f'(x),\ 1)}{\sqrt{1+(f'(x))^2}} \\ \\ \mathbf{N} \text{ at } (x,0) = (0,-1). \end{cases} \tag{16}$$

Both normal vectors in (16) are pointing outwards. Writing $\mathbf{W}(x,y) = (u(x,y), v(x,y))$, the divergence theorem is equivalent to

$$\begin{aligned} &\iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy, \quad \operatorname{div}\mathbf{W} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ &= \int_{\partial_1\Omega} (u(x,y), v(x,y)) \cdot \frac{(-f'(x),\ 1)}{\sqrt{1+(f'(x))^2}} ds + \int_{\partial_2\Omega} (u(x,y), v(x,y)) \cdot (0,-1)\, ds, \end{aligned} \tag{17}$$

where $ds$ is the **arc length** differential. Note that the boundary line integral $\int_{\partial\Omega} ds$ in (17) has **no orientation**. Clearly we have

$$\begin{cases} y = f(x), \quad ds = \sqrt{1+(f'(x))^2}dx \quad \text{on} \quad \partial_1\Omega \\ \\ y = 0, \quad ds = dx \quad \text{on} \quad \partial_2\Omega \end{cases} \tag{18}$$

and so the RHS (right-hand side) of (17) becomes

$$\begin{aligned} &\int_a^b (u(x,f(x)), v(x,f(x))) \cdot (-f'(x), 1)\, dx + \int_a^b (u(x,0), v(x,0)) \cdot (0,-1)\, dx \\ &= \int_a^b [-u(x,f(x)) f'(x) + v(x,f(x))]\, dx - \int_a^b v(x,0)\, dx \\ &= -\overbrace{\int_a^b u(x,f(x)) f'(x)\, dx} + \underbrace{\int_a^b [v(x,f(x)) - v(x,0)]\, dx}. \end{aligned} \tag{19}$$

Also the LHS of (17) is

$$\iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy = \overbrace{\int_a^b \int_0^{f(x)} \left(\frac{\partial u}{\partial x}\right) dydx} + \underbrace{\int_a^b \int_0^{f(x)} \left(\frac{\partial v}{\partial y}\right) dydx}. \tag{20}$$

Now we have

$$\underbrace{\int_a^b \int_0^{f(x)} \left(\frac{\partial v}{\partial y}\right) dydx} = \underbrace{\int_a^b [v(x,f(x)) - v(x,0)]\, dx}. \tag{21}$$

Hence it suffices to show

$$\overbrace{\int_a^b u(x,f(x)) f'(x)\, dx + \int_a^b \int_0^{f(x)} \frac{\partial u}{\partial x}(x,y)\, dydx} = 0. \tag{22}$$

Note that the LHS (left-hand side) of (22) is

$$\int_a^b \left( \underline{u\left(x, f\left(x\right)\right) f'\left(x\right) + \int_0^{f(x)} \frac{\partial u}{\partial x}\left(x, y\right) dy} \right) dx, \tag{23}$$

where the integrand in (23) can be written as

$$\underline{u\left(x, f\left(x\right)\right) f'\left(x\right) + \int_0^{f(x)} \frac{\partial u}{\partial x}\left(x, y\right) dy} = \underline{\frac{d}{dx}\left( \int_0^{f(x)} u\left(x, y\right) dy \right)}.$$

Hence we have

$$\begin{aligned} &\int_a^b \left( \underline{u\left(x, f\left(x\right)\right) f'\left(x\right) + \int_0^{f(x)} \frac{\partial u}{\partial x}\left(x, y\right) dy} \right) dx \\ &= \int_a^b \underline{\frac{d}{dx}\left( \int_0^{f(x)} u\left(x, y\right) dy \right)} dx = \int_0^{f(b)} u\left(b, y\right) dy - \int_0^{f(a)} u\left(a, y\right) dy = 0, \end{aligned}$$

due to $f\left(a\right) = f\left(b\right) = 0$. The proof is done. □

## 1.1 Averaging property of the Laplace operator.

To understand the "**averaging property of the Laplace operator**", it suffices to do the following interesting problem:

**Problem 1.9** *Let $u\left(x_1, x_2\right) : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function and let $\{v, w\} \subset \mathbb{R}^2$ be **any** orthonormal basis in $\mathbb{R}^2$. Let $p \in \mathbb{R}^2$ be a fixed point. By definition, we have*

$$\left(\triangle u\right)\left(p\right) = \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2}\left(p\right) = \left.\frac{d^2}{dt^2}\right|_{t=0} u\left(p + te_1\right) + \left.\frac{d^2}{dt^2}\right|_{t=0} u\left(p + te_2\right), \quad t \in \left(-\varepsilon, \varepsilon\right), \tag{24}$$

*where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. Show that we also have the identity*

$$\left(\triangle u\right)\left(p\right) = \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2}\left(p\right) = \left.\frac{d^2}{dt^2}\right|_{t=0} u\left(p + tv\right) + \left.\frac{d^2}{dt^2}\right|_{t=0} u\left(p + tw\right), \quad t \in \left(-\varepsilon, \varepsilon\right). \tag{25}$$

**Remark 1.10** *Does the quantity $\left.\frac{d}{dt}\right|_{t=0} u\left(p + tv\right) + \left.\frac{d}{dt}\right|_{t=0} u\left(p + tw\right)$ have similar property?*

**Remark 1.11** *Similar result holds for the case $u\left(x_1, \ \dots \ , x_n\right) : \mathbb{R}^n \to \mathbb{R}$ and $\{v_1, \ \dots \ , v_n\}$ is **any** orthonormal basis in $\mathbb{R}^n$. The proof is similar to the case $n = 2$.*

The Laplace operator has the following important geometric meaning: let $u\left(x, y\right)$ be a $C^2$ function defined on $\mathbb{R}^2$ and let $O = \left(0, 0\right)$ be the origin. Any line $L$ passing through $O$ with direction $v\left(\theta\right) = \left(\cos\theta, \sin\theta\right), \ \theta \in \left[0, 2\pi\right]$, has the parametric form

$$L = \{\left(t\cos\theta, t\sin\theta\right) : t \in \left(-\infty, \infty\right), \ \theta \in \left[0, 2\pi\right]\}.$$

The function $u\left(x, y\right)$ restricted on $L$ becomes a function of $t$, i.e., we have (here the angle $\theta$ is fixed)

$$h\left(t\right) := u\left(tv\right), \text{ where } v = v\left(\theta\right) = \left(\cos\theta, \sin\theta\right), \quad t \in \left(-\infty, \infty\right).$$

We note that

$$h'\left(0\right) = \lim_{t\to 0} \frac{u\left(tv\right) - u\left(0, 0\right)}{t} = D_v u\left(O\right) = \left\langle \nabla u\left(O\right), v \right\rangle, \tag{26}$$

which is the **directional derivative** of $u$ at $O=(0,0)$ in the direction $v=(\cos\theta,\sin\theta)$. If we compute the second derivative of $h(t)$ at $t=0$ and average it among all possible directions (i.e. among all possible angle $\theta\in[0,2\pi]$), we get

$$\frac{1}{2\pi}\int_0^{2\pi} h''(0)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\left.\frac{d^2}{dt^2}\right|_{t=0} u(tv(\theta))\right)d\theta$$
$$=\frac{1}{2}(\triangle u)(O),\quad v(\theta)=(\cos\theta,\sin\theta),\quad \theta\in[0,2\pi]. \tag{27}$$

Thus the quantity $\frac{1}{2}(\triangle u)(O)$ is the average of the second derivatives of $u$ (among all possible directions $v(\theta)$). To see (27), by the chain rule we have

$$h'(0)=D_\nu u(O)=\langle\nabla u(O),v\rangle=\frac{\partial u}{\partial x}(O)\cos\theta+\frac{\partial u}{\partial y}(O)\sin\theta$$
$$h''(0)=\frac{\partial^2 u}{\partial x^2}(O)\cos^2\theta+2\frac{\partial^2 u}{\partial x\partial y}(O)\cos\theta\sin\theta+\frac{\partial^2 u}{\partial y^2}(O)\sin^2\theta$$

and (27) follows due to

$$\frac{1}{2\pi}\int_0^{2\pi} h''(0)\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial^2 u}{\partial x^2}(O)\cos^2\theta+2\frac{\partial^2 u}{\partial x\partial y}(O)\cos\theta\sin\theta+\frac{\partial^2 u}{\partial y^2}(O)\sin^2\theta\right)d\theta$$
$$=\frac{1}{2\pi}\left[\frac{\partial^2 u}{\partial x^2}(O)\cdot\pi+\frac{\partial^2 u}{\partial y^2}(O)\cdot\pi\right]=\frac{1}{2}(\triangle u)(O). \tag{28}$$

**Remark 1.12** *One can use* ***rotating orthonormal frame*** *in* $\mathbb{R}^2$ *to see why we have the coefficient* $1/2$ *in (28).*

## 1.2 Green identities.

There are many useful consequence of the **divergence theorem** (equivalent to the **Green Theorem** if we are in $\mathbb{R}^2$). Among the most important are the **Green identities**: Assume $\Omega$ is a $C^1$ bounded domain in $\mathbb{R}^2$ (or $\mathbb{R}^n$, $n\geq 3$) and $u,\ v\in C^2(\Omega)\bigcap C^1(\bar{\Omega})$ ($u,\ v:\Omega\to\mathbb{R}$). We have the **Green 1st identity:**

$$\int_\Omega v\triangle u\,dx+\int_\Omega\nabla u\cdot\nabla v\,dx=\int_{\partial\Omega} v\frac{\partial u}{\partial\mathbf{N}}d\sigma,\qquad \nabla u=\text{gradient of } u \tag{29}$$

and **Green 2nd identity:**

$$\int_\Omega (v\triangle u-u\triangle v)\,dx=\int_{\partial\Omega}\left(v\frac{\partial u}{\partial\mathbf{N}}-u\frac{\partial v}{\partial\mathbf{N}}\right)d\sigma, \tag{30}$$

where in (29), $\nabla u\cdot\nabla v$ denotes the inner product of the two gradient vectors $\nabla u$, $\nabla v$, and $\frac{\partial u}{\partial\mathbf{N}}$ is the directional derivative of $u$ on $\partial\Omega$ along the outward unit normal vector $\mathbf{N}$ which, by the chain rule, is equal to

$$\frac{\partial u}{\partial\mathbf{N}}=\nabla u\cdot\mathbf{N},\quad \frac{\partial u}{\partial\mathbf{N}}(p)=\lim_{t\to 0^-}\frac{u(p+t\mathbf{N})-u(p)}{t},\quad p\in\partial\Omega. \tag{31}$$

**Remark 1.13** *We also call* $\frac{\partial u}{\partial\mathbf{N}}$ *as the* ***outward normal derivative*** *of* $u$ *on* $\partial\Omega$.

We note that (29) is a consequence of the **divergence theorem** and the identity

$$\operatorname{div}(v\nabla u)=v\triangle u+\nabla v\cdot\nabla u,\quad u,\ v\in C^2(\Omega)\bigcap C^1(\bar{\Omega}). \tag{32}$$

By the divergence theorem, we have

$$\int_{\Omega} v \triangle u dx + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} \operatorname{div}(v \nabla u)\, dx = \int_{\partial\Omega} (v \nabla u) \cdot \mathbf{N} d\sigma = \int_{\partial\Omega} v \frac{\partial u}{\partial \mathbf{N}} d\sigma, \tag{33}$$

which gives (29).

In particular, when $u = v$ in (29), we get the identity

$$\int_{\Omega} u \triangle u dx + \int_{\Omega} |\nabla u|^2 \, dx = \int_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{N}} d\sigma, \tag{34}$$

and when $v \equiv 1$ in (29), we get the identity

$$\int_{\Omega} \triangle u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{N}} d\sigma = \int_{\partial\Omega} \nabla u \cdot \mathbf{N} d\sigma. \tag{35}$$

Physically, the quantity $\int_{\partial\Omega} \nabla u \cdot \mathbf{N} d\sigma$ is called the **flux** of the vector field $\nabla u$ across the boundary $\partial\Omega$ of $\Omega$. In particular, if $u \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$ is harmonic function on $\Omega$, we have $\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{N}} d\sigma = 0$ (zero flux across $\partial\Omega$).

## 1.3 Radial harmonic functions in $\mathbb{R}^n$.

In this section, we look at the Laplace equation on $\mathbb{R}^n$ :

$$\triangle u(x) = 0, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{36}$$

and we want to look for a special solution of the Laplace equation in $\mathbb{R}^n$ which is **radial**, i.e., it has the form

$$u(x) = v(r), \quad r = |x| = \sqrt{x_1^2 + \cdots + x_n^2},$$

where $v(r)$ is chosen so that it satisfies $\triangle u(x) = 0$ for all $x \in \mathbb{R}^n$. **Intuitively speaking, since Laplace equation has symmetry among all directions** $e_1, e_2, \ldots, e_n$**, such a radial solution should exists.**

**Remark 1.14** *At this moment, we do not know if such function $v(r)$ exists or not; we shall see that the function $v(r)$ does exist, but* ***cannot*** *be defined at $r = 0$, which implies that the radial function $u(x)$ cannot be defined at $x = 0$.*

**Instead of solving a PDE for** $u(x)$, **we only have to solve an ODE for** $v(r)$. By the chain rule, for $r > 0$ we have

$$\frac{\partial u}{\partial x_i}(x) = v'(r) \frac{x_i}{r}, \quad \frac{\partial^2 u}{\partial x_i^2}(x) = v''(r) \frac{x_i^2}{r^2} + v'(r) \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right), \quad 1 \leq i \leq n,$$

which gives (sum over $i = 1, 2, 3, \ldots, n$)

$$\triangle u(x) = v''(r) + \frac{n-1}{r} v'(r) = 0, \quad r > 0 \tag{37}$$

and we obtain a second-order ODE for $v(r)$ over the domain $r \in (0, \infty)$. Multiplying equation (37) by $r^{n-1}$, one can verify that the general solution of (37) is given by

$$v(r) = v(|x|) = u(x) = \begin{cases} Ar^{2-n} + B, & n > 2, \quad r = |x| \in (0, \infty) \\ A \log r + B, & n = 2, \quad r = |x| \in (0, \infty) \end{cases} \tag{38}$$

where $A,\ B$ are integration constants. Since $v(r)$ is **not** defined at $r=0$, the above radial solution $u(x)$ is not defined at $x=0$. The corresponding $u(x)=u(x_1,\ \dots,x_n)$ lies in the space $C^\infty(\mathbb{R}^n\backslash\{0\})$, given by

$$u(x_1,\ \dots,x_n)=\begin{cases} A\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{\frac{2-n}{2}}+B, & n>2,\\ A\log\sqrt{x_1^2+x_2^2}+B=\tilde{A}\log\left(x_1^2+x_2^2\right)+B, \quad \tilde{A}=\frac{A}{2}, & n=2.\end{cases} \tag{39}$$

**Remark 1.15** *(**Interesting observation.**) Why does the Laplace equation have radial solutions? This is because the Laplace operator has **symmetry** in it. If we change the Laplace operator into a **non-symmetric form**, for example, the form:*

$$\tilde{\triangle}=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}+\frac{\partial}{\partial x_1}=\triangle+\frac{\partial}{\partial x_1}\ \textit{(or other forms, say } \triangle+\frac{\partial^2}{\partial x_1^2}\textit{)},$$

*then if $u(x)$ has the radial form $v(r)$, $r=|x|$, we have*

$$\tilde{\triangle}u(x)=\triangle u(x)+\frac{\partial u}{\partial x_1}(x)=v''(r)+\frac{n-1}{r}v'(r)+\underbrace{v'(r)\frac{x_1}{r}},$$

*which **cannot produce a self-contained equation (ODE)** for $v(r)$ due to the term $v'(r)\frac{x_1}{r}$. Thus for the new operator $\tilde{\triangle}$, it has **no radial** solution at all (except the trivial constant solutions).*

We can conclude the following:

**Lemma 1.16** *Consider the Laplace equation on $\mathbb{R}^n$, given by*

$$(\triangle u)(x)=0,\quad x=(x_1,\ \dots,x_n)\in\mathbb{R}^n. \tag{40}$$

*Its **radial solution** (defined only on $\mathbb{R}^n\backslash\{0\}$) is given by*

$$u(x)=\begin{cases} A|x|^{2-n}+B, & n>2, \quad x\in\mathbb{R}^n\backslash\{0\}\\ A\log|x|+B, & n=2, \quad x\in\mathbb{R}^2\backslash\{0\}\end{cases} \tag{41}$$

*and no others. Here $A,\ B$ are two arbitrary constants.*

**Definition 1.17** *The radial function $u(x)$ given by (41) is also called the **fundamental solution** of the Laplace equation. It is a **harmonic function** defined on $\mathbb{R}^n\backslash\{0\}$.*

In $\mathbb{R}^3$ we have (now we denote $x\in\mathbb{R}^3\backslash\{0\}$ as $(x,y,z)\in\mathbb{R}^3\backslash\{0\}$) the **radial solution** of the Laplace equation:

$$u(x,y,z)=\frac{A}{r}+B=\frac{A}{\sqrt{x^2+y^2+z^2}}+B,\quad (x,y,z)\in\mathbb{R}^3\backslash\{0\}$$

and then

$$\begin{aligned}&(\nabla u)(x,y,z)\\ &=A\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}},\ \frac{-y}{(x^2+y^2+z^2)^{3/2}},\ \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)\\ &=\frac{A}{|\overrightarrow{r}|^2}\left(\frac{-\overrightarrow{r}}{|\overrightarrow{r}|}\ \text{(this is unit vector)}\right),\quad \text{where } \overrightarrow{r}=(x,y,z)\in\mathbb{R}^3\backslash\{0\}.\end{aligned} \tag{42}$$

For suitable constant $A$, (42) describes **the force field of the earth gravity** with point mass at the origin. Note that each component function of $(\nabla u)(x,y,z)$ is also harmonic on $\mathbb{R}^3\backslash\{0\}$.

