PDE Teaching for Spring, 2023

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1 Laplace equation on $\mathbb{R}^n$ (for $n = 2,\ 3$) and the divergence theorem.

From now on, we shall focus on second order elliptic equation with constant coefficients. If $u(x,y)$ is a function of two variables, its canonical form is

$$u_{xx}(x,y) + u_{yy}(x,y) + (\text{lower order terms}) = f(x,y),$$

where $f(x,y)$ is a given continuous function defined on some domain $\Omega \subseteq \mathbb{R}^2$. To begin with, for simplicity, we shall look at the 2-dimensional **Laplace equation:**

$$(\triangle u)(x,y) \ \ (\text{also write it as } \ \triangle u(x,y)\,) := u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad (x,y) \in \Omega \subseteq \mathbb{R}^2. \tag{1}$$

The purpose is to find C^2 solutions $u(x,y) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$. If the domain Ω is not specified in advance, we want the solutions $u(x,y)$ to be defined on some open set Ω in $\mathbb{R}^2$, as large as possible. A C^2 solution u of (1), defined on Ω, is called a **harmonic function** on Ω. In case $u = u(x,y,z)$ is a three-variable function, then the above becomes

$$\triangle u(x,y,z) := u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z) = 0, \quad (x,y,z) \in \Omega \subseteq \mathbb{R}^3. \tag{2}$$

Example 1.1 *The following functions*

$$u(x,y) = x, \quad y, \quad x^2 - y^2, \quad 2xy, \quad x^3 - 3xy^2, \quad y^3 - 3x^2y, \quad e^x \cos y, \quad e^x \sin y \tag{3}$$

are all harmonic functions defined on the whole plane $\mathbb{R}^2$. The functions

$$u(x,y) = \frac{x}{x^2+y^2}, \quad \frac{-y}{x^2+y^2}$$

*are harmonic functions defined on $\mathbb{R}^2 \backslash \{(0,0)\}$. In the above, the polynomials $x,\ y,\ x^2-y^2,\ 2xy,\ x^3-3xy^2,\ y^3-3x^2y,\ \ldots$ etc. are called **harmonic polynomials** with degree 1, 2, 3, ... etc.*

Definition 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a domain (open and connected) and let $V(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 **vector field** on Ω given by*

$$V(x) = (V_1(x),\ \ldots, V_n(x)), \quad V_i(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R} \text{ is a } C^1 \text{ function.}$$

*Its **divergence**, denoted as $(\operatorname{div} V)(x)$ (or just $\operatorname{div} V(x)$): $\Omega \subset \mathbb{R}^n \to \mathbb{R}$, is a **scalar function** defined as*

$$\operatorname{div} V(x) = \frac{\partial V_1}{\partial x_1}(x) + \cdots + \frac{\partial V_n}{\partial x_n}(x), \quad x = (x_1,\ \ldots, x_n) \in \Omega.$$

Note that $\operatorname{div} V(x)$ is a continuous function on Ω. In physics, we also denote $\operatorname{div} V(x)$ as

$$\operatorname{div} V(x) = (\nabla \cdot V)(x),$$

where ∇ denotes the vector operator $\nabla = \left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \ldots,\ \frac{\partial}{\partial x_n}\right)$.

Remark 1.3 *Note that we can also express $\operatorname{div} V(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ as*

$$\operatorname{div} V(x) = Tr(DV)(x), \quad x \in \Omega, \tag{4}$$

where DV is an $n \times n$ matrix, which is the derivative of the map $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. $Tr(DV)(x)$ means $Tr((DV)(x))$.

We first note that for C^2 function $u(x,y) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, $\triangle u(x,y)$ can be decomposed as

$$\triangle u(x,y) = (\operatorname{div}(\nabla u))(x,y) = (\nabla \cdot \nabla u)(x,y) = Tr\left(\underbrace{D(\nabla u)}\right)(x,y), \quad (x,y) \in \Omega. \tag{5}$$

Similar identity holds for $\triangle u(x)$, $x \in \Omega \subseteq \mathbb{R}^n$, $n \geq 3$. Note that the matrix $(D(\nabla u))(x,y)$ is actually the **Hessian matrix** of $u(x,y)$.

The following **divergence theorem** (general version of the Fundamental Theorem of Calculus) will be needed often later on:

Theorem 1.4 *(**Divergence theorem.**) Let Ω be a C^1 bounded domain in $\mathbb{R}^n$ (which means that its boundary $\partial\Omega$ is a C^1 $(n-1)$-dimensional surface in $\mathbb{R}^n$) and $\mathbf{W} : \bar{\Omega} \to \mathbb{R}^n$ is a vector field on $\bar{\Omega}$ with $\mathbf{W} \in C^1(\Omega) \bigcap C^0(\bar{\Omega})$. We have the identity*

$$\int_\Omega \operatorname{div} \mathbf{W} dx \ \textit{(volume integral in } \mathbb{R}^n, \ dx \textit{ means } dx_1 \cdots dx_n\textit{)}$$
$$= \int_{\partial\Omega} \mathbf{W} \cdot \mathbf{N} d\sigma \ \textit{((n-1)-dimensional } \textbf{surface integral} \textit{ in } \mathbb{R}^n\textit{)} \tag{6}$$