## 1.4 Laplace equation in polar coordinates $(r,\theta)$; radial and angular harmonic functions in $\mathbb{R}^2$.

**Remark 1.18** *A major purpose of expressing Laplace operator in polar coordinates $(r,\theta)$ is to find some important special solutions, in particular, the **radial** solution $U(r)$ and the **angular** solution $U(\theta)$. In particular, we can use it to solve the Dirichlet problem of the Laplace equation on the **disc** in $\mathbb{R}^2$ or on the **ball** in $\mathbb{R}^3$.*

The polar coordinates $(r,\theta)$ in $\mathbb{R}^2$ and the Euclidean coordinates $(x,y)$ in $\mathbb{R}^2$ are related by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r>0, \quad 0<\theta<2\pi, \tag{43}$$

where the change of variables is a **diffeomorphism** between the following two **open sets**:

$$\mathbb{R}^2\setminus\{(x,0): x\geq 0\} \subset xy\text{-plane} \longleftrightarrow (r,\theta)\in(0,\infty)\times(0,2\pi)\subset r\theta\text{-plane}. \tag{44}$$

For convenience, we denote the above open in $xy$-plane set as $\tilde{\mathbb{R}}^2$ and denote the above open set in $r\theta$-plane as $\Sigma$ in this section. Let $u(x,y):\tilde{\mathbb{R}}^2\to\mathbb{R}$ be a $C^2$ function. Under the above change of variables $u(x,y)$ becomes a $C^2$ function $U(r,\theta):\Sigma\to\mathbb{R}$, i.e., $u(r\cos\theta, r\sin\theta)=U(r,\theta)$. What is the expression $u_{xx}(x,y)+u_{yy}(x,y)$ under polar coordinates $(r,\theta)$? The answer is:

$$\triangle u(x,y) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y) = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)U(r,\theta). \tag{45}$$

**Example 1.19** *Let $u(x,y)=x^2y$. Then $U(r,\theta)=r^3\cos^2\theta\sin\theta$. We have*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y)=2y.$$

*On the other hand, we also have*

$$\begin{aligned}&\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)U(r,\theta)\\ &=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(r^3\cos^2\theta\sin\theta\right)=2r\sin\theta.\end{aligned}$$

*Since $y=r\sin\theta$, both sides of (45) are equal.*

To derive (45), for the first derivatives, we have the relation:

$$\begin{aligned}\frac{\partial U}{\partial r}&=\frac{\partial}{\partial r}[u(r\cos\theta,r\sin\theta)]=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\\ &=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta=\frac{1}{r}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right), \quad r=\sqrt{x^2+y^2}\end{aligned} \tag{46}$$

and

$$\begin{aligned}\frac{\partial U}{\partial\theta}&=\frac{\partial}{\partial\theta}[u(r\cos\theta,r\sin\theta)]=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta}\\ &=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta)=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}.\end{aligned} \tag{47}$$

We can rewrite the above as the system:

$$r\frac{\partial U}{\partial r}=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}, \qquad \frac{\partial U}{\partial\theta}=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y} \tag{48}$$

or equivalently, the **operator relation:**

$$r\frac{\partial}{\partial r}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\qquad \frac{\partial}{\partial \theta}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}. \tag{49}$$

**It says that the operator $r\frac{\partial}{\partial r}$ is comparable to $\frac{\partial}{\partial \theta}$.** In the matrix form, we have the operator identity

$$\begin{pmatrix} r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta}\end{pmatrix} \text{ (acting on } U(r,\theta)\text{ )} = \begin{pmatrix} x & y \\ -y & x\end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y}\end{pmatrix} \text{ (acting on } u(x,y)\text{ )}, \tag{50}$$

and so

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y}\end{pmatrix} = \frac{1}{x^2+y^2}\begin{pmatrix} x & -y \\ y & x\end{pmatrix}\begin{pmatrix} r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta}\end{pmatrix} = \frac{1}{r}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta}\end{pmatrix}. \tag{51}$$

More precisely, the above gives

$$\frac{\partial u}{\partial x}=\cos\theta\frac{\partial U}{\partial r}-\frac{\sin\theta}{r}\frac{\partial U}{\partial \theta},\qquad \frac{\partial u}{\partial y}=\sin\theta\frac{\partial U}{\partial r}+\frac{\cos\theta}{r}\frac{\partial U}{\partial \theta}. \tag{52}$$

In particular, we have two different ways to express the **gradient vector** of $u$ (for clarity, we look at the vector $\nabla u$ at a particular point $(x_0, y_0)$):

$$\begin{aligned}\nabla u(x_0,y_0) &= \frac{\partial u}{\partial x}(x_0,y_0)\begin{pmatrix}1\\0\end{pmatrix}+\frac{\partial u}{\partial y}(x_0,y_0)\begin{pmatrix}0\\1\end{pmatrix}\\ &= \frac{\partial U}{\partial r}(r_0,\theta_0)\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix}+\frac{1}{r_0}\frac{\partial U}{\partial \theta}(r_0,\theta_0)\begin{pmatrix}-\sin\theta_0\\ \cos\theta_0\end{pmatrix},\end{aligned} \tag{53}$$

where $(r_0,\theta_0)$ is the polar coordinates corresponding to $(x_0,y_0)$. The above says that we can also express the gradient vector $\nabla u(x_0,y_0)$ in terms of the **orthonormal basis** $(\cos\theta_0,\sin\theta_0)$, $(-\sin\theta_0,\cos\theta_0)$, with the coefficients given by $\frac{\partial U}{\partial r}(r_0,\theta_0)$ and $\frac{1}{r_0}\frac{\partial U}{\partial \theta}(r_0,\theta_0)$.

**Remark 1.20** *Draw a picture for the vector $\nabla u(x_0,y_0)$ and the two orthonormal frames*

$$\{(1,0),\ (0,1)\},\qquad \{(\cos\theta_0,\sin\theta_0),\ (-\sin\theta_0,\cos\theta_0)\},$$

*where we note that the vector $(\cos\theta_0,\sin\theta_0)$ is pointing in the* ***radial direction*** *and the vector $(-\sin\theta_0,\cos\theta_0)$ is pointing in the* ***angular direction****.*

Keep going and use (49) to get

$$\begin{aligned}\left(r\frac{\partial}{\partial r}\right)^2 U &:= \left(r\frac{\partial}{\partial r}\right)\left[\left(r\frac{\partial}{\partial r}\right)U\right] = \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]\\ &= x\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]+y\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]\\ &= x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}.\end{aligned} \tag{54}$$

Similarly

$$\begin{aligned}\left(\frac{\partial}{\partial \theta}\right)^2 U &:= \frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial \theta}\right) = -y\frac{\partial}{\partial x}\left[-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}\right]+x\frac{\partial}{\partial y}\left[-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}\right]\\ &= y^2\frac{\partial^2 u}{\partial x^2}+x^2\frac{\partial^2 u}{\partial y^2}-2xy\frac{\partial^2 u}{\partial x\partial y}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}.\end{aligned} \tag{55}$$

Add (54) and (55) to get the beautiful identity:

$$\left(r\frac{\partial}{\partial r}\right)^2 U+\left(\frac{\partial}{\partial \theta}\right)^2 U=\left(x^2+y^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{56}$$

Finally, one can simplify $\left(r\frac{\partial}{\partial r}\right)^2 u$ as

$$\left(r\frac{\partial}{\partial r}\right)^2 U=\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial U}{\partial r}\right)=r^2\frac{\partial^2 U}{\partial r^2}+r\frac{\partial U}{\partial r} \tag{57}$$

and conclude the identity:

**Lemma 1.21** *(**Laplace operator in polar coordinates** $(r,\theta)$ **of** $\mathbb{R}^2$.) For any $C^2$ function $u(x,y)=u(r\cos\theta,r\sin\theta)=U(r,\theta)$ defined on $\mathbb{R}^2$, then on the two open sets (44), we have the identity*

$$\left(x^2+y^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\left(r\frac{\partial}{\partial r}\right)^2 U+\left(\frac{\partial}{\partial \theta}\right)^2 U, \tag{58}$$

*which is the same as*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\underbrace{\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}},\quad r>0,\quad 0<\theta<2\pi. \tag{59}$$

**Remark 1.22** *In particular, if $u(x,y)=U(r)$ is a radial function, (59) becomes*

$$U''(r)+\frac{1}{r}U'(r)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\quad r>0, \tag{60}$$

*which matches with (37).*

**Remark 1.23** *(**Omit this in class.**) (**Another method.**) The following more **straightforward method** can also be used, but it involves messier computations. By (51), we have the operator relation:*

$$\frac{\partial}{\partial x}=(\cos\theta)\frac{\partial}{\partial r}-\left(\frac{1}{r}\sin\theta\right)\frac{\partial}{\partial \theta},\qquad \frac{\partial}{\partial y}=(\sin\theta)\frac{\partial}{\partial r}+\left(\frac{1}{r}\cos\theta\right)\frac{\partial}{\partial \theta}. \tag{61}$$

*Then we have*

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=(\cos\theta)\frac{\partial}{\partial r}\left[(\cos\theta)\frac{\partial U}{\partial r}-\left(\frac{1}{r}\sin\theta\right)\frac{\partial U}{\partial \theta}\right]-\left(\frac{1}{r}\sin\theta\right)\frac{\partial}{\partial \theta}\left[(\cos\theta)\frac{\partial U}{\partial r}-\left(\frac{1}{r}\sin\theta\right)\frac{\partial U}{\partial \theta}\right]\\&=\left\{\begin{array}{l}\left(\cos^2\theta\right)\frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2}(\cos\theta\sin\theta)\frac{\partial U}{\partial \theta}-\frac{1}{r}(\cos\theta\sin\theta)\frac{\partial^2 U}{\partial r\partial \theta}\\+\frac{1}{r}\sin^2\theta\frac{\partial U}{\partial r}-\frac{1}{r}(\cos\theta\sin\theta)\frac{\partial^2 U}{\partial \theta\partial r}+\frac{1}{r^2}(\cos\theta\sin\theta)\frac{\partial U}{\partial \theta}+\frac{1}{r^2}\left(\sin^2\theta\right)\frac{\partial^2 U}{\partial \theta^2}\end{array}\right.\end{aligned} \tag{62}$$

*and*

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}&=(\sin\theta)\frac{\partial}{\partial r}\left[(\sin\theta)\frac{\partial U}{\partial r}+\left(\frac{1}{r}\cos\theta\right)\frac{\partial U}{\partial \theta}\right]+\left(\frac{1}{r}\cos\theta\right)\frac{\partial}{\partial \theta}\left[(\sin\theta)\frac{\partial U}{\partial r}+\left(\frac{1}{r}\cos\theta\right)\frac{\partial U}{\partial \theta}\right]\\&=\left\{\begin{array}{l}\left(\sin^2\theta\right)\frac{\partial^2 U}{\partial r^2}-\frac{1}{r^2}(\cos\theta\sin\theta)\frac{\partial U}{\partial \theta}+\frac{1}{r}(\cos\theta\sin\theta)\frac{\partial^2 U}{\partial r\partial \theta}\\+\frac{1}{r}\cos^2\theta\frac{\partial U}{\partial r}+\frac{1}{r}(\cos\theta\sin\theta)\frac{\partial^2 U}{\partial \theta\partial r}-\frac{1}{r^2}(\cos\theta\sin\theta)\frac{\partial U}{\partial \theta}+\frac{1}{r^2}\left(\cos^2\theta\right)\frac{\partial^2 U}{\partial \theta^2}.\end{array}\right.\end{aligned} \tag{63}$$

*Adding (62) and (63) together, we get*

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\\
&=\left\{\begin{array}{l}
\left\{\begin{array}{l}
\left(\cos^2\theta\right)\frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2}\left(\cos\theta\sin\theta\right)\frac{\partial U}{\partial\theta}-\frac{1}{r}\left(\cos\theta\sin\theta\right)\frac{\partial^2 U}{\partial r\partial\theta}\\
+\frac{1}{r}\sin^2\theta\frac{\partial U}{\partial r}-\frac{1}{r}\left(\cos\theta\sin\theta\right)\frac{\partial^2 U}{\partial\theta\partial r}+\frac{1}{r^2}\left(\cos\theta\sin\theta\right)\frac{\partial U}{\partial\theta}+\frac{1}{r^2}\left(\sin^2\theta\right)\frac{\partial^2 U}{\partial\theta^2}
\end{array}\right.\\
+\left\{\begin{array}{l}
\left(\sin^2\theta\right)\frac{\partial^2 U}{\partial r^2}-\frac{1}{r^2}\left(\cos\theta\sin\theta\right)\frac{\partial U}{\partial\theta}+\frac{1}{r}\left(\cos\theta\sin\theta\right)\frac{\partial^2 U}{\partial r\partial\theta}\\
+\frac{1}{r}\cos^2\theta\frac{\partial U}{\partial r}+\frac{1}{r}\left(\cos\theta\sin\theta\right)\frac{\partial^2 U}{\partial\theta\partial r}-\frac{1}{r^2}\left(\cos\theta\sin\theta\right)\frac{\partial U}{\partial\theta}+\frac{1}{r^2}\left(\cos^2\theta\right)\frac{\partial^2 U}{\partial\theta^2}
\end{array}\right.
\end{array}\right.\\
&=\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2},
\end{aligned}$$

*which gives the same formula.*

**Example 1.24** *(**Radial harmonic function on** $\mathbb{R}^2\backslash\{(0,0)\}$.) If a function $u(x,y)=U(r)$, $r=\sqrt{x^2+y^2}$, is **radial**, then by (58) we have*

$$\left(r\frac{\partial}{\partial r}\right)^2 U=\left(x^2+y^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{64}$$

*Thus a **radial harmonic function** $u(r)$ (defined on $\mathbb{R}^2\backslash\{(0,0)\}$) satisfies*

$$\left(r\frac{\partial}{\partial r}\right)^2 U=r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)=0,$$

*i.e.,*

$$r\frac{\partial U}{\partial r}=const.$$

*Hence*

$$u(x,y)=U(r)=a\ln r+b=a\log\sqrt{x^2+y^2}+b,\quad (x,y)\neq(0,0). \tag{65}$$

*for some constants $a$, $b$. Note that $u(x,y)$ is defined only on $\mathbb{R}^2\backslash\{(0,0)\}$ and*

$$\lim_{(x,y)\to(0,0)} u(x,y)=\infty\quad (\text{if } a>0).$$

*Its **gradient vector is pointing in the radial direction**, given by*

$$\nabla u(x,y)=\left(\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y)\right)=a\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right),\quad (x,y)\neq(0,0). \tag{66}$$

**Example 1.25** *(**Angular harmonic function on** $\mathbb{R}^2\backslash\{(x,y):x\geq 0\}$.) If a function $u(x,y)=U(\theta)$ depends only on angle $\theta\in(0,2\pi)$, then by (58) we have*

$$\left(\frac{\partial}{\partial\theta}\right)^2 U=\left(x^2+y^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{67}$$

*Thus an **angular harmonic function** $U(\theta)$ (defined on $\tilde{\mathbb{R}}^2=\mathbb{R}^2\backslash\{(x,y):x\geq 0\}$) satisfies $U''(\theta)=0$, i.e.*

$$u(x,y)=U(\theta)=c\theta+d=c\tan^{-1}\frac{y}{x}+d\ (\text{if } x\neq 0 \text{ and } (x,y) \text{ is in the first quadrant}) \tag{68}$$

*for some constants c, d. Its* ***gradient vector is perpendicular to the radial direction****, given by*

$$\nabla u(x,y)=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=c\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right),\quad (x,y)\neq(0,0). \tag{69}$$

*Since the derivative of a harmonic function is still harmonic, the functions*

$$\frac{x}{x^2+y^2},\qquad \frac{y}{x^2+y^2}$$

*are both harmonic in* $\mathbb{R}^2\setminus\{(0,0)\}$.

**Example 1.26** *(**Two important harmonic functions.**) By (59), one can check that* ***for any*** $n\in\mathbb{Z}$ *the two functions* $r^n\cos n\theta$, $r^n\sin n\theta$ *are both* ***harmonic functions*** *defined on* $\mathbb{R}^2\setminus\{(0,0)\}$ *(not just on* $\mathbb{R}^2\setminus\{(x,0):x\geq 0\}$*). For* $n\in\mathbb{N}$*, the functions are actually defined on the whole plane* $\mathbb{R}^2$*. The corresponding functions* $u(x,y)$ *are polynomials in the variables x and y with degree n, and are defined on the whole* $\mathbb{R}^2$*. They are called* ***harmonic polynomials*** *on* $\mathbb{R}^2$*. For example, when* $n=1$*, we get* $r\cos\theta=x$, $r\sin\theta=y$ *and for* $n=2$*, we get*

$$r^2\cos 2\theta=x^2-y^2,\qquad r^2\sin 2\theta=2xy,$$

*, etc. For* $n=-1$*, we get the familiar ones:*

$$r^{-1}\cos(-\theta)=\frac{r\cos\theta}{r^2}=\frac{x}{x^2+y^2},\qquad r^{-1}\sin(-\theta)=-\frac{r\sin\theta}{r^2}=\frac{-y}{x^2+y^2}.$$

## 1.5 Laplace equation is invariant under radial inversion in $\mathbb{R}^2$.

**Lemma 1.27** *(**Laplace equation is invariant under radial inversion on** $\mathbb{R}^2$.) Assume* $u(x,y)$ *is* ***harmonic*** *on* $\mathbb{R}^2$ *and let* $U(r,\theta)=u(r\cos\theta,r\sin\theta)$, $(r,\theta)\in(0,\infty)\times(0,2\pi)$*. Then the* ***radial inversion*** *function (i.e.* $r\to\frac{1}{r}$*)*

$$\tilde{U}(r,\theta)=U\left(\frac{1}{r},\theta\right),\quad (r,\theta)\in(0,\infty)\times(0,2\pi) \tag{70}$$

*also satisfies the equation*

$$\frac{\partial^2\tilde{U}}{\partial r^2}(r,\theta)+\frac{1}{r}\frac{\partial\tilde{U}}{\partial r}(r,\theta)+\frac{1}{r^2}\frac{\partial^2\tilde{U}}{\partial\theta^2}(r,\theta)=0 \tag{71}$$

*on* $(r,\theta)\in(0,\infty)\times(0,2\pi)$*. Therefore, if* $u(x,y)$ *is* ***harmonic*** *on* $\mathbb{R}^2$*, then the function*

$$\tilde{u}(x,y)=u\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right) \tag{72}$$

*is also harmonic on* $\mathbb{R}^2\setminus\{(0,0)\}$*. Note that the length of the vector* $(x/(x^2+y^2),y/(x^2+y^2))$ *is equal to* $1/\sqrt{x^2+y^2}$*.*

**Remark 1.28** *Lemma 1.27 is a special case of the* ***Kelvin transformation*** *in* $\mathbb{R}^n$*.*

**Proof.** Assume $(x,y)$ is **harmonic** on $\mathbb{R}^2$ and let $U(r,\theta)=u(r\cos\theta,r\sin\theta)$. It satisfies

$$\frac{\partial^2 U}{\partial r^2}(r,\theta)+\frac{1}{r}\frac{\partial U}{\partial r}(r,\theta)+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}(r,\theta)=0 \tag{73}$$

on the domain $(r,\theta)\in(0,\infty)\times(0,2\pi)$. We compute (write $U(r,\theta)$ as $U(s,\theta)$ to avoid confusion)

$$
\begin{aligned}
\tilde{U}(r,\theta) &= U(s,\theta), \quad s=\frac{1}{r} \\
\tilde{U}_r(r,\theta) &= U_s(s,\theta)\left(\frac{-1}{r^2}\right) = -s^2 U_s(s,\theta) \\
\tilde{U}_{rr}(r,\theta) &= U_{ss}(s,\theta)\left(\frac{1}{r^4}\right) + U_s(s,\theta)\left(\frac{2}{r^3}\right) = s^4 U_{ss}(s,\theta) + 2s^3 U_s(s,\theta) \\
\tilde{U}_{\theta\theta}(r,\theta) &= U_{\theta\theta}(s,\theta)
\end{aligned}
$$

and get

$$
\begin{aligned}
&\frac{\partial^2\tilde{U}}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial\tilde{U}}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2\tilde{U}}{\partial\theta^2}(r,\theta) \\
&= \left[s^4 U_{ss}(s,\theta) + 2s^3 U_s(s,\theta)\right] + s\left(-s^2 U_s(s,\theta)\right) + s^2 U_{\theta\theta}(s,\theta) \\
&= s^4\left\{\underbrace{U_{ss}(s,\theta) + \frac{1}{s}U_s(s,\theta) + \frac{1}{s^2}U_{\theta\theta}(s,\theta)}\right\} = 0. \qquad (74)
\end{aligned}
$$

The proof is done. □

**Example 1.29** *We know $x^2-y^2=r^2\cos 2\theta$ and $2xy=r^2\sin 2\theta$ are harmonic on $\mathbb{R}^2$. By (72) in Lemma 1.27, the functions*

$$
\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{1}{r^2}\cos 2\theta, \qquad \frac{2xy}{(x^2+y^2)^2} = \frac{1}{r^2}\sin 2\theta
$$

*are also harmonic on $\mathbb{R}^2\backslash\{(0,0)\}$.*

## 1.6 Laplace operator in spherical coordinates $(r,\theta,\varphi)$ of $\mathbb{R}^3$.

**Remark 1.30** *There are several different methods to derive the Laplace operator in spherical coordinates $(r,\theta,\varphi)$ of $\mathbb{R}^3$. Here we only provide the most straightforward method. For more discussions and details, see the file "**Laplace-equation-in-polar-and-spherical-coordinates-2019-4-28.tex**".*

The sphere coordinates in $\mathbb{R}^3$ is given by $(r,\theta,\varphi)$ and its relation with respect to the Euclidean coordinates is

$$
x = r\sin\varphi\cos\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\varphi, \quad x^2+y^2 = r^2\sin^2\varphi, \qquad (75)
$$

where $r>0,\ \theta\in(0,2\pi),\ \phi\in(0,\pi)$. We have

$$
r=\sqrt{x^2+y^2+z^2}, \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \sin\varphi\cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\varphi\sin\theta, \quad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\varphi \qquad (76)
$$

and

$$
\begin{cases}
\theta = \tan^{-1}\frac{y}{x}, \\
\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{r\sin\varphi\sin\theta}{r^2\sin^2\varphi} = -\frac{\sin\theta}{r\sin\varphi}, \\
\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{r\sin\varphi\cos\theta}{r^2\sin^2\varphi} = \frac{\cos\theta}{r\sin\varphi}, \\
\frac{\partial\theta}{\partial z} = 0,
\end{cases}
$$

and

$$\begin{cases} \varphi = \cos^{-1}\frac{z}{r}, \\ \frac{\partial \varphi}{\partial x} = \frac{-1}{\sqrt{1-\left(\frac{z}{r}\right)^2}}\frac{-z}{r^2}\frac{\partial r}{\partial x} = \frac{z}{r\sqrt{r^2-z^2}}\frac{\partial r}{\partial x} = \frac{r\cos\varphi}{r\sqrt{r^2\sin^2\varphi}}\sin\varphi\cos\theta = \frac{\cos\varphi\cos\theta}{r} \\ \frac{\partial \varphi}{\partial y} = \frac{r\cos\varphi}{r\sqrt{r^2\sin^2\varphi}}\sin\varphi\sin\theta = \frac{\cos\varphi\sin\theta}{r} \\ \frac{\partial \varphi}{\partial z} = \frac{-1}{\sqrt{1-\left(\frac{z}{r}\right)^2}}\left(\frac{1}{r}-\frac{z}{r^2}\frac{\partial r}{\partial z}\right) = \frac{-1}{\sqrt{r^2-z^2}}\left(1-\cos^2\varphi\right) = -\frac{\sin^2\varphi}{\sqrt{r^2\sin^2\varphi}} = -\frac{\sin\varphi}{r}. \end{cases}$$

With the above, we conclude the **first order operator relation:**

$$\begin{cases} \frac{\partial}{\partial x} = (\sin\varphi\cos\theta)\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\varphi}\frac{\partial}{\partial\theta} + \frac{\cos\varphi\cos\theta}{r}\frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial y} = (\sin\varphi\sin\theta)\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\varphi}\frac{\partial}{\partial\theta} + \frac{\cos\varphi\sin\theta}{r}\frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial z} = (\cos\varphi)\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}. \end{cases} \tag{77}$$

By (77), we have the following gradient vector relation similar to (53):

$$\begin{aligned} \nabla u(x,y,z) &= \frac{\partial u}{\partial x}(x,y,z)(1,0,0) + \frac{\partial u}{\partial y}(x,y,z)(0,1,0) + \frac{\partial u}{\partial z}(x,y,z)(0,0,1) \\ &= \frac{\partial U}{\partial r}\begin{pmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{pmatrix} + \frac{1}{r\sin\varphi}\frac{\partial U}{\partial\theta}\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} + \frac{1}{r}\frac{\partial U}{\partial\varphi}\begin{pmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{pmatrix} \\ &= \frac{\partial U}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\varphi}\frac{\partial U}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial U}{\partial\varphi}\mathbf{e}_\varphi, \end{aligned} \tag{78}$$

where the orthonormal basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$ is given by

$$\begin{cases} \mathbf{e}_r = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \\ \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0) \\ \mathbf{e}_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi). \end{cases} \tag{79}$$

**Theorem 1.31** *(**Laplace operator in spherical coordinates** $(r,\theta,\varphi)$ **of** $\mathbb{R}^3$.) Under the **spherical coordinates** $(r,\theta,\varphi)$ in $\mathbb{R}^3$, we have the identity:*

$$\begin{aligned} &\left[\left(r\frac{\partial}{\partial r}\right)^2 U + \left(r\frac{\partial}{\partial r}\right)U\right] + \frac{1}{\sin^2\varphi}\left[\left(\frac{\partial}{\partial\theta}\right)^2 U + \left((\sin\varphi)\frac{\partial}{\partial\varphi}\right)^2 U\right] \\ = &\left(x^2+y^2+z^2\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = r^2 \triangle u(x,y,z), \end{aligned} \tag{80}$$

*which is the same as*

$$\triangle u(x,y,z) = \left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 U}{\partial\theta^2} + \frac{1}{r^2}\left(\frac{\partial^2 U}{\partial\varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{\partial U}{\partial\varphi}\right). \tag{81}$$

**Proof.** This is a long but routine computation. By the operator relation (77), we need to check the following (**it is a tedious computation, but not difficult at all !!**):

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}(x,y,z)+\frac{\partial^2 u}{\partial y^2}(x,y,z)+\frac{\partial^2 u}{\partial z^2}(x,y,z)\\
&=\left\{\begin{array}{l}
\left[(\sin\varphi\cos\theta)\frac{\partial}{\partial r}-\frac{\sin\theta}{r\sin\varphi}\frac{\partial}{\partial\theta}+\frac{\cos\varphi\cos\theta}{r}\frac{\partial}{\partial\varphi}\right]\left((\sin\varphi\cos\theta)\frac{\partial U}{\partial r}-\frac{\sin\theta}{r\sin\varphi}\frac{\partial U}{\partial\theta}+\frac{\cos\varphi\cos\theta}{r}\frac{\partial U}{\partial\varphi}\right)\\
+\left[(\sin\varphi\sin\theta)\frac{\partial}{\partial r}+\frac{\cos\theta}{r\sin\varphi}\frac{\partial}{\partial\theta}+\frac{\cos\varphi\sin\theta}{r}\frac{\partial}{\partial\varphi}\right]\left((\sin\varphi\sin\theta)\frac{\partial U}{\partial r}+\frac{\cos\theta}{r\sin\varphi}\frac{\partial U}{\partial\theta}+\frac{\cos\varphi\sin\theta}{r}\frac{\partial U}{\partial\varphi}\right)\\
+\left[(\cos\varphi)\frac{\partial}{\partial r}-\frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}\right]\left((\cos\varphi)\frac{\partial U}{\partial r}-\frac{\sin\varphi}{r}\frac{\partial U}{\partial\varphi}\right)
\end{array}\right.\\
&=\left(\frac{\partial^2 U}{\partial r^2}+\frac{2}{r}\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2 U}{\partial\theta^2}+\frac{1}{r^2}\left(\frac{\partial^2 U}{\partial\varphi^2}+\frac{\cos\varphi}{\sin\varphi}\frac{\partial U}{\partial\varphi}\right).
\end{aligned}$$

□

**Remark 1.32** ***(Interesting observation.)*** *In case we have $u(x,y,z)=h(x,y)$ for some function $h$, i.e. the function $u(x,y,z)$ is independent of $z$, then the function $U(r,\theta,\varphi)$ is given by*

$$U(r,\theta,\varphi)=h(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta),\qquad r=\sqrt{x^2+y^2+z^2},\tag{82}$$

*which still depends on each variable $r$, $\theta$, and $\varphi$. In this case, it is* ***better*** *to use* ***cylindrical coordinates*** *$(R,\theta,z)$ in $\mathbb{R}^3$, where now $R=\sqrt{x^2+y^2}$, $x=R\cos\theta$, $y=R\sin\theta$, $z=z$. By this, we obtain*

$$u(x,y,z)=h(x,y)=h(R\cos\theta,R\sin\theta)\ \ (\textit{denote it as } H(R,\theta)\,),\tag{83}$$

*where by the formula (59) for the Laplace operator in the plane $\mathbb{R}^2$ under polar coordinates $(R,\theta)$, we have the identity*

$$\begin{aligned}
\triangle u(x,y,z)&=\frac{\partial^2 h}{\partial x^2}(x,y)+\frac{\partial^2 h}{\partial y^2}(x,y)\\
&=\frac{\partial^2 H}{\partial R^2}+\frac{1}{R}\frac{\partial H}{\partial R}+\frac{1}{R^2}\frac{\partial^2 H}{\partial\theta^2},\quad \textit{where}\quad R=\sqrt{x^2+y^2}.
\end{aligned}\tag{84}$$

*On the other hand, if we use $U(r,\theta,\varphi)$ in (82), then we must use the identity in (81).*

If $u(x,y,z)=U(r)$ is a radial function only, then **a radial harmonic function** $U(r)$ (defined on $\mathbb{R}^3\backslash\{(0,0,0)\}$) satisfies

$$U''(r)+\frac{2}{r}U'(r)=0\quad\text{on}\quad(0,\infty).\tag{85}$$

Its solution is given by

$$U(r)=\frac{a}{r}+b,\quad r\in(0,\infty),\quad a,\ b\text{ are constants}\tag{86}$$

or

$$u(x,y,z)=\frac{a}{\sqrt{x^2+y^2+z^2}}+b,\qquad (x,y,z)\in\mathbb{R}^3\backslash\{(0,0,0)\},\tag{87}$$

which has a singularity at the origin $O=(0,0,0)\in\mathbb{R}^3$ with

$$\lim_{(x,y,z)\to(0,0,0)}u(x,y,z)=\infty\quad(\text{if } a>0).\tag{88}$$

On the other hand, if $u(x,y,z)=U(\theta)$, i.e. it depends only on the angle $\theta$, we have $U''(\theta)=0$, $U(\theta)=c\theta+d$ for some constants $c$, $d$. This gives the **$\theta$-angular harmonic function:** (here we use the representation $\theta=\tan^{-1}\frac{y}{x}$)

$$u(x,y,z)=c\tan^{-1}\frac{y}{x}+d\quad(\text{if } (x,y,z)\text{ lies in the first octant of }\mathbb{R}^3).$$

The domain of $u(x,y,z) = U(\theta)$ is $\mathbb{R}^3 \backslash \{(x,0,z) : x \geq 0,\ z \in \mathbb{R}\}$ (same as $\theta \in (0, 2\pi)$). Note that the set $\{(x,0,z) : x \geq 0,\ z \in \mathbb{R}\}$ has **measure zero** in $\mathbb{R}^3$. Any point $p$ on this measure zero set has angle $\theta = 0$ or $\theta = 2\pi$.

Finally, if $u(x,y,z) = U(\varphi)$, i.e. it depends only on the angle $\varphi$, we have $U'(\varphi) = \frac{c}{\sin\varphi}$ for some constant $c$ and then

$$U(\varphi) = c\int \frac{1}{\sin\varphi} d\varphi + d = c\int \csc\varphi d\varphi + d = c\log|\csc\varphi - \cot\varphi| + d,$$

which gives the **$\varphi$-angular harmonic function:** (here we use the representation $\varphi = \cos^{-1}\frac{z}{r}$)

$$\begin{aligned} u(x,y,z) &= c\log|\csc\varphi - \cot\varphi| + d \\ &= c\log\left|\csc\left(\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \cot\left(\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right| + d, \end{aligned}$$

where $c,\ d$ are constants. The domain of $u(x,y,z) = U(\varphi)$ is $\mathbb{R}^3$ minus the $z$-axis (same as $\varphi \in (0,\pi)$). Any point $p = (0,0,z)$ on $z$-axis has angle $\varphi = 0$ or $\varphi = \pi$ and $\log|\csc\varphi - \cot\varphi|$ is undefined at $\varphi = 0$ and $\varphi = \pi$.

To end this section, we have the following invariant result similar to that in Lemma 1.27.

**Lemma 1.33** ***(Put this as a HW problem.)*** *Let $(r,\theta,\varphi)$ be the spherical coordinates in $\mathbb{R}^3$. Assume that $U(r,\theta,\varphi)$ is* ***harmonic*** *on the domain $(r,\theta,\varphi) \in (a,b)\times(0,2\pi)\times(0,\pi),\ 0 < a < b$. Show that the function*

$$V(r,\theta,\varphi) = \frac{1}{r}U\left(\frac{1}{r},\theta,\varphi\right)$$

*is also harmonic on the domain $(r,\theta,\varphi) \in \left(\frac{1}{b},\frac{1}{a}\right)\times(0,2\pi)\times(0,\pi)$.*

**Remark 1.34** *Lemma 1.33 is a special case of the* ***Kelvin transformation*** *in $\mathbb{R}^n$.*

**Proof.** For convenience, we denote $U(r,\theta,\varphi)$ as $U(s,\theta,\varphi)$ and it satisfies the equation

$$\frac{\partial^2 U}{\partial s^2} + \frac{2}{s}\frac{\partial U}{\partial s} + \frac{1}{s^2\sin^2\varphi}\left[\frac{\partial^2 U}{\partial\theta^2} + \left(\sin\varphi\frac{\partial}{\partial\varphi}\right)^2 U\right] = 0, \quad U = U(s,\theta,\varphi)$$

on the domain $(s,\theta,\varphi) \in (a,b)\times[0,2\pi]\times(0,\pi)$. Now we have

$$V(r,\theta,\varphi) = sU(s,\theta,\varphi), \quad s = \frac{1}{r} \in (a,b), \quad r \in \left(\frac{1}{b},\frac{1}{a}\right), \quad \frac{\partial s}{\partial r} = \frac{-1}{r^2} = -s^2$$

and by the chain rule we have

$$\frac{\partial V}{\partial r}(r,\theta,\varphi) = \frac{-1}{r^2}U(s,\theta,\varphi) + s\frac{\partial U}{\partial s}(s,\theta,\varphi)\frac{-1}{r^2} = -s^2 U(s,\theta,\varphi) - s^3\frac{\partial U}{\partial s}(s,\theta,\varphi)$$

and

$$\begin{aligned} &\frac{\partial^2 V}{\partial r^2}(r,\theta,\varphi) \\ &= -2s\left(-s^2\right)U(s,\theta,\varphi) - s^2\frac{\partial U}{\partial s}(s,\theta,\varphi)\cdot\left(-s^2\right) - 3s^2\left(-s^2\right)\frac{\partial U}{\partial s}(s,\theta,\varphi) - s^3\frac{\partial^2 U}{\partial s^2}(s,\theta,\varphi)\cdot\left(-s^2\right) \\ &= 2s^3 U(s,\theta,\varphi) + 4s^4\frac{\partial U}{\partial s}(s,\theta,\varphi) + s^5\frac{\partial^2 U}{\partial s^2}(s,\theta,\varphi) \end{aligned}$$

and

$$\frac{\partial^2 V}{\partial \theta^2}(r,\theta,\varphi) = s\frac{\partial^2 U}{\partial \theta^2}(s,\theta,\varphi), \quad \left(\sin\varphi\frac{\partial}{\partial\varphi}\right)^2 V(r,\theta,\varphi) = s\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)^2 U(s,\theta,\varphi).$$

Combining all of the above, we obtain

$$\begin{aligned}
&\underbrace{\frac{\partial^2 V}{\partial r^2}(r,\theta,\varphi)}+\underline{\frac{2}{r}\frac{\partial V}{\partial r}(r,\theta,\varphi)}+\frac{1}{r^2\sin^2\varphi}\left[\frac{\partial^2 V}{\partial \theta^2}(r,\theta,\varphi)+\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)^2 V(r,\theta,\varphi)\right]\\
&=\begin{cases}\underbrace{2s^3U(s,\theta,\varphi)+4s^4\frac{\partial U}{\partial s}(s,\theta,\varphi)+s^5\frac{\partial^2 U}{\partial s^2}(s,\theta,\varphi)}\\
+\underline{2s\left(-s^2U(s,\theta,\varphi)-s^3\frac{\partial U}{\partial s}(s,\theta,\varphi)\right)}\\
+\frac{1}{r^2\sin^2\varphi}\left[s\frac{\partial^2 U}{\partial\theta^2}(s,\theta,\varphi)+s\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)^2U(s,\theta,\varphi)\right], \quad s=\frac{1}{r}\end{cases}\\
&=\underbrace{2s^4\frac{\partial U}{\partial s}(s,\theta,\varphi)+s^5\frac{\partial^2 U}{\partial s^2}(s,\theta,\varphi)}+\frac{s^3}{\sin^2\varphi}\left[\frac{\partial^2 U}{\partial\theta^2}(s,\theta,\varphi)+\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)^2U(s,\theta,\varphi)\right]\\
&=s^5\left\{\frac{\partial^2 U}{\partial s^2}(s,\theta,\varphi)+\frac{2}{s}\frac{\partial U}{\partial s}(s,\theta,\varphi)+\frac{1}{s^2\sin^2\varphi}\left[\frac{\partial^2 U}{\partial\theta^2}(s,\theta,\varphi)+\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)^2U(s,\theta,\varphi)\right]\right\}=0.
\end{aligned}$$

The proof is done. □

## 1.7 The application of Green identities to Dirichlet problem for Poisson equation on bounded domains.

When $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, the most important question related to the Laplace operator is the Dirichlet problem for Poisson equation on $\Omega$. Let $\Omega \subseteq \mathbb{R}^2$ ($\Omega \subseteq \mathbb{R}^n$ is also OK ..) be a bounded $C^1$ domain and let $f(x)$, $h(x)$ be continuous functions on $\Omega$ and $\partial\Omega$ respectively. The **Dirichlet problem for Poisson equation** has the form

$$\begin{cases}\triangle u(x) = f(x) & \text{in} \quad \Omega \subseteq \mathbb{R}^2\\ u(x) = h(x) & \text{on} \quad \partial\Omega.\end{cases} \tag{89}$$

One can use (34) to show that (89) has a **unique** solution (we will not discuss the **existence** of a solution here). In PDE theory, the boundary condition $u(x) = h(x)$ on $\partial\Omega$ is also called **Dirichlet condition**.