*where $\mathbf{N}$ is the unit **outward** normal to $\partial\Omega$. Here $\operatorname{div}\mathbf{W} : \Omega \to \mathbb{R}$ is the **divergence** of the vector field $\mathbf{W}$ and $d\sigma$ is the "**surface measure**" on $\partial\Omega$.*

Remark 1.5 *Ω is a C^1 bounded domain in $\mathbb{R}^n$ means that its boundary $\partial\Omega$ is locally a C^1 graph everywhere. For example, if Ω is a C^1 bounded domain in $\mathbb{R}^3$, then near any $p \in \partial\Omega$, the boundary $\partial\Omega$ can be expressed as a graph $z = f(x,y)$ for some C^1 function $f(x,y)$ defined on some open subset of $\mathbb{R}^2$. Therefore the boundary $\partial\Omega$ is a C^1 surface in $\mathbb{R}^3$.*

Remark 1.6 *(**Important.**) Be careful that the divergence theorem **is valid only when** $\Omega \subset \mathbb{R}^n$ **is bounded**. For example, let $\Omega = \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ be the upper half-space of $\mathbb{R}^n$ and let $\mathbf{W}$ be the smooth vector field on $\bar{\Omega}$ given by*

$$\mathbf{W}(x) = (0, 0, ..., 0, x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \bigcup \partial\mathbb{R}^n_+.$$

*Then we have $\operatorname{div}\mathbf{W} \equiv 1$ on $\mathbb{R}^n_+$ and $\mathbf{W} \equiv 0$ on $\partial\mathbb{R}^n_+$. In such a case, the identity (6) clearly fails. In view of this, the **Green identities also fail on unbounded domains**.*

Remark 1.7 *If $n = 2$, then $d\sigma$ means ds, where ds is the **arc length differential** and the above theorem is the same as the familiar **Green Theorem** for plane region enclosed by a **simple closed curve** Γ. More precisely, let $C \subset \mathbb{R}^2$ be a **counterclockwise** simple closed curve parametrized by $\alpha(t) = (x(t), y(t))$, $t \in [a,b]$, where t is an arbitrary parameter (**not necessarily the arc length parameter**) and let $\Omega \subset \mathbb{R}^2$ be the open region enclosed by C with $\partial\Omega = C$ ($\partial\Omega$ means the boundary of Ω). Let $\mathbf{W} \in C^1(\Omega) \cap C^0(\bar{\Omega})$ be a vector field on $\bar{\Omega}$ given by*

$$\mathbf{W}(x,y) = (p(x,y), q(x,y)), \quad (x,y) \in \bar{\Omega}.$$

Now we first have

$$\int_\Omega \operatorname{div}\mathbf{W} dx \ \textit{(volume integral in } \mathbb{R}^2\textit{)} = \iint_\Omega \left(\frac{\partial p}{\partial x}(x,y) + \frac{\partial q}{\partial y}(x,y)\right) dxdy \tag{7}$$

*and the unit **outward** normal $\mathbf{N}$ to $(x(t), y(t)) \in \partial\Omega$, $t \in [a,b]$, is given by*

$$\mathbf{N}(x(t), y(t)) = \left(\frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}, \frac{-x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}\right), \quad t \in [a,b]$$

and we also have

$$d\sigma = ds = \sqrt{\left(x'\left(t\right)\right)^2 + \left(y'\left(t\right)\right)^2}dt, \quad t \in [a,b].$$

Hence we conclude

$$\begin{aligned}
&\int_{\partial\Omega} \mathbf{W}\cdot\mathbf{N}d\sigma \\
&= \int_a^b \left[\left(p\left(x\left(t\right),y\left(t\right)\right),q\left(x\left(t\right),y\left(t\right)\right)\right)\cdot\left(\frac{y'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}, \frac{-x'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}\right)\right]\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt \\
&= \int_a^b \left[\left(p\left(x\left(t\right),y\left(t\right)\right)y'\left(t\right) - q\left(x\left(t\right),y\left(t\right)\right)\right)x'\left(t\right)\right]dt = \int_C -qdx + pdy \text{ (this is line integral).} \qquad (8)
\end{aligned}$$

By (7) and (8), we conclude the identity

$$\iint_\Omega \left(\frac{\partial p}{\partial x}(x,y) + \frac{\partial q}{\partial y}(x,y)\right)dxdy = \int_C -qdx + pdy, \tag{9}$$

***which is exactly the familiar Green Theorem (in a slightly different way)**. Note that the value of the line integral $\int_C -qdx + pdy$ is **independent of** parametrization.*

Remark 1.8 *If $n = 3$, then the surface measure $d\sigma$ means*

$$d\sigma = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}dxdy \tag{10}$$

*if the surface is represented by **the graph of a function** $z = f(x,y)$. Therefore, we are doing **surface integrals** in $\mathbb{R}^3$. Also, if the surface in $\mathbb{R}^3$ is given by the **parametrization form***

$$X(u,v) : (u,v) \in U \subset \mathbb{R}^2 \to (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3,$$

then

$$d\sigma = |X_u \times X_v|\,dudv, \quad \text{where } \times \text{ is the cross product in } \mathbb{R}^3. \tag{11}$$