**Lemma 1.35** *(**Uniqueness of solution for Dirichlet problem of Poisson equation.**) Let $u,\ v \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$ be two solutions of (89) on $\Omega$, where $f \in C^0(\Omega)$ and $h \in C^1(\partial\Omega)$ are given. Then we must have $u \equiv v$ on $\bar{\Omega}$.*

**Proof.** Set $w = u - v \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$. It satisfies

$$\begin{cases}\triangle w(x) = 0 & \text{in} \quad \Omega \subseteq \mathbb{R}^2\\ w(x) = 0 & \text{on} \quad \partial\Omega.\end{cases} \tag{90}$$

By (34), we have the identity

$$\int_\Omega w \triangle w dx + \int_\Omega |\nabla w|^2 dx = \int_{\partial\Omega} w\frac{\partial w}{\partial \mathbf{N}} d\sigma \ (d\sigma = ds \text{ here since we are in } \mathbb{R}^2), \tag{91}$$

which, together with (90), gives the identity

$$\int_{\Omega} |\nabla w|^2 \, dx = 0, \tag{92}$$

where we also know that $|\nabla w|^2$ is a continuous function on $\Omega$ with $|\nabla w|^2 \geq 0$ everywhere. Hence we conclude $|\nabla w|^2 \equiv 0$ on $\Omega$ and $w(x)$ must be a constant function on $\Omega$. As $w(x) = 0$ on $\partial\Omega$, we must have $w(x) \equiv 0$ on $\bar{\Omega}$. The proof is done. □

**Remark 1.36** ***(Important.)*** *The above uniqueness result does not hold on* ***unbounded domains****. This is because the divergence theorem and the Green identities* ***are valid only on bounded domains*** $\Omega \subset \mathbb{R}^n$*. See Remark 1.6. For example, the two functions*

$$w(x,y) \equiv 0, \quad v(x,y) = y, \quad (x,y) \in \mathbb{R}^2_{y+} = \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\}$$

*all satisfy the equation* $\triangle u = 0$ *on* $\mathbb{R}^2_{y+}$ *with* $u \equiv 0$ *on* $\partial\mathbb{R}^2_{y+}$.

## 1.8 Comparing the Laplace equation and the wave equation.

There are many striking differences between the Laplace equation and the wave equation. One can notice this from many aspects. For simplicity, we just look at one interesting example. For the convenience of constructing a solution for the Laplace equation, here we take $\Omega$ to be unbounded, with $\Omega = \mathbb{R}^2$, and consider the following two initial value problems (here for $u(x,y)$ we view $y$ as time and denote it as $t$):

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\partial^2 u}{\partial t^2}(x,t) = 0, & (x,t) \in \mathbb{R}^2, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in (-\infty,\infty). \end{cases} \tag{93}$$

and

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial^2 u}{\partial t^2}(x,t) = 0, & (x,t) \in \mathbb{R}^2, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in (-\infty,\infty) \end{cases} \tag{94}$$

We know that the equation in problem (94) is a **wave equation** and the ivp has a **unique solution** given by:

$$u(x,t) = \frac{1}{2}\left[\phi(x+t) + \phi(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\, ds, \quad \forall\ (x,t) \in \mathbb{R}^2. \tag{95}$$

From it we can see that the problem is **well-posed** (the solution **exists**, is **unique**, and depends "**continuously**" (in some distance sense) on the data $\phi(x),\ \psi(x)$.

On the other hand, the equation in problem (93) is a **Laplace equation** and it is **not well-posed**. To see this, in both examples we take $\phi(x) = 0$ and $\psi(x) = \frac{\sin nx}{n}$, $x \in (-\infty,\infty)$, where $n \in \mathbb{N}$ is a positive integer. Clearly, the function

$$u_n(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \frac{\sin ns}{n} ds = \frac{1}{2}\left[-\frac{\cos n(x+t)}{n^2} + \frac{\cos n(x-t)}{n^2}\right], \quad (x,t) \in \mathbb{R}^2 \tag{96}$$

is a solution of the **wave equation** problem (94). As $n \to \infty$, the initial data $\psi(x)$, although oscillates a lot, satisfies

$$\lim_{n\to\infty} |\psi(x)| = \lim_{n\to\infty} \left|\frac{\sin nx}{n}\right| = 0 \text{ **uniformly** in } x \in (-\infty,\infty). \tag{97}$$

For wave equation solution $u_n(x,t)$, given by (96), it also satisfies

$$\lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} \frac{1}{2}\left[-\frac{\cos n(x+t)}{n^2} + \frac{\cos n(x-t)}{n^2}\right] = 0 \text{ **uniformly** in } (x,t) \in \mathbb{R}^2. \tag{98}$$

On the other hand, one can verify that the function

$$u_n(x,t) = \frac{(\sinh nt)(\sin nx)}{n^2}, \quad (x,t) \in \mathbb{R}^2 \tag{99}$$

is a solution of the **Laplace equation** problem (93) with

$$u_n(x,0) = 0, \quad (u_n)_t(x,0) = \frac{\sin nx}{n}, \quad \forall\, x \in \mathbb{R}. \tag{100}$$

However, note that

$$u_n\left(\frac{\pi}{2n}, \frac{1}{\sqrt{n}}\right) = \frac{\left(\sinh n\left(\frac{1}{\sqrt{n}}\right)\right)\left(\sin n\left(\frac{\pi}{2n}\right)\right)}{n^2} = \frac{\sinh\sqrt{n}}{n^2} \to \infty \quad \text{as} \quad n \to \infty. \tag{101}$$

By (101), we see that the solution to problem (93) does not depend on the initial data $\phi(x)$ and $\psi(x)$ **in a continuous way**. Therefore, the initial value problem (93) for the Laplace equation is **not well-posed**. As $n \to \infty$, we have $\frac{\sin nx}{n} \to 0$ uniformly on $x \in \mathbb{R}$. So naturally we expect the solution $u_n(x,t)$ to be uniformly small too. However, there exists a sequence $(x_n, t_n) \in \mathbb{R}^2$ with $\lim_{n\to\infty}(x_n, t_n) = (0,0)$ such that $\lim_{n\to\infty} u_n(x_n, y_n) = \infty$.

**Remark 1.37** ***(Important observation.)*** *On the other hand, if we take $\Omega \subset \mathbb{R}^2$ (the $(x,t)$-plane) to be a $C^1$* ***bounded domain*** *and look at the following two* ***Dirichlet problems (boundary value problems):***

$$\begin{cases} \triangle u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\partial^2 u}{\partial t^2}(x,t) = f(x,t) & \text{in} \quad \Omega \subseteq \mathbb{R}^2 \\ u(x,t) = h(x,t) & \text{on} \quad \partial\Omega \end{cases} \tag{102}$$

*and*

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial^2 u}{\partial t^2}(x,t) = f(x,t) & \text{in} \quad \Omega \subseteq \mathbb{R}^2 \\ u(x,t) = h(x,t) & \text{on} \quad \partial\Omega, \end{cases} \tag{103}$$

*where $f \in C^0(\Omega)$ and $h \in C^1(\partial\Omega)$ are given (to be safe, maybe take $\Omega \subset \mathbb{R}^2$ to be a* ***smooth bounded domain*** *and $f$ and $h$ to be two* ***smooth*** *functions on $\bar{\Omega}$ and $\partial\Omega$ respectively; with this, the existence of a solution for (102) has no problem), then the problem (102) is* ***well-posed*** *and the problem (103) is* ***not well-posed****. For example, take $\Omega$ to be the unit disc in $\mathbb{R}^2$ and $f = h \equiv 0$, then by Lemma 1.35, problem (102) has unique solution $u(x,t) \equiv 0$, but problem (103)* ***does not*** *have unique solution. The two functions (note that the general solution of the wave equation has the form $F(x+t) + G(x-t)$)*

$$u(x,t) \equiv 0, \quad u(x,t) = (x+t)^2 + (x-t)^2 - 2 = 2\left(x^2 + t^2\right) - 2 \tag{104}$$

*are both solutions. From this, one can see that problem (103) is* ***not well-posed****.*

**Example 1.38** ***(Important.) (Omit this in class !!)*** *Laplace equation is also not well-posed for* ***Dirichlet problem (boundary value problem)*** *on "****unbounded****" domains. For example, we take $\Omega = \mathbb{R}^2$, Consider the equation*

$$\begin{cases} \triangle u(x,y) = 0 & \text{for} \quad y > 0 \\ u(x,0) = \frac{\sin nx}{n}, & x \in \mathbb{R} \end{cases}$$

*and note that for each $n \in \mathbb{N}$ the function*

$$u_n(x,y) = \frac{\sin nx \cdot e^{ny}}{n}, \quad n \in \mathbb{N}$$

*is a harmonic function in the upper half plane with* $u_n(x,0) = \frac{\sin nx}{n}$ *for all* $x \in \mathbb{R}$*. For large* $n$*,* $u_n(x,0)$ *is close to* 0 *(but oscillate a lot when* $n$ *is large) uniformly in* $x \in \mathbb{R}$*, but we have*

$$\sup_{x\in\mathbb{R},\ y\in[0,\varepsilon)} |u_n(x,y)| = \sup_{x\in\mathbb{R},\ y\in[0,\varepsilon)} \left|\frac{\sin nx \cdot e^{ny}}{n}\right| \to \infty \quad as \quad n\to\infty$$

*for any* $\varepsilon > 0$*. We conclude: Laplace equation is also* ***not well-posed*** *for "****Dirichlet problems on unbounded domains****". Another easy example is that the problem*

$$\begin{cases} \triangle u(x,y) = 0 \quad for \ \ y>0 \\ u(x,0) = 0, \quad x\in\mathbb{R} \end{cases}$$

*has a nonzero solution* $u(x,y) = y$ *on the space* $\mathbb{R}^2_+ = \{(x,y)\in\mathbb{R}^2 : y>0\}$*.*

## 1.9 The weak maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.

From now on, $\Omega$ will denote a **bounded domain** in $\mathbb{R}^2$ or $\mathbb{R}^3$. However, the maximum/minimum principle for harmonic functions on any bounded domain $\Omega\subset\mathbb{R}^n$ is also valid.

**Lemma 1.39** *(****Weak maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.****) Let* $\Omega$ *be a* ***bounded domain*** *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. Assume* $u\in C^2(\Omega)\bigcap C^0(\bar\Omega)$*. If* $\triangle u \geq 0$ $(\leq 0)$ *everywhere in* $\Omega$*, then*

$$\max_{\bar\Omega} u = \max_{\partial\Omega} u \quad \left(\min_{\bar\Omega} u = \min_{\partial\Omega} u\right). \tag{105}$$

*Consequently for a* ***harmonic*** *function* $u\in C^2(\Omega)\bigcap C^0(\bar\Omega)$*, we have*

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u, \quad \forall\ x\in\bar\Omega. \tag{106}$$

**Remark 1.40** *The condition that* $\Omega$ *is a* ***bounded*** *domain is* ***essential*** *in the above lemma. Lemma 1.39 is* ***false*** *on unbounded domains. For example, take* $u(x,y) = y$ *on* $\mathbb{R}^2_{y+}$*. It does not satisfy the maximum principle.*

**Remark 1.41** *By the above lemma, we conclude that, if* $\triangle u\geq 0$ *in* $\Omega$*, it satisfies the* ***maximum principle****; and if* $\triangle u \leq 0$ *in* $\Omega$*, it satisfies the* ***minimum principle.***

**Remark 1.42** *(****An interesting simple example.****) Here is a simple example in the 1-dimensional case. Let* $f(x) = x^2,\ x\in(-1,1)$*. It satisfies* $f''(x) = 2>0$ *everywhere in* $\Omega = (-1,1)\subset\mathbb{R}$*. The maximum value of* $f(x)$ *on* $\bar\Omega$ *is* 1*, attained at* $x=\pm1\in\partial\Omega$*. Similarly, the function* $g(x) = -x^2,\ x\in(-1,1)$ *satisfies* $g''(x) = -2<0$ *everywhere in* $\Omega=(-1,1)\subset\mathbb{R}$*. The minimum value of* $g(x)$ *on* $\bar\Omega$ *is* $-1$*, attained at* $x=\pm1\in\partial\Omega$*. Finally, the function* $h(x) = ax+b$ *(*$a$*,* $b$ *are any two numbers) is* ***harmonic*** *on* $\Omega=(-1,1)$*. Its maximum value and minimum value attained at* $x=\pm1\in\partial\Omega$ *respectively. Unless* $h(x)$ *is a* ***constant*** *function, otherwise, it is impossible for the harmonic function* $h(x)$ *to attain its maximum value (or minimum value) at some* $x_0\in\Omega = (-1,1)$ *(this is the* ***strong maximum/minimum principle****, to be proved later on).*

**Proof.** Assume first that $\triangle u>0$ everywhere in $\Omega$. Then since $u$ is continuous on $\bar\Omega$ (compact set), there is some point $p\in\bar\Omega$ such that $u(p) = \max_{\bar\Omega} u$. If $p\in\partial\Omega$, the result follows. If $p\in\Omega$ (interior point), we get

$$\triangle u(p) = \frac{\partial^2 u}{\partial x^2}(p) + \frac{\partial^2 u}{\partial y^2}(p) \leq 0, \tag{107}$$

a contradiction. Hence $p$ must lie on the boundary of $\Omega$ and so $\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u$ (note that we always have $\max_{\bar{\Omega}} u \ge \max_{\partial\Omega} u$).

Next, assume that $\triangle u \ge 0$ everywhere in $\Omega$. We can use a small perturbation argument. Let

$$v(x,y) = u(x,y) + \varepsilon\left(x^2+y^2\right), \quad (x,y) \in \bar{\Omega},$$

where $\varepsilon > 0$ is a constant. We have

$$\triangle v(x,y) = \triangle u(x,y) + 4\varepsilon \ge 0 + 4\varepsilon > 0 \quad \text{everywhere in} \quad \Omega.$$

Hence we have

$$\max_{\bar{\Omega}} u \le \max_{\bar{\Omega}} v = \max_{\partial\Omega} v \le \left(\max_{\partial\Omega} u\right) + \varepsilon \max_{\bar{\Omega}}\left(x^2+y^2\right). \tag{108}$$

As $\varepsilon > 0$ is arbitrary and $\Omega$ is a **bounded** domain (hence $\max_{\bar{\Omega}}(x^2+y^2)$ is finite), letting $\varepsilon \to 0^+$ in (108), we conclude

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u.$$

On the other hand, we also have $\max_{\bar{\Omega}} u \ge \max_{\partial\Omega} u$ and so (105) is verified. The proof of the minimum case is similar. □

### 1.9.1 Application of the weak maximum/minimum principle.

Let $f \in C^0(\Omega)$ and $h \in C^0(\partial\Omega)$. We can consider the following problem on bounded domain $\Omega$ :

$$\begin{cases} \triangle u(x) = f(x) & \text{in} \quad \Omega \\ u(x) = h(x) & \text{on} \quad \partial\Omega. \end{cases} \tag{109}$$

This problem is **well-posed** and we have the following **uniqueness** property due to the maximum principle:

**Lemma 1.43** ***(Uniqueness of solution for Dirichlet problem of Poisson equation.)*** *The problem (109) has at most one solution $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$.*

**Remark 1.44** *Recall that we have used Green identity to prove Lemma 1.43 before for the case $u \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$. See Lemma 1.35.*

**Proof.** Assume there are two solutions $u,\ v \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$. Then the function $w = u - v \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$ satisfies

$$\begin{cases} \triangle w(x) = 0 & \text{in} \quad \Omega \\ w(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{110}$$

By the weak maximum/minimum principle, we have $w \equiv 0$ in $\Omega$. Hence $u \equiv v$ in $\Omega$ (and so on $\bar{\Omega}$). □

## 1.10 Poisson integral formula in the plane; Dirichlet problem on a disc.

Let $B_a(0) \subset \mathbb{R}^2$ be the open disc centered at the origin $(0,0)$ with radius $a > 0$. We want to solve the **Dirichlet problem** (this is a **well-posed problem**):

$$\begin{cases} \triangle u(x,y) = 0 & \text{in} \quad (x,y) \in B_a(0) \subset \mathbb{R}^2 \\ u(x,y) = h(x,y) & \text{on} \quad (x,y) \in \partial B_a(0) \ \text{(the boundary of } B_a(0)\text{)}, \end{cases} \tag{111}$$

where $h(x,y)$ is a given **continuous** function defined on $\partial B_a(0)$.

Due to the **symmetry** of the domain and the **symmetry** of the Laplace operator, there is a **solution formula** for this problem. The solution lies in the space $C^2(B_a(0)) \bigcap C^0\left(\overline{B_a(0)}\right)$. Moreover, by the maximum principle, the solution in the function space $C^2(B_a(0)) \bigcap C^0\left(\overline{B_a(0)}\right)$ is **unique**.