In case the surface in $\mathbb{R}^3$ is given by the equation

$$\varphi(x,y,z) = 0$$

for some smooth function $\varphi(x,y,z) : O \subset \mathbb{R}^3 \to \mathbb{R}$, then (assuming the surface can be expressed as $z = f(x,y)$ for $(x,y) \in U \subset \mathbb{R}^2$) we have

$$X(x,y) = (x,y,f(x,y)) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$$

and by chain rule we have

$$d\sigma = |X_x \times X_y|\,dxdy = \sqrt{1 + f_x^2 + f_y^2}dxdy = \frac{\sqrt{\varphi_x^2 + \varphi_y^2 + \varphi_z^2}}{|\varphi_z|}dxdy. \tag{12}$$

The first identity in (12) is clear, to see the second identity, note that by $\varphi(x,y,f(x,y)) = 0$ for all $(x,y) \in U \subset \mathbb{R}^2$, we have

$$f_x = -\frac{\varphi_x}{\varphi_z}, \quad f_y = -\frac{\varphi_y}{\varphi_z} \tag{13}$$

and so

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \left(\frac{\varphi_x}{\varphi_z}\right)^2 + \left(\frac{\varphi_y}{\varphi_z}\right)^2} = \frac{\sqrt{\varphi_x^2 + \varphi_y^2 + \varphi_z^2}}{|\varphi_z|}. \tag{14}$$

Proof. (Read this by yourself.) (Special case only.) We will give a proof of the theorem only for the case $n = 2$ and assuming that Ω takes the simple form:

$$\Omega = \{(x, y) : \ a \leq x \leq b, \quad 0 \leq y \leq f(x), \quad f(a) = f(b) = 0\} \tag{15}$$

where $f(x)$ is a C^1 function defined on $[a, b]$. Although Ω may not be smooth at $(a, f(a))$ and $(b, f(b))$, divergence theorem still holds for such Ω as shown below. Its boundary $\partial\Omega$ has two parts: the graph $y = f(x)$ and the segment $(x, 0)$, $a \leq x \leq b$. Call them $\partial_1\Omega$ and $\partial_2\Omega$ respectively. We have

$$\begin{cases} \mathbf{N} \ \text{at} \ (x, f(x)) \ = \dfrac{(-f'(x), \ 1)}{\sqrt{1 + (f'(x))^2}} \\ \\ \mathbf{N} \ \text{at} \ (x, 0) \ = (0, -1). \end{cases} \tag{16}$$

Both normal vectors in (16) are pointing outwards. Writing $\mathbf{W}(x, y) = (u(x, y), v(x, y))$, the divergence theorem is equivalent to

$$\begin{aligned} &\iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy, \quad \operatorname{div} \mathbf{W} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ &= \int_{\partial_1\Omega} (u(x, y), v(x, y)) \cdot \frac{(-f'(x), \ 1)}{\sqrt{1 + (f'(x))^2}} ds + \int_{\partial_2\Omega} (u(x, y), v(x, y)) \cdot (0, -1)\, ds, \end{aligned} \tag{17}$$

where ds is the **arc length** differential. Note that the boundary line integral $\int_{\partial\Omega} ds$ in (17) has **no orientation**. Clearly we have

$$\begin{cases} y = f(x), \quad ds = \sqrt{1 + (f'(x))^2} dx \quad \text{on} \quad \partial_1\Omega \\ \\ y = 0, \quad ds = dx \quad \text{on} \quad \partial_2\Omega \end{cases} \tag{18}$$

and so the RHS (right-hand side) of (17) becomes

$$\begin{aligned} &\int_a^b (u(x, f(x)), v(x, f(x))) \cdot (-f'(x), 1)\, dx + \int_a^b (u(x, 0), v(x, 0)) \cdot (0, -1)\, dx \\ &= \int_a^b [-u(x, f(x)) f'(x) + v(x, f(x))]\, dx - \int_a^b v(x, 0)\, dx \\ &= -\overbrace{\int_a^b u(x, f(x)) f'(x)\, dx} + \underbrace{\int_a^b [v(x, f(x)) - v(x, 0)]\, dx}. \end{aligned} \tag{19}$$

Also the LHS of (17) is

$$\iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy = \overbrace{\int_a^b \int_0^{f(x)} \left(\frac{\partial u}{\partial x}\right) dydx} + \underbrace{\int_a^b \int_0^{f(x)} \left(\frac{\partial v}{\partial y}\right) dydx}. \tag{20}$$

Now we have

$$\underbrace{\int_a^b \int_0^{f(x)} \left(\frac{\partial v}{\partial y}\right) dydx} = \underbrace{\int_a^b [v(x, f(x)) - v(x, 0)]\, dx}. \tag{21}$$

Hence it suffices to show

$$\overbrace{\int_a^b u(x, f(x)) f'(x)\, dx + \int_a^b \int_0^{f(x)} \frac{\partial u}{\partial x}(x, y)\, dydx} = 0. \tag{22}$$

Note that the LHS (left-hand side) of (22) is

$$\int_a^b \left(\underline{u(x, f(x)) f'(x) + \int_0^{f(x)} \frac{\partial u}{\partial x}(x, y)\, dy} \right) dx, \tag{23}$$

where the integrand in (23) can be written as

$$\underline{u(x, f(x)) f'(x) + \int_0^{f(x)} \frac{\partial u}{\partial x}(x, y)\, dy} = \underline{\frac{d}{dx}\left(\int_0^{f(x)} u(x, y)\, dy \right)}.$$

Hence we have

$$\begin{aligned} &\int_a^b \left(\underline{u(x, f(x)) f'(x) + \int_0^{f(x)} \frac{\partial u}{\partial x}(x, y)\, dy} \right) dx \\ &= \int_a^b \underline{\frac{d}{dx}\left(\int_0^{f(x)} u(x, y)\, dy \right)} dx = \int_0^{f(b)} u(b, y)\, dy - \int_0^{f(a)} u(a, y)\, dy = 0, \end{aligned}$$

due to $f(a) = f(b) = 0$. The proof is done. □

1.1 Averaging property of the Laplace operator.

The Laplace operator has the following important geometric meaning: let $u(x, y)$ be a C^2 function defined on $\mathbb{R}^2$ and let $O = (0, 0)$ be the origin. Any line L passing through O with direction $v(\theta) = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$, has the parametric form

$$L = \{(t\cos\theta, t\sin\theta) : t \in (-\infty, \infty),\ \theta \in [0, 2\pi]\}.$$

The function $u(x, y)$ restricted on L becomes a function of t, i.e., we have (here the angle θ is fixed)

$$h(t) := u(O + tv) = u(tv), \text{ where } v = v(\theta) = (\cos\theta, \sin\theta), \quad t \in (-\infty, \infty).$$

We note that

$$\begin{aligned} h'(0) &= \left.\frac{d}{dt}\right|_{t=0} u(O + tv) = \lim_{t \to 0} \frac{u(O + tv) - u(O)}{t} \\ &= \lim_{t \to 0} \frac{u(tv) - u(0, 0)}{t} = D_v u(O) = \langle \nabla u(O), v \rangle, \end{aligned} \tag{24}$$

which is the **directional derivative** of u at $O = (0, 0)$ in the direction $v = (\cos\theta, \sin\theta)$. If we compute the second derivative of $h(t)$ at $t = 0$, by

$$\begin{cases} h'(t) = \frac{d}{dt} u(O + tv) = \langle \nabla u(O + tv), v \rangle = u_x(O + tv)\cos\theta + u_y(O + tv)\sin\theta, & t \in (-\infty, \infty) \\ h'(0) = D_\nu u(O) = \langle \nabla u(O), v \rangle = u_x(O)\cos\theta + u_y(O)\sin\theta, & O = (0, 0), \end{cases} \tag{25}$$

we obtain

$$\begin{cases} h''(t) = \left.\frac{d^2}{dt^2}\right|_{t=0} u(O + tv) = \left(\frac{d}{dt} u_x(O + tv)\right)\cos\theta + \left(\frac{d}{dt} u_y(O + tv)\right)\sin\theta \\ = (\cos\theta, \sin\theta) \begin{pmatrix} u_{xx}(O + tv) & u_{xy}(O + tv) \\ u_{yx}(O + tv) & u_{yy}(O + tv) \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad u_{xy}(O + tv) = u_{yx}(O + tv) \end{cases} \tag{26}$$

and then

$$h''(0) = u_{xx}(O)\cos^2\theta + 2u_{xy}(O)\cos\theta\sin\theta + u_{yy}(O)\sin^2\theta, \tag{27}$$

and if we **average** $h''(0)$ among all possible directions (i.e. among all possible angle $\theta \in [0, 2\pi]$), we get

$$\frac{1}{2\pi}\int_0^{2\pi} h''(0)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\left.\frac{d^2}{dt^2}\right|_{t=0} u(tv(\theta))\right)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial^2 u}{\partial x^2}(O)\cos^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}(O)\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}(O)\sin^2\theta\right)d\theta$$
$$= \frac{1}{2\pi}\left[\frac{\partial^2 u}{\partial x^2}(O)\cdot\pi + \frac{\partial^2 u}{\partial y^2}(O)\cdot\pi\right] = \frac{1}{2}(\triangle u)(O), \quad v(\theta) = (\cos\theta, \sin\theta), \quad \theta \in [0, 2\pi]. \tag{28}$$

Thus the quantity $\frac{1}{2}(\triangle u)(O)$ is the average of the second derivatives of u among all possible directions $v(\theta)$.