As the domain $B_a(0)$ is a disc, to solve the problem (111), it is natural to use **polar coordinates** $(r, \theta)$ instead of the Euclidean coordinates $(x, y)$. The Laplace equation under polar coordinates $(r, \theta)$ is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad u = u(r, \theta), \quad 0 < r < a, \quad 0 < \theta < 2\pi. \tag{112}$$

and we we want to solve it. Since in the problem (111) the solution $u(x, y)$ is to be defined at $(x, y) = (0, 0)$, we hope equation (112) can also be defined at $r = 0$ and the solution $u(r, \theta)$ **will not have a singularity at** $r = 0$ (i.e. $u(r, \theta)$ is **well-defined at** $r = 0$). In view of this, we multiply the equation by $r^2$ to get

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad u = u(r, \theta), \quad 0 < r < a, \quad 0 < \theta < 2\pi \tag{113}$$

and focus on equation (113). Moreover, we also want the solution $u(r, \theta)$ to be $2\pi$**-periodic in** $\theta \in (-\infty, \infty)$ so that the corresponding solution $u(x, y)$ can be defined on the whole open disc $B_a(0)$. Using polar coordinates $(r, \theta)$, the problem (111) can be expressed as (note that since the radius of $B_a(0)$ is fixed, the continuous boundary function $h(x, y)$ on $\partial B_a(0)$ is a $2\pi$**-periodic** continuous function depending only on $\theta \in [0, 2\pi]$)

$$\begin{cases} (1).\ r^2\frac{\partial^2 u}{\partial r^2}(r, \theta) + r\frac{\partial u}{\partial r}(r, \theta) + \frac{\partial^2 u}{\partial \theta^2}(r, \theta) = 0, & (r, \theta) \in [0, a) \times [0, 2\pi] \\ (2).\ u(a, \theta) = h(\theta), \quad \forall\, \theta \in [0, 2\pi]. \end{cases} \tag{114}$$

Recall that there is a family of **separable solutions** $\{u_n(r, \theta)\}_{n=0}^{\infty}$ for equation (1) in (114), which are defined on $(r, \theta) \in [0, a) \times [0, 2\pi]$, given by

$$u_n(r, \theta) = \begin{cases} r^n(A_n \cos n\theta + B_n \sin n\theta), & n \in \mathbb{N}, \quad (r, \theta) \in [0, a) \times [0, 2\pi] \\ A_0, & n = 0, \end{cases} \tag{115}$$

where $A_0,\ A_n,\ B_n$ are arbitrary constants. They satisfy the boundary condition

$$u_n(a, \theta) = a^n(A_n \cos n\theta + B_n \sin n\theta), \quad \forall\, \theta \in [0, 2\pi]. \tag{116}$$

Therefore, if $h(\theta)$ is a finite linear combination of $\cos n\theta$ and $\sin n\theta$ for $n \in \mathbb{N}$, then (114) can be easily solved.

**Remark 1.45** *Note that each corresponding function $u_n(x, y)$ is a* ***harmonic polynomial*** *on the whole disc $B_a(0)$ (or on $\mathbb{R}^2$).*

**Example 1.46** *Solve the problem on $B_a(0)$:*

$$\begin{cases} (1).\ r^2\frac{\partial^2 u}{\partial r^2}(r, \theta) + r\frac{\partial u}{\partial r}(r, \theta) + \frac{\partial^2 u}{\partial \theta^2}(r, \theta) = 0, & (r, \theta) \in [0, a) \times [0, 2\pi] \\ (2).\ u(a, \theta) = 3\sin\theta - 5\cos\theta + 7\cos(2\theta), \quad \forall\, \theta \in [0, 2\pi]. \end{cases} \tag{117}$$

<u>**Solution**</u>:

The function

$$u(r, \theta) = 3\left(\frac{r}{a}\right)\sin\theta - 5\left(\frac{r}{a}\right)\cos\theta + 7\left(\frac{r}{a}\right)^2\cos(2\theta), \quad (r, \theta) \in [0, a) \times [0, 2\pi]$$

is clearly a solution of (**??**). In terms of $(x,y)$-coordinates, the corresponding $u(x,y)$ is

$$u(x,y)=3\left(\frac{y}{a}\right)-5\left(\frac{x}{a}\right)+7\left(\frac{x^2-y^2}{a^2}\right),\quad (x,y)\in B_a(0)$$

and the corresponding function $h(x,y)$ on $\partial B_a(0)$ is given by (note that $\theta=\tan^{-1}\frac{y}{x}$)

$$h(x,y)=3\frac{y}{\sqrt{x^2+y^2}}-5\frac{x}{\sqrt{x^2+y^2}}+7\left(\frac{x^2-y^2}{x^2+y^2}\right),\quad (x,y)\in\partial B_a(0).$$

□

We can use the above observation as a **motivation** to solve the general Dirichlet problem (114) for **arbitrary continuous function** $h(\theta)$ **"if one can express** $h(\theta)$ **as a linear combination of all possible** $\cos n\theta$ **and** $\sin n\theta$ **for all possible** $n\in\mathbb{N}$". This is indeed possible if we assume $h(\theta)$ is a $2\pi$**-periodic** $C^1$ **function** on $\theta\in[0,2\pi]$. More precisely, we have the following result from **Fourier series theory**:

**Lemma 1.47** *(**Fourier series result.**) Assume $h(\theta)$ is a $2\pi$**-periodic** $C^1$ **function** defined on $\theta\in[0,2\pi]$ (or on $\theta\in\mathbb{R}$). Then the following series*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\theta+b_n\sin n\theta\right),\quad \theta\in[0,2\pi] \tag{118}$$

*converges **absolutely** and **uniformly** to $h(\theta)$ on $[0,2\pi]$. That is*

$$h(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\theta+b_n\sin n\theta\right),\quad \forall\,\theta\in[0,2\pi], \tag{119}$$

*where*

$$\begin{cases} a_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}h(\varphi)\cos(n\varphi)\,d\varphi, & n=0,\ 1,\ 2,\ 3,\ ...,\\ b_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}h(\varphi)\sin(n\varphi)\,d\varphi, & n=1,\ 2,\ 3,\ .... \end{cases} \tag{120}$$

*The series (118) is called the **Fourier series** of the function $h$ on $\theta\in[0,2\pi]$.*

**Remark 1.48** *The above lemma **fails** if we only assume $h(\theta)$ is a $2\pi$-periodic **continuous** function.*

**Proof.** Omit. □

We now assume that $h(\theta)$ is a $2\pi$-periodic $C^1$ **function** and see what we can do (in the problem (114) we only assume $h$ to be a continuous function). Motivated by the Fourier series, we now consider the sum of $u_n(r,\theta)$ from (115) for all $n=0,\ 1,\ 2,\ 3,\ ...$, and get a function of the form

$$u(r,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}r^n\left(A_n\cos n\theta+B_n\sin n\theta\right),\quad (r,\theta)\in[0,a)\times[0,2\pi], \tag{121}$$

where the constants $A_0,\ A_n,\ B_n$ will be chosen so that when $r=a$ it reduces to the Fourier series of $h(\theta)$ on $\theta\in[0,2\pi]$.

**Remark 1.49** *Since $A_0/2$ and each $r^n (A_n \cos n\theta + B_n \sin n\theta)$ are harmonic on $(r,\theta) \in [0,a) \times [0,2\pi]$, we "**expect**" that the series (121) to converge to a **harmonic function** on $[0,a) \times [0,2\pi]$. Moreover, if we choose the coefficients $A_0$, $A_n$ and $B_n$ suitably, then the sum of the series (i.e. the function $u(r,\theta)$ in (121)) will tend to $h(\theta)$ as $r \to a$ and the Dirichlet problem (114) can be solved. See below for details.*

To satisfy the boundary condition we need to require

$$u(a,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta) = h(\theta), \quad \forall\, \theta \in [0,2\pi]. \tag{122}$$

Hence, by (120), we must require

$$\begin{cases} A_0 = \frac{1}{\pi} \int_0^{2\pi} h(\varphi)\, d\varphi \\ A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \cos(n\varphi)\, d\varphi, \qquad B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \sin(n\varphi)\, d\varphi, \quad n \in \mathbb{N}. \end{cases} \tag{123}$$

**Remark 1.50** *It is **impossible** to have $u(a,\theta) = h(\theta)$ for **arbitrary** $h(\theta)$ if we only consider **finite sum** in (122).*

By the above, we arrive at a series of the form

$$\begin{aligned} u(r,\theta) &= \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} h(\varphi)\, d\varphi \\ + \sum_{n=1}^{\infty} r^n \left[ \left( \frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \cos(n\varphi)\, d\varphi \right) \cos n\theta + \left( \frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \sin(n\varphi)\, d\varphi \right) \sin n\theta \right] \end{cases} \\ &= \frac{1}{2\pi} \int_0^{2\pi} h(\varphi)\, d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} \underbrace{\left(\frac{r}{a}\right)^n h(\varphi) [\cos n\varphi \cos n\theta + \sin n\varphi \sin n\theta]}\, d\varphi \\ &= \frac{1}{2\pi} \left[ \int_0^{2\pi} h(\varphi)\, d\varphi + 2 \sum_{n=1}^{\infty} \int_0^{2\pi} \left( \underbrace{\left(\frac{r}{a}\right)^n \cos n(\theta - \varphi)} \right) h(\varphi)\, d\varphi \right], \end{aligned} \tag{124}$$

where $(r,\theta) \in [0,a) \times [0,2\pi]$.

The identity (124) gives us a **motivation** to look at the series

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \varphi), \tag{125}$$

which is **convergent** when $(r,\theta,\varphi) \in [0,a) \times [0,2\pi] \times [0,2\pi]$ and we can find its sum explicitly. Before we go on, we note the following comparison:

1. If we assume $h(\theta)$ is a $2\pi$-periodic $C^1$ **function,** then the series (124) **converges** at $r = a$ and it is equal to the Fourier series of $h(\theta)$.
2. On the other hand, the series (125) **diverges** at $r = a$ for any $\theta$, $\varphi$.

To go on, motivated by (124), we study the following series properties:

**Lemma 1.51** *The series*

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \varphi) \tag{126}$$

*converges **absolutely** on $(r,\theta,\varphi)\in[0,a)\times[0,2\pi]\times[0,2\pi]$ and **uniformly** on $(r,\theta,\varphi)\in[0,a-\varepsilon]\times[0,2\pi]\times[0,2\pi]$ for any small $\varepsilon>0$. We also have*

$$\frac{\partial^k}{\partial r^k}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right)=2\sum_{n=1}^{\infty}\frac{\partial^k}{\partial r^k}\left[\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right] \tag{127}$$

*and*

$$\frac{\partial^k}{\partial \theta^k}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right)=2\sum_{n=1}^{\infty}\frac{\partial^k}{\partial \theta^k}\left[\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right] \tag{128}$$

*on $(r,\theta,\varphi)\in[0,a)\times[0,2\pi]\times[0,2\pi]$ for all $k\in\mathbb{N}$. For each $k\in\mathbb{N}$, the series on the right hand side of (127) and (128) also converge **absolutely** on $[0,a)\times[0,2\pi]\times[0,2\pi]$ and **uniformly** on $[0,a-\varepsilon]\times[0,2\pi]\times[0,2\pi]$.*

**Proof.** The convergence result is a consequence of standard Series Theory. We omit its proof. □

By Lemma 1.51, we can move the summation $\sum_{n=1}^{\infty}$ into the integral sign as long as we confine $(r,\theta)\in[0,a)\times[0,2\pi]$, i.e.

**Corollary 1.52** ***(Commute the summation and the integral.)*** *We have*

$$\begin{aligned} u\left(r,\theta\right) &= \frac{1}{2\pi}\left[\int_0^{2\pi}h\left(\varphi\right)d\varphi+2\sum_{n=1}^{\infty}\int_0^{2\pi}\left(\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right)h\left(\varphi\right)d\varphi\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right)h\left(\varphi\right)d\varphi,\quad\forall\left(r,\theta\right)\in[0,a)\times[0,2\pi]. \end{aligned} \tag{129}$$

*Moreover, on the domain $(r,\theta)\in[0,a)\times[0,2\pi]$, we have the identities:*

$$\frac{\partial^k}{\partial r^k}u\left(r,\theta\right)=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial^k}{\partial r^k}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right)\right)h\left(\varphi\right)d\varphi \tag{130}$$

*and*

$$\frac{\partial^k}{\partial \theta^k}u\left(r,\theta\right)=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial^k}{\partial \theta^k}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)\right)\right)h\left(\varphi\right)d\varphi. \tag{131}$$

The lemma below says that we can simplify the series (126) and find its sum explicitly.

**Lemma 1.53** ***(Evaluating the series.)*** *We have the identity*

$$1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\left(\theta-\varphi\right)=\frac{a^2-r^2}{a^2-2ar\cos\left(\theta-\varphi\right)+r^2},\quad\forall\ \left(r,\theta,\varphi\right)\in[0,a)\times[0,2\pi]\times[0,2\pi]. \tag{132}$$

*In particular, by Lemma 1.51, we know that for fixed $a>0$ and $\varphi\in[0,2\pi]$, the function*

$$\frac{a^2-r^2}{a^2-2ar\cos\left(\theta-\varphi\right)+r^2} \tag{133}$$

*is a **harmonic function** on the domain $(r,\theta)\in[0,a)\times[0,2\pi]$.*

**Remark 1.54** ***(Be careful.)*** *Note that the series in (132) **diverges** when $r=a$ for any $\theta$, $\varphi$. At $r=a$, it has the form*

$$1+2\sum_{n=1}^{\infty}\cos n\left(\theta-\varphi\right)$$

*and note that $\lim_{n\to\infty}\cos n\left(\theta-\varphi\right)$ **does not converge to zero for any values of** $\theta$, $\varphi$ (recall that in Calculus, if a series $\sum_{n=1}^{\infty}a_n$ converges, we must have $\lim_{n\to\infty}a_n=0$).*

**Proof.** Since, by Lemma 1.51, we can **differentiate under the summation sign**, and each $\left(\frac{r}{a}\right)^n \cos n(\theta-\varphi)$ is a harmonic function on $B_a(0)$ (for fixed $\varphi$ and $a$), the sum is also a harmonic function on $B_a(0)$. To prove (132), you may have to use **Euler's formula for complex number** $z = re^{i\theta},\ r\in[0,\infty),\ \theta\in[0,2\pi]$, which is

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta), \quad \forall\, n\in\mathbb{N}.$$

Then look at the series

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n [\cos n(\theta-\varphi) + i\sin n(\theta-\varphi)]$$

and use the identity

$$1 + 2\sum_{n=1}^{\infty} z^n = \frac{2}{1-z} - 1, \quad \forall\, z\in\mathbb{C} \text{ with } |z|<1$$

to get

$$\begin{aligned}
&1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\varphi) = \mathrm{Re}\left[1 + 2\sum_{n=1}^{\infty} z^n\right], \quad z = \frac{r}{a}e^{i(\theta-\varphi)}, \quad r\in[0,a) \\
&= \mathrm{Re}\left[\frac{2}{1-z} - 1\right] = \mathrm{Re}\left[\frac{2}{1-\frac{r}{a}e^{i(\theta-\varphi)}} - 1\right] \\
&= \mathrm{Re}\left[\frac{2a\,[a - r\cos(\theta-\varphi) + ir\sin(\theta-\varphi)]}{[a - r\cos(\theta-\varphi) - ir\sin(\theta-\varphi)]\,[a - r\cos(\theta-\varphi) + ir\sin(\theta-\varphi)]} - 1\right] \\
&= \frac{2a\,[a - r\cos(\theta-\varphi)] - [a^2 - 2ar\cos(\theta-\varphi) + r^2]}{a^2 - 2ar\cos(\theta-\varphi) + r^2} = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}.
\end{aligned}$$

□

**Lemma 1.55** ***(Geometric way to express the function in (132).)*** *We have*

$$\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2} = \frac{a^2 - |\mathbf{x}|^2}{|\mathbf{x}-\mathbf{z}|^2}, \quad (r,\theta,\varphi)\in[0,a)\times[0,2\pi]\times[0,2\pi]. \tag{134}$$

*where* $\mathbf{x} = (r\cos\theta, r\sin\theta)\in B_a(0)$, $\mathbf{z} = (a\cos\varphi, a\sin\varphi)\in\partial B_a(0)$. *For each fixed* $\mathbf{z}\in\partial B_a(0)$, *the function*

$$\frac{a^2 - |\mathbf{x}|^2}{|\mathbf{x}-\mathbf{z}|^2}, \quad \mathbf{x}\in B_a(0) \tag{135}$$

*is* ***harmonic*** *in* $\mathbf{x}\in B_a(0)$.

**Proof.** This is a simple verification. We have $|\mathbf{x}|^2 = r^2$ and

$$|\mathbf{x}-\mathbf{z}|^2 = (r\cos\theta - a\cos\varphi)^2 + (r\sin\theta - a\sin\varphi)^2 = a^2 - 2ar\cos(\theta-\varphi) + r^2.$$

The proof is done. □

**Example 1.56** ***(Put this as a HW problem.)*** *Let* $B_a(0)$ *be the open disc in* $\mathbb{R}^2$ *centered at* $O = (0,0)$ *with radius* $a>0$. *For each fixed* $\mathbf{z} = (x_0,y_0)\in\partial B_a(0)$, *by* ***direct computation****, show that the function*

$$\frac{a^2 - |\mathbf{x}|^2}{|\mathbf{x}-\mathbf{z}|^2} = \frac{a^2 - (x^2+y^2)}{(x-x_0)^2 + (y-y_0)^2}, \quad \mathbf{x} = (x,y)\in B_a(0) \tag{136}$$

*is* ***harmonic*** *in* $\mathbf{x}\in B_a(0)$.