Remark 1.9 *One can* ***rotate the orthonormal basis*** $\{e_1, e_2\}$ *in* $\mathbb{R}^2$ *to see why we have the coefficient* $1/2$ *in (28). Because we have*

$$\int_0^{2\pi} h''(0)\,d\theta \text{ (rotation of } e_1 \text{ by } 2\pi) + \int_0^{2\pi} h''(0)\,d\theta \text{ (rotation of } e_2 \text{ by } 2\pi) = 2\pi\cdot(\triangle u)(O), \tag{29}$$

which gives (28).

1.2 Green identities.

There are many useful consequence of the **divergence theorem** (equivalent to the **Green Theorem** if we are in $\mathbb{R}^2$). Among the most important are the **Green identities**: Assume Ω is a C^1 bounded domain in $\mathbb{R}^2$ (or in $\mathbb{R}^n$, $n \geq 3$) and $u,\ v \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$ $(u,\ v : \bar{\Omega} \to \mathbb{R})$. We have the **Green 1st identity:**

$$\int_\Omega v \triangle u dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{N}} d\sigma, \quad \nabla u = \text{gradient of } u \tag{30}$$

and **Green 2nd identity:**

$$\int_\Omega (v \triangle u - u \triangle v)\,dx = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial \mathbf{N}} - u\frac{\partial v}{\partial \mathbf{N}}\right)d\sigma, \tag{31}$$

where in (30), $\nabla u \cdot \nabla v$ denotes the inner product of the two gradient vectors ∇u, ∇v, and $\frac{\partial u}{\partial \mathbf{N}}$ is the directional derivative of u on $\partial\Omega$ along the unit outward normal vector $\mathbf{N}$ which, by the chain rule, is equal to

$$\frac{\partial u}{\partial \mathbf{N}} = \nabla u \cdot \mathbf{N}, \quad \frac{\partial u}{\partial \mathbf{N}}(p) = \lim_{t \to 0^-}\frac{u(p + t\mathbf{N}) - u(p)}{t}, \quad p \in \partial\Omega. \tag{32}$$

Remark 1.10 *We call* $\frac{\partial u}{\partial \mathbf{N}}$ *the* ***outward normal derivative*** *of* u *on* $\partial\Omega$*. Note that* $\frac{\partial u}{\partial \mathbf{N}}$ *is defined only on* $\partial\Omega$ *since the vector* $\mathbf{N}$ *is defined only on* $\partial\Omega$

Remark 1.11 *We can use* ***Green identities*** *to prove the* ***uniqueness of solution*** *for Dirichlet boundary value problem.*

We note that (30) is a consequence of the **divergence theorem** and the identity

$$\text{div}(v\nabla u) = v \triangle u + \nabla v \cdot \nabla u, \quad u,\ v \in C^2(\Omega)\bigcap C^1(\bar{\Omega}). \tag{33}$$

By the divergence theorem, we have

$$\int_\Omega v\triangle u dx+\int_\Omega \nabla u\cdot\nabla v dx=\int_\Omega \operatorname{div}(v\nabla u)\,dx=\int_{\partial\Omega}(v\nabla u)\cdot\mathbf{N}d\sigma=\int_{\partial\Omega}v\frac{\partial u}{\partial\mathbf{N}}d\sigma, \tag{34}$$

which gives (30). As for (31), it is an easy consequence of (30).

In particular, when $u=v$ in (30), we get the identity

$$\int_\Omega u\triangle u dx+\int_\Omega |\nabla u|^2\,dx=\int_{\partial\Omega}u\frac{\partial u}{\partial\mathbf{N}}d\sigma, \tag{35}$$

and when $v\equiv 1$ in (30), we get the identity

$$\int_\Omega \triangle u dx=\int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{N}}d\sigma=\int_{\partial\Omega}\nabla u\cdot\mathbf{N}d\sigma. \tag{36}$$

Physically, the quantity $\int_{\partial\Omega}\nabla u\cdot\mathbf{N}d\sigma$ is called the **flux** of the vector field ∇u across the boundary $\partial\Omega$ of Ω. In particular, if $u\in C^2(\Omega)\bigcap C^1(\bar\Omega)$ is harmonic function on Ω, we have $\int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{N}}d\sigma=0$ (zero flux across $\partial\Omega$). Therefore, **a harmonic function is in the state of balance.**

An useful application of (35) is the following:

Lemma 1.12 *Let Ω be a C^1 bounded domain in $\mathbb{R}^n$, $n\geq 2$, and $u:\bar\Omega\to\mathbb{R}$, $u\in C^2(\Omega)\bigcap C^1(\bar\Omega)$. If we have*

$$\begin{cases}\triangle u=0 & in\quad \Omega\\ u=0 & on\quad \partial\Omega,\end{cases}$$

then we must have $u\equiv 0$ on $\bar\Omega$.