**Solution**:

Denote the function as $u(x,y)$. Compute

$$\frac{\partial u}{\partial x} = -\frac{2x}{(x-x_0)^2+(y-y_0)^2} - \frac{2(x-x_0)\left[a^2-(x^2+y^2)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} := A + B$$

and

$$\frac{\partial u}{\partial y} = -\frac{2y}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y-y_0)\left[a^2-(x^2+y^2)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} := C + D$$

and then

$$\triangle u(x,y) = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} + \frac{\partial D}{\partial y} = \left(\frac{\partial A}{\partial x} + \frac{\partial C}{\partial y}\right) + \left(\frac{\partial B}{\partial x} + \frac{\partial D}{\partial y}\right),$$

where

$$\begin{aligned}
\frac{\partial A}{\partial x} + \frac{\partial C}{\partial y} &= \left\{\begin{array}{l} -\frac{2}{(x-x_0)^2+(y-y_0)^2} + \frac{2x\cdot 2(x-x_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} \\ -\frac{2}{(x-x_0)^2+(y-y_0)^2} + \frac{2y\cdot 2(y-y_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} \end{array}\right. \\
&= \frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left\{\begin{array}{l} -2\left[(x-x_0)^2+(y-y_0)^2\right] + 4x(x-x_0) \\ -2\left[(x-x_0)^2+(y-y_0)^2\right] + 4y(y-y_0) \end{array}\right\} \\
&= \frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left\{\left(2x^2-2y^2-2a^2+4yy_0\right) + \left(2y^2-2x^2-2a^2+4xx_0\right)\right\} \\
&= \frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left(-4a^2+4xx_0+4yy_0\right). \qquad (137)
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{\partial B}{\partial x} &= \left\{\begin{array}{l} -\frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left\{2\left(a^2-(x^2+y^2)\right) + 2(x-x_0)(-2x)\right\} \\ +\frac{2\cdot 2(x-x_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^3}\left\{2(x-x_0)\left[a^2-(x^2+y^2)\right]\right\} \end{array}\right. \\
&= \left\{\begin{array}{l} -\frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left\{2\left(a^2-(x^2+y^2)\right) - 4x(x-x_0)\right\} \\ +\frac{8(x-x_0)^2\left[a^2-\left(x^2+y^2\right)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^3} \end{array}\right.
\end{aligned}$$

and

$$\frac{\partial D}{\partial y} = \left\{\begin{array}{l} -\frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left\{2\left(a^2-(x^2+y^2)\right) - 4y(y-y_0)\right\} \\ +\frac{8(y-y_0)^2\left[a^2-\left(x^2+y^2\right)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^3} \end{array}\right.$$

and then

$$\begin{aligned}
&\frac{\partial B}{\partial x} + \frac{\partial D}{\partial y} \\
&= \left\{\begin{array}{l} \frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left\{-4a^2+4\left(x^2+y^2\right)+4x(x-x_0)+4y(y-y_0)\right\} \\ +\frac{8\left[a^2-\left(x^2+y^2\right)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}. \end{array}\right. \\
&= \frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\left(4a^2-4xx_0-4yy_0\right) \qquad (138)
\end{aligned}$$

The proof is done due to (137) and (138). □

By Lemma 1.53 we conclude the important formula:

$$u(r,\theta)=\frac{1}{2\pi}\left[\int_0^{2\pi}h(\varphi)\,d\varphi+2\sum_{n=1}^{\infty}\int_0^{2\pi}\left(\underbrace{\left(\frac{r}{a}\right)^n\cos n(\theta-\varphi)}\right)h(\varphi)\,d\varphi\right] \tag{139}$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left(\underbrace{1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\varphi)}\right)h(\varphi)\,d\varphi \tag{140}$$

$$=\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi,\quad (r,\theta)\in[0,a)\times[0,2\pi], \tag{141}$$

which is known as the **Poisson Integral Formula.**

**Remark 1.57** *(**Important.**) Note that the interchange of the summation $\sum_{n=1}^{\infty}$ and the integral $\int_0^{2\pi}$ is valid only for $r\in[0,a)$, **not** for $r=a$. In case we assume $h(\theta)$ is a $C^1$ **function** on $\partial B_a(0)$, then the identity in (139) makes sense at $r=a$ and it gives*

$$u(a,\theta)$$
$$=\frac{1}{2\pi}\left[\int_0^{2\pi}h(\varphi)\,d\varphi+2\sum_{n=1}^{\infty}\int_0^{2\pi}\cos n(\theta-\varphi)\cdot h(\varphi)\,d\varphi\right]\ \text{(this is the \textbf{Fourier series} of } h(\theta)\text{)}$$
$$=h(\theta),\quad\forall\ \theta\in[0,2\pi]. \tag{142}$$

*On the other hand, we cannot let $r=a$ in the identity (140) since the series $\sum_{n=1}^{\infty}\cos n(\theta-\varphi)$ diverges for any $\theta$, $\varphi$. Also, we cannot let $r=a$ in the identity (141) since the integral*

$$\int_0^{2\pi}\frac{h(\varphi)}{1-\cos(\theta-\varphi)}d\varphi \tag{143}$$

*diverges too, and the quantity $\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi$ is of the form $0\cdot\infty$ as $r\to a$. However, the identity in (139) suggests that we have the limit (148) (or (149)) below as $(r,\theta)\to(a,\varphi_0)$ when $h(\theta)$ is a $C^1$ **function** on $\partial B_a(0)$. Moreover, the limit is still valid when $h(\theta)$ is a **continuous function** on $\partial B_a(0)$.*

By (134), one can also write $u(r,\theta)$ in a more **geometric way** as

$$u(\mathbf{x})=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_0^{2\pi}\frac{h(\varphi)}{|\mathbf{x}-\mathbf{z}|^2}a\,d\varphi=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds,\quad \mathbf{x}\in B_a(0), \tag{144}$$

where

$$\mathbf{x}=(r\cos\theta,r\sin\theta)\in B_a(0),\qquad \mathbf{z}=(a\cos\varphi,a\sin\varphi)\in\partial B_a(0), \tag{145}$$

and the integral on the right hand side of (144) is the **line integral with respect to arc length parameter** $s$ **on** $\partial B_a(0)$, where we know that $ds=a\,d\varphi$.

Before we go on, we summarize the following important properties again:

1. For each fixed $a>0$ and $\varphi\in[0,2\pi]$, the $(r,\theta)$ function, given by

$$\frac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}\quad\left(=1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\varphi)\right),\quad (r,\theta)\in[0,a)\times[0,2\pi]$$

is **harmonic** in $(r,\theta)\in[0,a)\times[0,2\pi]$.

2. Let $B_a(0)$ be the open disc in $\mathbb{R}^2$ centered at $O=(0,0)$ with radius $a>0$. For each fixed $\mathbf{z}=(x_0,y_0)\in\partial B_a(0)$, the function

$$\frac{a^2-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{z}|^2}=\frac{a^2-(x^2+y^2)}{(x-x_0)^2+(y-y_0)^2},\quad \mathbf{x}=(x,y)\in B_a(0) \tag{146}$$

is **harmonic** in $\mathbf{x}=(x,y)\in B_a(0)$.

The Poisson Integral Formula (141) is motivated by the fact that $h\in C^1$ (so that we can apply the Fourier series theory). However, to solve the Dirichlet problem (111), it suffices to assume that $h\in C^0$ (i.e. $h$ is a continuous function).

Our main result is the following:

**Theorem 1.58** *Let $h$ be a* ***continuous*** *function on $\partial B_a(0)$ and let*

$$u(\mathbf{x})=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds,\quad \mathbf{x}=(x,y)\in B_a(0),\quad |\mathbf{x}|<a. \tag{147}$$

*Then $u(\mathbf{x})=u(x,y)$ is* ***harmonic*** *in $B_a(0)$ and for each fixed $\mathbf{p}\in\partial B_a(0)$ we have the limit*

$$\lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)}u(\mathbf{x})=h(\mathbf{p}). \tag{148}$$

**Remark 1.59** ***(Important.)*** *In terms of $(r,\theta)$, the limit (148) is the same as (assume $\mathbf{p}\in\partial B_a(0)$ has angle $\varphi_0\in[0,2\pi]$)*

$$\begin{aligned}&\lim_{(r,\theta)\to(a,\varphi_0)}u(r,\theta)\\ &=\lim_{(r,\theta)\to(a,\varphi_0)}\left(\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi\right)=h(\varphi_0).\end{aligned} \tag{149}$$

**Remark 1.60** ***(Important observation ...)*** *Note that the integral*

$$\int_{|\mathbf{z}|=a}\frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds$$

*diverges if $\mathbf{x}=(r\cos\theta,r\sin\theta)\in\partial B_a(0)$ (i.e. at $r=a$). In such a case we have (for convenience, assume $\theta=0$)*

$$\begin{aligned}\int_{|\mathbf{z}|=a}\frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds&=\int_0^{2\pi}\frac{h(\varphi)}{2a^2-2a^2\cos(\theta-\varphi)}ad\varphi\\ &=\frac{1}{2a}\int_0^{2\pi}\frac{h(\varphi)}{1-\cos(\theta-\varphi)}d\varphi=\frac{1}{2a}\int_0^{2\pi}\frac{h(\varphi)}{1-\cos\varphi}d\varphi\end{aligned}$$

*and we know that the improper integral*

$$\int_0^{2\pi}\frac{1}{1-\cos\varphi}d\varphi$$

*diverges (near $\varphi=0$, $1-\cos\varphi$ is like $\frac{1}{2}\varphi^2$). However, as $\mathbf{x}\to\mathbf{p}\in\partial B_a(0)$, the term $\left(a^2-|\mathbf{x}|^2\right)/2\pi a$ will tend to zero. As a result of balance, we will get the limit (148).*

**Remark 1.61** *The integral in (147) is a* ***proper integral*** *when $\mathbf{x}\in B_a(0)$.*

**Proof.** In the integral (147), we have $|\mathbf{x}-\mathbf{z}| \neq 0$ for each $\mathbf{x} \in B_a(0)$ and $\mathbf{z} \in \partial B_a(0)$. Thus the integral is a **regular** integral (not an improper integral) and one can differentiate (with respect to $x$ or $y$) under the integral sign. Thus $u(\mathbf{x}) = u(x,y)$ is **harmonic** in $B_a(0)$.

Next we note that

$$\begin{aligned}
&\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{1}{|\mathbf{x}-\mathbf{z}|^2}ds \\
&= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{1}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\varphi)\right)d\varphi = 1, \quad \forall\, \mathbf{x}\in B_a(0),
\end{aligned} \tag{150}$$

where in the above we have used the identity

$$\int_0^{2\pi}\left(2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\varphi)\right)d\varphi = 2\sum_{n=1}^{\infty}\left(\left(\frac{r}{a}\right)^n\int_0^{2\pi}\cos n(\theta-\varphi)\,d\varphi\right) = 2\sum_{n=1}^{\infty}0 = 0.$$

Hence

$$\begin{aligned}
&|u(\mathbf{x})-h(\mathbf{p})| \\
&= \left|\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{h(\mathbf{z})-h(\mathbf{p})}{|\mathbf{x}-\mathbf{z}|^2}ds\right| \leq \frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2}ds.
\end{aligned}$$

Since $h$ is continuous at $\mathbf{p}$, for any $\varepsilon > 0$ there exists a small arc $C(\mathbf{p}) \subset \partial B_a(0)$ centered at $\mathbf{p}$ with length $2\delta$ such that

$$|h(\mathbf{z})-h(\mathbf{p})| < \varepsilon \quad \text{if} \quad \mathbf{z}\in C(\mathbf{p}).$$

Now

$$\begin{aligned}
|u(\mathbf{x})-h(\mathbf{p})| &\leq \begin{cases} \frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{\mathbf{z}\in C(\mathbf{p})}\frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2}ds \\ +\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{\mathbf{z}\in\partial B_a(0)\setminus C(\mathbf{p})}\frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2}ds \end{cases} \\
&\leq \varepsilon + \underbrace{\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{\mathbf{z}\in\partial B_a(0)\setminus C(\mathbf{p})}\frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2}ds}.
\end{aligned}$$

Now if $\mathbf{x}\in B_a(0)$ and $\mathbf{x}$ is close to $\mathbf{p}\in\partial B_a(0)$ with $|\mathbf{x}-\mathbf{p}| < \delta/2$, then by the triangle inequality

$$|\mathbf{z}-\mathbf{p}| \leq |\mathbf{x}-\mathbf{z}| + |\mathbf{x}-\mathbf{p}|,$$

we will have for $\mathbf{z}\in\partial B_a(0)\setminus C(\mathbf{p})$ the estimate

$$|\mathbf{x}-\mathbf{z}| \geq |\mathbf{z}-\mathbf{p}| - |\mathbf{x}-\mathbf{p}| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}. \tag{151}$$

Hence for $\mathbf{x}$ close to $\mathbf{p}\in\partial B_a(0)$ with $|\mathbf{x}-\mathbf{p}| < \delta/2$, we have the estimate

$$\int_{\mathbf{z}\in\partial B_a(0)\setminus C(\mathbf{p})}\frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2}ds \leq \int_{\mathbf{z}\in\partial B_a(0)\setminus C(\mathbf{p})}\frac{|h(\mathbf{x})-h(\mathbf{p})|}{\left(\frac{\delta}{2}\right)^2}ds \leq \frac{2M}{\left(\frac{\delta}{2}\right)^2}\cdot 2\pi a$$

where $M = \sup_{\mathbf{z}\in\partial B_a(0)}|h(\mathbf{z})|$. Hence

$$\lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)}|u(\mathbf{x})-h(\mathbf{p})| \leq \varepsilon + \lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)}\left(\frac{a^2-|\mathbf{x}|^2}{2\pi a}\cdot\frac{2M}{\left(\frac{\delta}{2}\right)^2}\cdot 2\pi a\right) = \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, we obtain $\lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)}|u(\mathbf{x})-h(\mathbf{p})| = 0$. The proof is done. □

We can summarize the following:

**Theorem 1.62** *(**Solution of the Dirichlet problem (111).**) Consider the Dirichlet problem for the Laplace equation on $B_a(0)$:*

$$\begin{cases} \triangle u(x,y) = 0 \quad in \quad (x,y) \in B_a(0) \\ u(x,y) = h(x,y) \quad on \quad (x,y) \in \partial B_a(0) \ \ (the\ boundary\ of\ B_a(0)\ ), \end{cases} \tag{152}$$

*where $h(x,y)$ is a given **continuous** function defined on $\partial B_a(0)$. The solution in the space*

$$C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$$

*is **unique** and is given by*

$$u(\mathbf{x}) = \begin{cases} \dfrac{a^2 - |\mathbf{x}|^2}{2\pi a} \displaystyle\int_{|\mathbf{z}|=a} \frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, & \mathbf{x} \in B_a(0) \\ h(\mathbf{x}), \quad \mathbf{x} \in \partial B_a(0). \end{cases} \tag{153}$$

**Remark 1.63** *(**Important observation.**) The representation formula (153) says that the values of $u$ at interior points $\mathbf{x} \in B_a(0)$ is completely determined by its boundary data. This matches with the maximum/minimum principle.*

**Remark 1.64** *In terms of the polar coordinates $(r,\theta)$, the function $u(\mathbf{x})$ in (153) can be written as (note that $ds = ad\varphi$)*

$$u(r,\theta) = \begin{cases} \dfrac{a^2 - r^2}{2\pi} \displaystyle\int_0^{2\pi} \frac{h(\varphi)}{a^2 - 2ar\cos(\theta - \varphi) + r^2} d\varphi, & (r,\theta) \in [0,a) \times [0,2\pi] \\ h(\theta), \quad r = a, \quad \theta \in [0,2\pi]. \end{cases} \tag{154}$$

## 1.11 The case when $h$ is a $C^1$ function on $\partial B_a(0)$.

In case $h$ is a $C^1$ function on $\partial B_a(0)$ in the Dirichlet problem (111), then $h(\theta)$ is a **$2\pi$-periodic $C^1$ function** defined on $\theta \in [0,2\pi]$ with Fourier series expansion

$$h(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \quad \forall\, \theta \in [0,2\pi], \tag{155}$$

where the coefficients $a_n,\ b_n$ are given by (120). In this situation, we have:

**Theorem 1.65** *(**Solution of the Dirichlet problem (111) when** $h$ **is** $C^1$ **on** $\partial B_a(0)$.) Assume $h(\theta)$ is a $2\pi$**-periodic** $C^1$ **function** defined on $\theta \in [0,2\pi]$. The solution of the Dirichlet problem (111) can also be expressed as*

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cdot \underbrace{\left(\frac{r}{a}\right)^n \cos n\theta} + b_n \cdot \underbrace{\left(\frac{r}{a}\right)^n \sin n\theta} \right], \quad (r,\theta) \in [0,a] \times [0,2\pi] \tag{156}$$

*where $a_n,\ b_n$ are the **Fourier series coefficients** of $h(\theta)$ on $\theta \in [0,2\pi]$.*

**Remark 1.66** *(**Important.**) Note that the series in (156) is defined at $r = a$ with sum equal to $h(\theta)$ for all $\theta \in [0,2\pi]$.*

**Proof.** We already know that the solution $u(r,\theta)$ is given by

$$\begin{aligned} u(r,\theta) &= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi, \quad (r,\theta)\in[0,a)\times[0,2\pi] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\varphi)\right)h(\varphi)\,d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n(\cos n\theta\cos n\varphi+\sin n\theta\sin n\varphi)\right)h(\varphi)\,d\varphi \end{aligned} \tag{157}$$

and we can change the order of integration and summation to get

$$\begin{aligned} &u(r,\theta) \\ &= \frac{1}{2\pi}\int_0^{2\pi}h(\varphi)\,d\varphi+\sum_{n=1}^{\infty}\left[\begin{array}{c}\left(\frac{1}{\pi}\int_0^{2\pi}h(\varphi)(\cos n\varphi)\,d\varphi\right)\underbrace{\left(\frac{r}{a}\right)^n\cos n\theta} \\ +\left(\frac{1}{\pi}\int_0^{2\pi}h(\varphi)(\sin n\varphi)\,d\varphi\right)\underbrace{\left(\frac{r}{a}\right)^n\sin n\theta}\end{array}\right] \\ &= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cdot\underbrace{\left(\frac{r}{a}\right)^n\cos n\theta}+b_n\cdot\underbrace{\left(\frac{r}{a}\right)^n\sin n\theta}\right] \end{aligned}$$

where $(r,\theta)\in[0,a)\times[0,2\pi]$. When $r=a$, $\theta\in[0,2\pi]$, the above becomes the Fourier series of $h(\theta)$, which is the same as $h(\theta)$. The proof is done. □

**Example 1.67** ***(Symmetry of** $h$ **implies symmetry of** $u$.)* *Assume $h(x,y):\partial B_a(0)\to\mathbb{R}$ is an **odd** (or **even**) continuous function with respect to $y$ (i.e. in the odd case we have $h(x,-y)=-h(x,y)$ for $(x,y)\in\partial B_a(0)$ and in the even case we have $h(x,-y)=h(x,y)$ for $(x,y)\in\partial B_a(0)$). Then by the **Poisson Integral Formula**, the solution $u(x,y)$ for*

$$\begin{cases} \triangle u(x,y)=0 \quad \text{in} \quad (x,y)\in B_a(0) \\ u(x,y)=h(x,y) \quad \text{on} \quad (x,y)\in\partial B_a(0) \ \text{(the boundary of } B_a(0)\text{)}, \end{cases} \tag{158}$$

*is also an **odd** (or **even**) function of $y$. In particular, for the odd case, we have $u(x,0)=0$ on the $x$-axis inside the open disc $B_a(0)$. To see this, note that for any $(r,\theta)\in[0,a)\times[0,2\pi]$ we have*

$$\begin{aligned} u(r,-\theta) &= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(-\theta-\varphi)+r^2}d\varphi \\ &= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\theta+\varphi)+r^2}d\varphi \\ &= \frac{a^2-r^2}{2\pi}\int_{-2\pi}^{0}\frac{h(-\sigma)}{a^2-2ar\cos(\theta-\sigma)+r^2}d\sigma, \quad \varphi=-\sigma \\ &= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(-\sigma)}{a^2-2ar\cos(\theta-\sigma)+r^2}d\sigma \\ &= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{-h(\sigma)}{a^2-2ar\cos(\theta-\sigma)+r^2}d\sigma=-u(r,\theta). \end{aligned} \tag{159}$$

*By (**??**), we will have $u(x,-y)=-u(x,y)$ for all $(x,y)\in B_a(0)$. The proof for the even case is similar.*

## 1.12 Representation formula for harmonic functions on a disc.

An important consequence of Theorem 1.62 is the following:

**Theorem 1.68** *(**Representation formula for harmonic functions on the disc** $B_a(0)$.) Assume*

$$u(\mathbf{x}) \in C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$$

*is a **harmonic function** on $B_a(0)$ and is **continuous up to the boundary** $\partial B_a(0)$. Then on $B_a(0)$ we have the identity*

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\, \mathbf{x} \in B_a(0), \tag{160}$$

*which is the same as*

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(\varphi)}{a^2 - 2ar\cos(\theta-\varphi) + r^2} d\varphi, \quad (r,\theta) \in [0,a) \times [0,2\pi], \tag{161}$$

*where $u(\varphi) = u(a\cos\varphi, a\sin\varphi)$, $\varphi \in [0, 2\pi]$.*

**Remark 1.69** *See Theorem 1.86 below for a similar result when $u(\mathbf{x})$ satisfying $\triangle u(\mathbf{x}) \geq 0$ on $B_a(0)$ (or $\triangle u(\mathbf{x}) \leq 0$ on $B_a(0)$).*

**Remark 1.70** *(**Important.**) On the disc $B_a(\mathbf{x}_0)$ centered at some $\mathbf{x}_0 \in \mathbb{R}^2$, the identity (160) becomes*

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\, \mathbf{x} \in B_a(\mathbf{x}_0). \tag{162}$$

*To see this, let $v(\mathbf{x}) = u(\mathbf{x}+\mathbf{x}_0)$, then $v$ is harmonic on $B_a(0)$ if and only if $u$ is harmonic on $B_a(\mathbf{x}_0)$. For $v$, we have*

$$v(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{v(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\, \mathbf{x} \in B_a(0), \tag{163}$$

*and back to $u$ we have*

$$u(\mathbf{x}+\mathbf{x}_0) = \frac{a^2 - |(\mathbf{x}+\mathbf{x}_0)-\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z}+\mathbf{x}_0)}{|(\mathbf{x}+\mathbf{x}_0)-(\mathbf{z}+\mathbf{x}_0)|^2} ds, \quad \forall\, \mathbf{x} \in B_a(0). \tag{164}$$

*If we let $\mathbf{y} = \mathbf{x}+\mathbf{x_0} \in B_a(\mathbf{x}_0)$ and $\tilde{\mathbf{z}} = \mathbf{z}+\mathbf{x}_0 \in \partial B_a(\mathbf{x}_0)$, then the above is the same as*

$$u(\mathbf{y}) = \frac{a^2 - |\mathbf{y}-\mathbf{x}_0|^2}{2\pi a} \int_{|\tilde{\mathbf{z}}-\mathbf{x}_0|=a} \frac{u(\tilde{\mathbf{z}})}{|\mathbf{y}-\tilde{\mathbf{z}}|^2} ds, \quad \forall\, \mathbf{y} \in B_a(\mathbf{x}_0), \tag{165}$$

*which gives (162).*

## 1.13 The strong maximum/minimum principle for harmonic functions on the disc $B_a(0)$.

**Theorem 1.71** *(**Strong maximum/minimum principle for harmonic functions on the disc** $B_a(0)$.) Assume that $u(\mathbf{x}) \in C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$, $\mathbf{x} = (x,y) \in \mathbb{R}^2$, is **harmonic** on $B_a(0)$. Let $M = \max_{\mathbf{x}\in\bar{B}_a(0)} u(\mathbf{x})$ and $m = \min_{\mathbf{x}\in\bar{B}_a(0)} u(\mathbf{x})$. If there exists $\mathbf{x}_0 \in B_a(0)$ ($\mathbf{x}_0$ is an interior point of $\bar{B}_a(0)$) such that $u(\mathbf{x}_0) = M$ (or $u(\mathbf{x}_0) = m$), then $u$ must be a **constant function** on $\bar{B}_a(0)$.*

**Remark 1.72** *Note that the strong maximum/minimum principle will imply the weak maximum/minimum principle.*

**Proof.** By the **Poisson *Integral Formula,*** we have

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\, \mathbf{x} = (x,y) \in B_a(0)$$

where $u(\mathbf{z}) \le M$ for all $\mathbf{z} \in \partial B_a(0)$. By the identity (see (150) also)

$$0 = u(\mathbf{x}_0) - M = \frac{a^2 - |\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z}) - M}{|\mathbf{x}_0 - \mathbf{z}|^2} ds, \quad \text{where } a^2 - |\mathbf{x}_0|^2 > 0, \tag{166}$$

we must have

$$\int_{|\mathbf{z}|=a} \frac{u(\mathbf{z}) - M}{|\mathbf{x}_0 - \mathbf{z}|^2} ds = 0, \tag{167}$$

where we also note that $u(\mathbf{z}) - M \le 0$ for all $\mathbf{z} \in \partial B_a(0)$ due to $M = \max_{\mathbf{x}\in \bar{B}_a(0)} u(\mathbf{x})$. Therefore we must have $u(\mathbf{z}) \equiv M$ on $\partial B_a(0)$. Hence $u(\mathbf{x})$ is harmonic in $B_a(0)$ and has constant value on $\partial B_a(0)$. By the **weak maximum principle,** $u(\mathbf{x})$ must be a constant function on $\bar{B}_a(0)$ with $u(\mathbf{x}) \equiv M$. Similar result holds for the case $u(\mathbf{x}_0) = m$. □

**Remark 1.73** *Draw a one-dimensional picture for the above theorem.*

## 1.14 Mean value formula for harmonic functions.

**Theorem 1.74** *(**Mean value formula; line integral version.**) Assume $u \in C^2(\Omega)$ is **harmonic** on some open set $\Omega \subset \mathbb{R}^2$, then for any open disc $B_a(\mathbf{x}_0) \subset\subset \Omega$, $a > 0$, we have the identity*

$$u(\mathbf{x}_0) = \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} u(\mathbf{z})\, ds \ \ (\textbf{line integral} \text{ on the circle } \partial B_a(\mathbf{x}_0)), \tag{168}$$

*i.e. the value of $u$ at the **center** $\mathbf{x}_0$ of the disc $B_a(\mathbf{x}_0)$ is equal to its average on the circumference $|\mathbf{z} - \mathbf{x}_0| = a$. The integral in (168) is the line integral with respect to arc length parameter $s$ on $\partial B_a(\mathbf{x}_0)$, $ds = a d\varphi$, $\varphi \in [0, 2\pi]$.*

**Remark 1.75** *For harmonic functions $u \in C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$ defined on an open disc $B_a(0) \subset \mathbb{R}^n$ for arbitrary $n \in \mathbb{N}$, there is also a mean value formula.*

**Proof.** On the disc $B_a(\mathbf{x}_0)$, by the representation formula (162), we have

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\, \mathbf{x} \in B_a(\mathbf{x}_0)$$

and so

$$\begin{aligned} u(\mathbf{x}_0) &= \frac{a^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x}_0-\mathbf{z}|^2} ds \\ &= \frac{a^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{a^2} ds = \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} u(\mathbf{z})\, ds. \end{aligned}$$

The proof is done. □

**Theorem 1.76** *(**Mean value formula; double integral version.**) Assume $u \in C^2(\Omega)$ is harmonic on some open set $\Omega \subset \mathbb{R}^2$, then for any open disc $B_a(\mathbf{x}_0) \subset\subset \Omega$, $a > 0$, we have the identity*

$$u(\mathbf{x}_0) = \frac{1}{\pi a^2} \iint_{B_a(\mathbf{x}_0)} u(\mathbf{x})\, d\mathbf{x} \text{ (\textbf{double integral} on the disc } B_a(\mathbf{x}_0)\text{)}. \tag{169}$$

**Proof.** By (168), we have

$$ru(\mathbf{x}_0) = \frac{1}{2\pi} \int_{|\mathbf{z}-\mathbf{x}_0|=r} u(\mathbf{z})\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(\mathbf{x}_0 + r(\cos\theta, \sin\theta)) \cdot r d\theta$$

for any $B_r(\mathbf{x}_0) \subset B_a(\mathbf{x}_0) \subset\subset \Omega$, $r \in [0, a]$, and if we integrate with respect to the radius $r$ from 0 to $a$, we get

$$\int_0^a ru(\mathbf{x}_0)\, dr = \frac{1}{2\pi} \int_0^a \left[ \int_0^{2\pi} u(\mathbf{x}_0 + r(\cos\theta, \sin\theta)) \cdot r d\theta \right] dr$$

and so

$$\frac{a^2}{2} u(\mathbf{x}_0) = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} \int_0^a u(\mathbf{x}_0 + r(\cos\theta, \sin\theta)) \cdot r dr d\theta}_{},$$

which, by the change of variables formula for double integral in the plane, we have

$$u(\mathbf{x}_0) = \frac{1}{\pi a^2} \underbrace{\iint_{B_a(\mathbf{x}_0)} u(\mathbf{x})\, d\mathbf{x}}_{}.$$

The proof is done. □

## 1.15 Gradient estimate and Liouville theorem for harmonic functions.

**Theorem 1.77** *(**Derivatives estimate of harmonic functions.**) Assume $u(x, y) \in C^2(\Omega)$ is harmonic on $\Omega \subset \mathbb{R}^2$. Then we have*

$$\left| \frac{\partial u}{\partial x}(x_0, y_0) \right| \left( or \left| \frac{\partial u}{\partial y}(x_0, y_0) \right| \right) \leq \frac{2}{a} \max_{\partial B_a(x_0, y_0)} |u| \tag{170}$$

*as long as $B_a(x_0, y_0) \subset\subset \Omega$.*

**Proof.** First note that $u \in C^\infty(\Omega)$ (note that a $C^2$ harmonic function is automatically a $C^\infty$ function; we can see this from the **Poisson Integral Formula**). In particular, the function $\frac{\partial u}{\partial x}$ $\left(\text{or } \frac{\partial u}{\partial y}\right)$ is also a **harmonic function** on $\Omega$. Hence by the mean value formula we have

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{1}{\pi a^2} \underbrace{\iint_{B_a(x_0, y_0)} \frac{\partial u}{\partial x}(x, y)\, dx dy}_{}, \tag{171}$$

where by the **divergence theorem**, we know

$$\underbrace{\iint_{B_a(x_0, y_0)} \frac{\partial u}{\partial x}(x, y)\, dx dy}_{}$$

$$= \iint_{B_a(x_0, y_0)} div\,(u(x, y), 0)\, dx dy = \int_{\partial B_a(x_0, y_0)} \langle (u(x, y), 0),\ \mathbf{N}_{out}(x, y) \rangle\, ds. \tag{172}$$

Therefore

$$\begin{aligned}\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| &\le \left|\frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)}\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle\, ds\right| \\ &\le \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)} |\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle|\, ds \\ &\le \frac{2\pi a}{\pi a^2}\max_{\partial B_a(x_0,y_0)} |(u(x,y),0)| = \frac{2}{a}\max_{\partial B_a(x_0,y_0)}|u|. \end{aligned} \tag{173}$$

The proof is done. □

We also have:

**Corollary 1.78** ***(Derivatives estimate of nonnegative harmonic functions.)*** *Assume $u(x,y)\in C^2(\Omega)$ is harmonic on $\Omega\subset\mathbb{R}^2$ with $u\ge 0$* ***everywhere*** *in $\Omega$. Then we have*

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \left(or\ \left|\frac{\partial u}{\partial y}(x_0,y_0)\right|\right) \le \frac{2}{a}u(x_0,y_0) \tag{174}$$

*as long as $B_a(x_0,y_0)\subset\subset\Omega$.*

**Proof.** By (171) and (172), we have

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)}\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle\, ds$$

and so

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \le \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)} |\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle|\, ds.$$

Since $u\ge 0$ everywhere in $\Omega$, we have

$$|\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle| \le |(u(x,y),0)| = u(x,y),\quad \forall\ (x,y)\in\partial B_a(x_0,y_0).$$

By this, we get (we use mean value property again)

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \le \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)} u(x,y)\, ds = \frac{2\pi a}{\pi a^2}\left(\frac{1}{2\pi a}\int_{\partial B_a(x_0,y_0)} u(x,y)\, ds\right) = \frac{2}{a}u(x_0,y_0).$$

The proof is done. □

**Example 1.79** *Let $\Omega\subset\mathbb{R}^2$ be a domain. Assume $u(x,y)\in C^2(\Omega)$ is harmonic on $\Omega\subset\mathbb{R}^2$ with $u\ge -13$ everywhere in $\Omega$. Assume $B_a(x_0,y_0)\subset\subset\Omega$ and $u(x_0,y_0)=-5$ (here $B_a(x_0,y_0)$ is an open disc in $\mathbb{R}^2$ centered at $(x_0,y_0)$ with radius $a>0$). Then we have the following derivatives estimate:*

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \left(or\ \left|\frac{\partial u}{\partial y}(x_0,y_0)\right|\right) \le \underline{\hspace{4cm}}.$$

**Solution**:

The function $v(x,y)=u(x,y)+13$ is harmonic on $\Omega\subset\mathbb{R}^2$ with $v\ge 0$ everywhere in $\Omega$. Hence we have

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| = \left|\frac{\partial v}{\partial x}(x_0,y_0)\right| \le \frac{2}{a}v(x_0,y_0) = \frac{2}{a}(u(x_0,y_0)+13) = \frac{2}{a}(-5+13) = \frac{16}{a}.$$

□

**Theorem 1.80** *(**Liouville theorem of harmonic functions on entire space.**) If $u$ is harmonic on $\mathbb{R}^2$ and is **bounded either above or below** on $\mathbb{R}^2$, then $u$ must be a constant function.*

**Remark 1.81** *Note that here $u(x,y)$ is defined on the **whole space** $\mathbb{R}^2$. This is essential.*

**Remark 1.82** *The same property holds if $u$ is harmonic on $\mathbb{R}^n$, $n > 2$, and is **bounded either above or below** on $\mathbb{R}^n$.*

**Proof.** Without loss of generality, assume $u$ is bounded below (if $u$ is bounded above, we can look at $-u$). By adding a large constant if necessary, we may assume $u \geq 0$ on $\mathbb{R}^2$. Apply (174) to $u(x,y)$ with $\Omega = \mathbb{R}^2$, $a \to \infty$, to get

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial u}{\partial y}(x_0, y_0) \equiv 0, \quad \forall\ (x_0, y_0) \in \mathbb{R}^2.$$

As the point $(x_0, y_0)$ is arbitrary, we are done. $\square$

## 1.16 $C^2$ Harmonic functions are automatically $C^\infty$ functions.

**Theorem 1.83** *Let $\Omega \subset \mathbb{R}^2$ be any open set (may not be bounded) and $u \in C^2(\Omega)$ is harmonic on $\Omega$. Then we must have $u \in C^\infty(\Omega)$.*

**Proof.** For any $\mathbf{x}_0 \in \Omega$ one can find a small open disc $B_a(\mathbf{x}_0) \subset\subset \Omega$ and we have the identity

$$\begin{aligned} u(\mathbf{x}) &= \frac{a^2 - |\mathbf{x} - \mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x} - \mathbf{z}|^2} ds \\ &= \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \underbrace{\frac{a^2 - |\mathbf{x} - \mathbf{x}_0|^2}{|\mathbf{x} - \mathbf{z}|^2}} u(\mathbf{z})\, ds, \quad \forall\ \mathbf{x} \in B_a(\mathbf{x}_0) \end{aligned} \tag{175}$$

Note that, for $\mathbf{z} \in \partial B_a(\mathbf{x}_0)$, the function $\left(a^2 - |\mathbf{x} - \mathbf{x}_0|^2\right) / |\mathbf{x} - \mathbf{z}|^2$ is a $C^\infty$ function of $\mathbf{x} \in B_a(\mathbf{x}_0)$ and at any fixed $\mathbf{x}_* \in B_a(\mathbf{x}_0)$ the line integral

$$\frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \left( \underbrace{\left.\frac{\partial^{m+n}}{\partial x^m \partial y^n}\right|_{\mathbf{x}=\mathbf{x}_*} \left( \frac{a^2 - |\mathbf{x} - \mathbf{x}_0|^2}{|\mathbf{x} - \mathbf{z}|^2} \right)} \right) u(\mathbf{z})\, ds, \quad \forall\ m,\ n \in \mathbb{N} \bigcup \{0\}$$

still converges, by standard theorem in Advanced Calculus (see any textbook), the function $u(\mathbf{x})$ is a $C^\infty$ function of $\mathbf{x} \in B_a(\mathbf{x}_0)$ and one can differentiate under the integral sign, i.e.

$$\begin{aligned} &\left.\frac{\partial^{m+n}}{\partial x^m \partial y^n}\right|_{\mathbf{x}=\mathbf{x}_*} u(\mathbf{x}) \\ &= \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \left( \underbrace{\left.\frac{\partial^{m+n}}{\partial x^m \partial y^n}\right|_{\mathbf{x}=\mathbf{x}_*} \left( \frac{a^2 - |\mathbf{x} - \mathbf{x}_0|^2}{|\mathbf{x} - \mathbf{z}|^2} \right)} \right) u(\mathbf{z})\, ds, \quad \forall\ m,\ n \in \mathbb{N} \bigcup \{0\}. \end{aligned}$$

The proof is done. $\square$

## 1.17 Mean value inequality for subharmonic and superharmonic functions.

We first note that the "**name**" for subharmonic and superharmonic functions are due to the following properties:

**Lemma 1.84** *Assume* $\Omega$ *is a* ***bounded domain*** *and assume* $u,\ v \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$ *with*

$$(1).\ \triangle u \geq 0 \ \ in\ \ \Omega. \quad (2).\ \triangle v = 0 \ \ in\ \ \Omega. \quad (3).\ u \leq v \ \ on\ \ \partial\Omega, \tag{176}$$

*then* $u \leq v$ *in* $\Omega$ *(this is why we call* $v$ *a* ***subharmonic*** *function). Similarly, if we have*

$$(1).\ \triangle u \leq 0 \ \ in\ \ \Omega. \quad (2).\ \triangle v = 0 \ \ in\ \ \Omega. \quad (3).\ u \geq v \ \ on\ \ \partial\Omega, \tag{177}$$

*then* $u \geq v$ *in* $\Omega$ *(this is why we call* $v$ *a* ***superharmonic*** *function).*

**Remark 1.85** *Draw an one-dimensional picture for this.*

**Proof.** This is a consequence of the weak maximum/minimum principle. We prove the first case. Let $w = u - v$. It satisfies

$$\triangle w \geq 0 \ \text{ in } \ \Omega \qquad \text{and} \qquad w \leq 0 \ \text{ on } \ \partial\Omega.$$

Therefore, by the weak maximum principle, we have $\max_{\bar{\Omega}} w = \max_{\partial\Omega} w \leq 0$, which implies that $w = u - v \leq 0$ in $\Omega$. For the second case, let $w = u - v$ and apply the weak minimum principle. □