Proof. By (35), we have

$$\int_\Omega u\triangle u dx+\int_\Omega |\nabla u|^2\,dx=\int_{\partial\Omega}u\frac{\partial u}{\partial\mathbf{N}}d\sigma,$$

and so $\int_\Omega |\nabla u|^2\,dx=0$, which implies $\nabla u\equiv 0$ in Ω. Since Ω is open and connected, u must be a constant on Ω. This constant must be zero since $u=0$ on $\partial\Omega$. □

1.3 Radial harmonic functions in $\mathbb{R}^n$.

In this section, we look at the Laplace equation on $\mathbb{R}^n\backslash\{0\}$:

$$\triangle u(x)=0,\quad x=(x_1,\ \ldots\ ,x_n)\in\mathbb{R}^n\backslash\{0\}, \tag{37}$$

and we want to look for a **special important solution** of the Laplace equation in $\mathbb{R}^n$ which is **radial**, i.e., it has the form

$$u(x)=v(r),\quad r=|x|=\sqrt{x_1^2+\cdots+x_n^2}>0,$$

where $v(r)$ is chosen so that it satisfies $\triangle u(x)=0$ for all $x\in\mathbb{R}^n\backslash\{0\}$. **Intuitively speaking, since Laplace equation has symmetry among all directions** $e_1,\ e_2,\ \ldots,\ e_n$**, such a radial solution should exists.**

The following observation is also useful to see why there is a radius harmonic function. It is know that if $u(x):\mathbb{R}^n\to\mathbb{R}$ is a harmonic func. on $\mathbb{R}^n$, then for any **orthogonal matrix** $A\in O(n)$ ($O(n)$ is the space of all $n\times n$ orthogonal matrices), the func.

$$w(x):=u(Ax):\mathbb{R}^n\to\mathbb{R}^n$$

is also a harmonic func. on $\mathbb{R}^n$. **An interesting question is:** Is there a **harmonic func.** $u(x) : \mathbb{R}^n \to \mathbb{R}$ satisfying the property

$$u(x) = u(Ax), \quad \textbf{for all } x \in \mathbb{R}^n \text{ and } \textbf{for all } A \in O(n). \tag{38}$$

In linear algebra, if $x_0 \neq y_0 \in \mathbb{R}^n$ has the same length $|x_0| = |y_0| = r_0 > 0$, then there exists an orthogonal matrix $A \in O(n)$ such that $Ax_0 = y_0$. Therefore, if $u(x)$ satisfies property (38), it also satisfies the property

$$u(x) = u(y), \quad \text{for all } x,\ y \in \mathbb{R}^n \text{ with the same length,} \tag{39}$$

which means that $u(x) : \mathbb{R}^n \to \mathbb{R}$ must be a **radial harmonic func.**, i.e. it has the form

$$u(x) = v(r), \quad r = |x| = \sqrt{x_1^2 + \cdots + x_n^2} \geq 0 \tag{40}$$

for some func. $v(r) : [0, \infty) \to \mathbb{R}$.

Remark 1.13 *At this moment, we do not know if such function $v(r)$ exists or not; we shall see that the function $v(r)$ does exist, but **cannot** be defined at $r = 0$ (has a **singularity** at $r = 0$), which implies that the radial function $u(x)$ cannot be defined at $x = 0$.*

Instead of solving a PDE for $u(x)$, we only have to solve an ODE for $v(r)$. By the chain rule, for $r > 0$ we have

$$\frac{\partial u}{\partial x_i}(x) = v'(r)\frac{x_i}{r}, \quad \frac{\partial^2 u}{\partial x_i^2}(x) = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right), \quad 1 \leq i \leq n,$$

which gives (sum over $i = 1,\ 2,\ 3,\ ...,\ n$)

$$\triangle u(x) = v''(r) + \frac{n-1}{r}v'(r) = 0, \quad r > 0 \tag{41}$$

and we obtain a **second-order ODE** for $v(r)$ over the domain $r \in (0, \infty)$. Multiplying equation (41) by r^{n-1}, one can verify that the general solution of (41) is given by

$$v(r) = v(|x|) = u(x) = \begin{cases} Ar^{2-n} + B, & n > 2, \quad r = |x| \in (0, \infty) \\ A \log r + B, & n = 2, \quad r = |x| \in (0, \infty) \end{cases} \tag{42}$$

where $A,\ B$ are integration constants. Since $v(r)$ is **not** defined at $r = 0$, the above radial solution $u(x)$ is not defined at $x = 0$. The corresponding $u(x) = u(x_1,\ ...\ , x_n)$ lies in the space $C^\infty(\mathbb{R}^n \backslash \{0\})$, given by

$$u(x_1,\ ...\ , x_n) = \begin{cases} A\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{2-n}{2}} + B, & n > 2, \\ A \log \sqrt{x_1^2 + x_2^2} + B = \tilde{A} \log\left(x_1^2 + x_2^2\right) + B, \quad \tilde{A} = \frac{A}{2}, & n = 2. \end{cases} \tag{43}$$

Remark 1.14 *(**Interesting observation.**) Why does the Laplace equation have radial solutions? This is because the Laplace operator has **symmetry** in it. If we change the Laplace operator into a **non-symmetric form**, for example, the form:*

$$\tilde{\triangle} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial}{\partial x_1} = \triangle + \frac{\partial}{\partial x_1} \quad \textit{(or other forms, say } \triangle + \frac{\partial^2}{\partial x_1^2}\textit{)},$$

then if $u(x)$ has the radial form $v(r)$, $r = |x|$, we have

$$\tilde{\triangle} u(x) = \triangle u(x) + \frac{\partial u}{\partial x_1}(x) = v''(r) + \frac{n-1}{r}v'(r) + \underbrace{v'(r)\frac{x_1}{r}},$$

*which **cannot produce a self-contained equation (ODE)** for $v(r)$ due to the term $v'(r)\frac{x_1}{r}$. Thus for the new operator $\tilde{\triangle}$, it has **no radial** solution at all (except the trivial constant solutions).*

We can conclude the following:

Lemma 1.15 *Consider the Laplace equation on* $\mathbb{R}^n$*, given by*

$$(\triangle u)(x) = 0, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{44}$$

Its ***radial solution*** *(defined only on* $\mathbb{R}^n \setminus \{0\}$*) is given by*

$$u(x) = \begin{cases} A|x|^{2-n} + B, & n > 2, \quad x \in \mathbb{R}^n \setminus \{0\} \\ A \log|x| + B, & n = 2, \quad x \in \mathbb{R}^2 \setminus \{0\} \end{cases} \tag{45}$$

and no others. Here $A,\ B$ *are two arbitrary constants with* $A \neq 0.$

Definition 1.16 *The radial function* $u(x)$ *given by (45) is also called the* ***fundamental solution*** *of the Laplace equation. It is a* ***harmonic function*** *defined on* $\mathbb{R}^n \setminus \{0\}$*. Moreover, it has a singularity at* $x = 0$.

Remark 1.17 *In the one-dimensional* 1*-dimensional case, i.e.* $n = 1$*, the* ***fundamental solution*** *of the Laplace equation* $u''(x) = 0$ *is given by the radial function* $u(x) = A|x| + B$*, where* $A \neq 0,\ B$ *are two arbitrary constants. Unlike the case* $n \geq 2$*, it is continuous at the origin* $x = 0$ *(but not differentiable). It is not so interesting.*

In $\mathbb{R}^3$ we have (now we denote $x \in \mathbb{R}^3 \setminus \{0\}$ as $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$) the **radial solution** of the Laplace equation:

$$u(x, y, z) = \frac{A}{r} + B = \frac{A}{\sqrt{x^2 + y^2 + z^2}} + B, \quad (x, y, z) \in \mathbb{R}^3 \setminus \{0\}$$

and then

$$\begin{aligned} &(\nabla u)(x, y, z) \\ &= A\left(\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\right) \\ &= \frac{A}{|\overrightarrow{r}|^2}\left(\frac{-\overrightarrow{r}}{|\overrightarrow{r}|} \text{ (this is unit vector)}\right), \quad \text{where } \overrightarrow{r} = (x, y, z) \in \mathbb{R}^3 \setminus \{0\}. \end{aligned} \tag{46}$$

For suitable constant A, (46) describes **the force field of the earth gravity** with point mass at the origin. Note that each component function of $(\nabla u)(x, y, z)$ is also harmonic on $\mathbb{R}^3 \setminus \{0\}$.

1.4 Laplace equation in polar coordinates (r, θ); radial and angular harmonic functions in $\mathbb{R}^2$.

Remark 1.18 *A major purpose of expressing Laplace operator in polar coordinates* (r, θ) *is to find some important special solutions, in particular, the* ***radial*** *solution* $U(r)$ *and the* ***angular*** *solution* $U(\theta)$*. In particular, we can use it to solve the Dirichlet problem of the Laplace equation on the* ***disc*** *in* $\mathbb{R}^2$ *or on the* ***ball*** *in* $\mathbb{R}^3$.

The polar coordinates (r, θ) in $\mathbb{R}^2$ and the Euclidean coordinates (x, y) in $\mathbb{R}^2$ are related by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r > 0, \quad 0 < \theta < 2\pi, \tag{47}$$

where the change of variables is a **diffeomorphism** between the following two **open sets**:

$$\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\} \subset xy\text{-plane} \longleftrightarrow (r, \theta) \in (0, \infty) \times (0, 2\pi) \subset r\theta\text{-plane}. \tag{48}$$

For convenience, we denote the above open set in xy-plane as $\tilde{\mathbb{R}}^2$ and denote the above open set in $r\theta$-plane as Σ in this section. Let $u(x,y) : \tilde{\mathbb{R}}^2 \to \mathbb{R}$ be a C^2 function. Under the above change of variables $u(x,y)$ becomes a C^2 function $U(r,\theta) : \Sigma \to \mathbb{R}$, i.e., $u(r\cos\theta, r\sin\theta) = U(r,\theta)$. What is the expression $u_{xx}(x,y) + u_{yy}(x,y)$ under polar coordinates (r,θ)? The answer is:

$$\triangle u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) U(r,\theta). \tag{49}$$

Example 1.19 *Let $u(x,y) = x^2 y$. Then $U(r,\theta) = r^3\cos^2\theta\sin\theta$. We have*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = 2y.$$

On the other hand, we also have

$$\begin{aligned} &\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) U(r,\theta) \\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(r^3\cos^2\theta\sin\theta\right) = 2r\sin\theta. \end{aligned}$$

Since $y = r\sin\theta$, both sides of (49) are equal.