**Theorem 1.86** *(**Poisson integral inequality for subharmonic and superharmonic functions.**) Let* $B_a(0)$ *be the open disc in* $\mathbb{R}^2$ *centered at* $O = (0,0)$ *with radius* $a > 0$*. Assume* $u(\mathbf{x})$ *is a function defined on* $\bar{B}_a(0)$*, lying in the space* $C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$*, and satisfies* $\triangle u(\mathbf{x}) \geq 0$ *on* $B_a(0)$ *(i.e. it is subharmonic on* $B_a(0)$*). Then we have*

$$u(\mathbf{x}) \leq \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\ \mathbf{x} \in B_a(0). \tag{178}$$

*Similarly, if* $u(\mathbf{x})$ *satisfies* $\triangle u(\mathbf{x}) \leq 0$ *on* $B_a(0)$*, then we have*

$$u(\mathbf{x}) \geq \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\ \mathbf{x} \in B_a(0). \tag{179}$$

**Remark 1.87** *In case the disc is* $B_a(\mathbf{x}_0)$*, (178) and (179) become*

$$u(\mathbf{x}) \leq \frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\ \mathbf{x} \in B_a(\mathbf{x}_0) \tag{180}$$

*and*

$$u(\mathbf{x}) \geq \frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \forall\ \mathbf{x} \in B_a(\mathbf{x}_0). \tag{181}$$

**Proof.** This is a consequence of Lemma 1.84. Let

$$v(\mathbf{x}) = \begin{cases} \dfrac{a^2 - |\mathbf{x}|^2}{2\pi a} \displaystyle\int_{|\mathbf{z}|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, & \mathbf{x} \in B_a(0) \\ u(\mathbf{x}), & \mathbf{x} \in \partial B_a(0). \end{cases}$$

We know that $v(\mathbf{x}) \in C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$ is **harmonic** in $B_a(0)$ with $v(\mathbf{x}) = u(\mathbf{x})$ on $\partial B_a(0)$. If we have $\triangle u(\mathbf{x}) \geq 0$ on $B_a(0)$, then Lemma 1.84 implies $u(\mathbf{x}) \leq v(\mathbf{x})$ in $\Omega$, which gives (178). The proof of (179) is similar. □

**Theorem 1.88** ***(Mean value inequality for subharmonic and superharmonic functions.)*** *For any open set $\Omega \subset \mathbb{R}^2$, if $u \in C^2(\Omega)$ is* ***subharmonic*** *on $\Omega$ (i.e. $\triangle u \geq 0$ on $\Omega$), then for any open disc $B_a(\mathbf{x}_0) \subset\subset \Omega$, $a > 0$, we have the following* ***mean value inequality***

$$u(\mathbf{x}_0) \leq \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} u(\mathbf{z})\, ds \ \textit{(line integral on the circle } \partial B_a(\mathbf{x}_0)\textit{)} \tag{182}$$

*and*

$$u(\mathbf{x}_0) \leq \frac{1}{\pi a^2} \iint_{B_a(\mathbf{x}_0)} u(\mathbf{x})\, d\mathbf{x} \ \textit{(double integral on the disc } B_a(\mathbf{x}_0)\textit{).} \tag{183}$$

*Similarly, if $u \in C^2(\Omega)$ is* ***superharmonic*** *on $\Omega$ (i.e. $\triangle u \leq 0$ on $\Omega$), then we have (182) and (183) with " $\leq$ " replaced by " $\geq$ ".*

**Proof.** Assume $u$ is **subharmonic**. For any open disc $B_a(\mathbf{x}_0) \subset\subset \Omega$, by (180) (evaluated at $\mathbf{x} = \mathbf{x}_0$), we have

$$u(\mathbf{x}_0) \leq \frac{a^2 - |\mathbf{x}_0 - \mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x}_0 - \mathbf{z}|^2} ds = \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} u(\mathbf{z})\, ds,$$

which proves (182).

For (183), we note that the inequality (182) is valid for any radius $r > 0$ as long as $B_r(\mathbf{x}_0) \subset B_a(\mathbf{x}_0) \subset\subset \Omega$. Hence we have

$$r u(\mathbf{x}_0) \leq \frac{1}{2\pi} \int_{|\mathbf{z}-\mathbf{x}_0|=r} u(\mathbf{z})\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(\mathbf{x}_0 + r(\cos\theta, \sin\theta)) \cdot r d\theta, \quad \forall\, r \in [0, a]$$

and if we integrate with respect to the radius $r$ from $0$ to $a$, we get

$$\int_0^a r u(\mathbf{x}_0)\, dr \leq \frac{1}{2\pi} \int_0^a \left[ \int_0^{2\pi} u(\mathbf{x}_0 + r(\cos\theta, \sin\theta)) \cdot r d\theta \right] dr$$

and so

$$\frac{a^2}{2} u(\mathbf{x}_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^a u(\mathbf{x}_0 + r(\cos\theta, \sin\theta)) \cdot r dr d\theta = \frac{1}{2\pi} \iint_{B_a(\mathbf{x}_0)} u(\mathbf{x})\, d\mathbf{x},$$

where the last identity in the above is due to **the change of variables formula for double integral in the plane**. The proof is done for the subharmonic case. The proof for the superharmonic case is similar. □

## 1.18 The strong maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.

Recall that we already have the weak maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains; see Lemma 1.39. Now we can use **mean value inequality** to prove the following **strong** maximum/minimum principle :

**Theorem 1.89** ***(Strong maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.)*** *Let $\Omega \subset \mathbb{R}^2$ be a* ***bounded domain (open and connected)*** *in $\mathbb{R}^2$ and $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$. Assume $u$ is* ***subharmonic*** *on $\Omega$ (i.e. $\triangle u \geq 0$ on $\Omega$) and there exists a point $\mathbf{p} \in \Omega$ such that $u(\mathbf{p}) = \max_{\bar{\Omega}} u$, then $u$ must be a* ***constant function*** *on $\Omega$. Similarly, if $u$ is* ***superharmonic*** *on $\Omega$ (i.e. $\triangle u \leq 0$ on $\Omega$) and there exists a point $\mathbf{p} \in \Omega$ such that $u(\mathbf{p}) = \min_{\bar{\Omega}} u$, then $u$ must be a* ***constant function*** *on $\Omega$. In particular, if $u$ is* ***harmonic*** *on $\Omega$ and there exists a point $\mathbf{p} \in \Omega$ such that $u(\mathbf{p}) = \max_{\bar{\Omega}} u$ or $u(\mathbf{p}) = \min_{\bar{\Omega}} u$, then $u$ must be a* ***constant function*** *on $\Omega$.*

**Proof.** Assume $u$ is **subharmonic** on $\Omega$ and we have $u(\mathbf{p}) = \max_{\bar{\Omega}} u$ (call this value $M$) for some $\mathbf{p} \in \Omega$. Since $\Omega$ is an open set, one can find an open disc $B_a(\mathbf{p}) \subset\subset \Omega$ for some $a > 0$. By the mean value inequality, we have

$$0 = u(\mathbf{p}) - M \leq \frac{1}{\pi a^2} \iint_{B_a(\mathbf{p})} (u(\mathbf{x}) - M)\, d\mathbf{x} \leq 0, \quad \text{where } u(\mathbf{x}) - M \leq 0 \text{ on } B_a(\mathbf{p}), \tag{184}$$

which implies $u(\mathbf{x}) \equiv M$ on $B_a(\mathbf{p})$. Therefore, the nonempty set

$$D = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = M\} \subset \Omega, \quad \mathbf{p} \in D$$

is **open** in $\Omega$. We claim that $D \subset \Omega$ is also **closed** in $\Omega$. Let $\mathbf{q}_n \in D$ be a sequence in $D$ with $\lim_{n\to\infty} \mathbf{q}_n = \mathbf{q}_* \in \Omega$ (i.e. the sequence $q_n \in D$ has a limit point $\mathbf{q}_* \in \Omega$). We claim that $\mathbf{q}_* \in D$. To see this, since $u \in C^0(\bar{\Omega})$, we have

$$M = \lim_{n\to\infty} u(\mathbf{q}_n) = u\left(\lim_{n\to\infty} \mathbf{q}_n\right) = u(\mathbf{q}_*), \quad \mathbf{q}_* \in \Omega,$$

which implies that $\mathbf{q}_* \in D$. Therefore, by definition, this means that $D \subset \Omega$ is also **closed** in $\Omega$. As $\Omega$ is **connected**, the only set which is **both open and closed in** $\Omega$ is either the empty set $\varnothing$ or the whole set $\Omega$. Since $D$ is not empty (because $\mathbf{p} \in D$), we must have $D = \Omega$. That is, $u \equiv M$ on all of $\Omega$.

The proof for the **superharmonic** case is similar since one can apply the above argument to the subharmonic function $-u$. The proof is done. □

## 1.19 Application of the weak maximum/minimum principle.

The following result is important and easy to prove using the weak maximum/minimum principle.

**Lemma 1.90** *Let $\Omega \subset \mathbb{R}^n$ be a **bounded domain** and $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$ satisfy*

$$\triangle u = f \quad in \quad \Omega, \qquad u = g \quad on \quad \partial\Omega, \tag{185}$$

*where $f$ and $g$ are continuous function on $\Omega$ and $\partial\Omega$ respectively. Let $B_R(x_0)$ be a ball centered at **some** $x_0 \in \Omega$ such that $\Omega \subset B_R(x_0)$. Then we have the estimate*

$$\max_{\bar{\Omega}} |u| \leq \max_{\partial\Omega} |g| + \frac{R^2}{2n} \sup_{\Omega} |f|. \tag{186}$$

**Remark 1.91** *We will use the weak maximum/minimum principle to prove the above lemma. The weak maximum/minimum principle is actually valid on a **bounded domain** $\Omega$ in $\mathbb{R}^n$ for any $n \in \mathbb{N}$.*

**Remark 1.92** *Since we assume $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$, the function $f \in C^0(\Omega)$ is defined only on $\Omega$ and, in general, may not be continuous up to $\bar{\Omega}$. Therefore, $\max_{\bar{\Omega}} |f|$ may not exist in general. In case $f$ is continuous on $\bar{\Omega}$, we replace $\sup_{\Omega} |f|$ in (186) by $\max_{\bar{\Omega}} |f|$.*

**Remark 1.93** *To make the estimate (186) as best as possible, one may choose $x_0 \in \Omega$ with $R > 0$ as smallest as possible.*

**Remark 1.94** *From (186) we see that the diameter $R$ of $\Omega$ comes into play. For example, take $\Omega = (-a, a)$ to be an interval in $\mathbb{R}$, and solve*

$$\triangle u = 1 \quad in \quad \Omega, \qquad u = 0 \quad on \quad \partial\Omega.$$

*The unique solution is $u(x) = (x^2 - a^2)/2$ with*

$$\max_{\bar{\Omega}} |u| = \frac{a^2}{2}, \qquad \max_{\partial\Omega} |g| + \frac{a^2}{2n} \sup_{\Omega} |f| = \frac{a^2}{2}, \quad n = 1$$

*and see **the diameter comes into play**. In this example, we actually have **equality** in (186). This is because we choose $x_0 = 0$ and the radius for the ball $B_R(x_0)$ is smallest, i.e. $R = a$.*

**Proof.** Let $A = \max_{\partial\Omega} |g|$, $B = \sup_{\Omega} |f|$. We may assume $A,\ B < \infty$. Define

$$w(x) = \left[A + \frac{B}{2n}\left(R^2 - |x - x_0|^2\right)\right] \pm u(x), \quad x \in \bar{\Omega} = \Omega \bigcup \partial\Omega, \tag{187}$$

then

$$\begin{aligned} &(1).\ w \geq 0 \quad \text{on} \quad \partial\Omega \\ &(2).\ \triangle w = -B \pm \triangle u \leq 0 \quad \text{on} \quad \Omega \\ &(3).\ w \in C^2(\Omega) \bigcap C^0(\bar{\Omega}). \end{aligned} \tag{188}$$

Maximum principle implies that $w \geq 0$ in $\Omega$ and so

$$\begin{aligned} |u(x)| &\leq A + \frac{B}{2n}\left(R^2 - |x - x_0|^2\right) \\ &\leq A + \frac{B}{2n}R^2 = \max_{\partial\Omega} |g| + \frac{R^2}{2n} \sup_{\Omega} |f|, \quad \forall\ x \in \Omega. \end{aligned}$$

The proof is done. □

Another application of the weak maximum/minimum principle is the following:

**Theorem 1.95** ***(Removable singularity of harmonic functions.)*** *Assume $n \geq 2$. Let $B_R \subset \mathbb{R}^n$ be the open ball centered at $x = 0$ with radius $R > 0$. Suppose $u$ is* ***harmonic*** *in $B_R \backslash \{0\}$ and satisfies*

$$u(x) = \begin{cases} o(\log|x|), & n = 2 \\ o\left(|x|^{2-n}\right), & n \geq 3 \end{cases} \quad \text{as} \quad |x| \to 0, \tag{189}$$

*which means that*

$$\begin{cases} \lim_{r\to 0}\left(\dfrac{\max_{\partial B_r}|u|}{\log r}\right) = 0, & n = 2 \\ \lim_{r\to 0}\left(\dfrac{\max_{\partial B_r}|u|}{r^{2-n}}\right) = 0, & n \geq 3, \end{cases} \tag{190}$$

*then $u$ can be defined at $x = 0$ such that it is $C^2$ and harmonic in $B_R$.*

**Remark 1.96** *(Important.)* ***The above says that any harmonic function with a singularity growing slower than the fundamental solution is in fact removable !***

<u>**Solution**</u>:

We first look at the case $n \geq 3$. By (189), we have

$$\lim_{r\to 0}\left(\frac{\max_{\partial B_r}|u|}{\log r}\right) = 0, \quad n = 2; \qquad \lim_{r\to 0}\left(\frac{\max_{\partial B_r}|u|}{r^{2-n}}\right) = 0, \quad n \geq 3. \tag{191}$$

Without loss of generality, we may assume $u$ is continuous on $0 < |x| \leq R$, i.e. continuous up to $\partial B_R$ (otherwise we can look at $u$ on $0 < |x| \leq R - \varepsilon$ for some small $\varepsilon > 0$).

Let $v$ solve the equation

$$\begin{cases} \triangle v = 0 & \text{in} \quad B_R \\ v = u & \text{on} \quad \partial B_R. \end{cases}$$

We will prove $u = v$ on $B_R \backslash \{0\}$ (with this, $u$ can be defined at $x = 0$ such that it is $C^2$ and harmonic in $B_R$).

Set $w = v - u$ in $B_R \setminus \{0\}$ and $M_r = \max_{\partial B_r} |w|$, $0 < r < R$. Clearly we have $w$ is harmonic in the region $\Omega_r = B_R \setminus \{0\} - B_r \setminus \{0\}$. The function $|x|^{2-n} / r^{2-n}$ is also **harmonic** in $\Omega_r$. By the maximum principle, we have (note that $w \equiv 0$ on $\partial B_R$)

$$|w(x)| \leq M_r \frac{|x|^{2-n}}{r^{2-n}} \quad \text{on} \quad \Omega_r. \tag{192}$$

On the other hand, we have (note that $v$ is harmonic on $B_R$, continuous up to $\partial B_R$)

$$M_r = \max_{\partial B_r} |v - u| \leq \underbrace{\max_{\partial B_r} |v|} + \max_{\partial B_r} |u| \leq \underbrace{\max_{\partial B_R} |v|} + \max_{\partial B_r} |u| = \max_{\partial B_R} |u| + \max_{\partial B_r} |u|.$$

From (192) we get

$$|w(x)| \leq \left( \max_{\partial B_R} |u| + \max_{\partial B_r} |u| \right) \frac{|x|^{2-n}}{r^{2-n}}, \quad \forall\, x \in \Omega_r. \tag{193}$$

By the assumption, we know that $\lim_{r \to 0} \left( \max_{\partial B_r} |u| \right) / r^{2-n} = 0$; hence if we let $r \to 0$ in (193), we get

$$w(x) \equiv 0 \quad \text{in} \quad B_R \setminus \{0\}.$$

The proof is done.

For $n = 2$, we may assume $R < 1$ and $u$ is continuous on $0 < |x| \leq R$. Now we replace (192) by

$$|w(x)| \leq M_r \frac{\log |x|}{\log r} \quad \text{on} \quad \Omega_r, \tag{194}$$

where the harmonic function $\log |x| / \log r$ has value 1 on $\partial B_r$ and has **positive** value $\log R / \log r$ on $\partial B_R$ (note that $w \equiv 0$ on $\partial B_R$). Therefore, we conclude

$$|w(x)| \leq \left( \max_{\partial B_R} |u| + \max_{\partial B_r} |u| \right) \frac{\log |x|}{\log r}, \quad \forall\, x \in \Omega_r \tag{195}$$

and if we let $r \to 0$ in the above, we get

$$w(x) \equiv 0 \quad \text{in} \quad B_R \setminus \{0\}.$$

The proof is done. □

**Corollary 1.97** *Assume $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ is an open set and $p_1, \ ..., \ p_k$, $k \in \mathbb{N}$, are finite distinct points in $\Omega$. If $u \in C^2(\Omega \setminus \{p_1, \ ..., \ p_k\}) \bigcap C^0(\Omega)$ and is **harmonic** on $\Omega \setminus \{p_1, \ ..., \ p_k\}$, then $u \in C^2(\Omega)$ and is **harmonic** on $\Omega$.*

**Remark 1.98** *(**Important.**) The above result is not correct in the case $n = 1$. For example, take $u(x) = |x|$, $x \in \Omega = (-\infty, \infty)$.*

**Example 1.99** *The function*

$$u(x, y) = \frac{x}{x^2 + y^2} \ \left(or \ \frac{y}{x^2 + y^2}\right), \quad (x, y) \neq (0, 0) \in \mathbb{R}^2$$

*is harmonic on $\mathbb{R}^2 \setminus \{(0,0)\}$ and has a **singularity** at $(0,0)$. Along each line $y = mx$, $m \in (-\infty, \infty)$, we have*

$$\lim_{x \to 0} u(x, mx) = \lim_{x \to 0} \frac{x}{(1 + m^2) x^2} = \frac{1}{1 + m^2} \lim_{x \to 0} \frac{1}{x} = \pm \infty.$$

*It is **impossible** to define $u(x, y)$ at $(0,0)$ so that it becomes harmonic on the whole plane $\mathbb{R}^2$. One can also see that the condition (190) cannot be satisfied, i.e.*

$$\lim_{r \to 0} \left( \frac{\max_{\partial B_r} |u|}{\log r} \right) = \lim_{r \to 0} \left( \frac{\max_{\theta \in [0, 2\pi]} \left| \frac{r \cos \theta}{r^2} \right|}{\log r} \right) = \lim_{r \to 0} \left( \frac{1}{r \cdot \log r} \right) = -\infty \neq 0. \tag{196}$$

This is the end of elliptic equations.

The above will be the coverage of the final exam on 2022/6/13