To derive (49), for the first derivatives, we have the relation:

$$\begin{aligned} \frac{\partial U}{\partial r} &= \frac{\partial}{\partial r}\left[u(r\cos\theta, r\sin\theta)\right] = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{1}{r}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right), \quad r = \sqrt{x^2+y^2} \end{aligned} \tag{50}$$

and

$$\begin{aligned} \frac{\partial U}{\partial \theta} &= \frac{\partial}{\partial \theta}\left[u(r\cos\theta, r\sin\theta)\right] = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}. \end{aligned} \tag{51}$$

We can rewrite the above as the system:

$$\underbrace{r\frac{\partial U}{\partial r} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}}, \qquad \underbrace{\frac{\partial U}{\partial \theta} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}} \tag{52}$$

or equivalently, the **operator relation:**

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{53}$$

It says that the operator $r\frac{\partial}{\partial r}$ is comparable to $\frac{\partial}{\partial \theta}$. In the matrix form, we have the operator identity

$$\begin{pmatrix} r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} (\text{acting on } U(r,\theta)) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} (\text{acting on } u(x,y)), \tag{54}$$

and so

$$\begin{aligned} &\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} (\text{acting on } u(x,y)) = \frac{1}{x^2+y^2}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\ &= \frac{1}{r}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} (\text{acting on } U(r,\theta)). \end{aligned} \tag{55}$$

More precisely, the above gives

$$\frac{\partial u}{\partial x}=\cos\theta\frac{\partial U}{\partial r}-\frac{\sin\theta}{r}\frac{\partial U}{\partial\theta},\qquad \frac{\partial u}{\partial y}=\sin\theta\frac{\partial U}{\partial r}+\frac{\cos\theta}{r}\frac{\partial U}{\partial\theta}. \tag{56}$$

In particular, we have two different ways to express the **gradient vector** of u (for clarity, we look at the vector ∇u at a particular point (x_0, y_0)):

$$\begin{aligned}\nabla u\left(x_0,y_0\right) &= \frac{\partial u}{\partial x}\left(x_0,y_0\right)\begin{pmatrix}1\\0\end{pmatrix}+\frac{\partial u}{\partial y}\left(x_0,y_0\right)\begin{pmatrix}0\\1\end{pmatrix}\\ &= \frac{\partial U}{\partial r}\left(r_0,\theta_0\right)\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix}+\frac{1}{r_0}\frac{\partial U}{\partial\theta}\left(r_0,\theta_0\right)\begin{pmatrix}-\sin\theta_0\\ \cos\theta_0\end{pmatrix},\end{aligned} \tag{57}$$

where (r_0,θ_0) is the polar coordinates corresponding to (x_0,y_0). The above says that we can also express the gradient vector $\nabla u\left(x_0,y_0\right)$ in terms of the **orthonormal basis** $(\cos\theta_0,\sin\theta_0)$, $(-\sin\theta_0,\cos\theta_0)$, with the coefficients given by $\frac{\partial U}{\partial r}\left(r_0,\theta_0\right)$ and $\frac{1}{r_0}\frac{\partial U}{\partial\theta}\left(r_0,\theta_0\right)$.

Remark 1.20 *Draw a picture for the vector $\nabla u\left(x_0,y_0\right)$ and the two orthonormal frames*

$$\{(1,0)\,,\ (0,1)\}\,,\qquad \{(\cos\theta_0,\sin\theta_0)\,,\ (-\sin\theta_0,\cos\theta_0)\}\,,$$

where we note that the vector $(\cos\theta_0,\sin\theta_0)$ is pointing in the ***radial direction*** *and the vector $(-\sin\theta_0,\cos\theta_0)$ is pointing in the* ***angular direction****.*

Keep going and use (53) to get

$$\begin{aligned}\underbrace{\left(r\frac{\partial}{\partial r}\right)^2}U &:= \left(r\frac{\partial}{\partial r}\right)\left[\left(r\frac{\partial}{\partial r}\right)U\right]=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]\\ &= x\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]+y\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]\\ &= \underbrace{x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}}.\end{aligned} \tag{58}$$

Similarly

$$\begin{aligned}\underbrace{\left(\frac{\partial}{\partial\theta}\right)^2}U &:= \frac{\partial}{\partial\theta}\left(\frac{\partial U}{\partial\theta}\right)=-y\frac{\partial}{\partial x}\left[-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}\right]+x\frac{\partial}{\partial y}\left[-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}\right]\\ &= \underbrace{y^2\frac{\partial^2 u}{\partial x^2}+x^2\frac{\partial^2 u}{\partial y^2}-2xy\frac{\partial^2 u}{\partial x\partial y}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}}.\end{aligned} \tag{59}$$

Add (58) and (59) to get the beautiful identity:

$$\underbrace{\left(r\frac{\partial}{\partial r}\right)^2U+\left(\frac{\partial}{\partial\theta}\right)^2U=\left(x^2+y^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)}. \tag{60}$$

Finally, one can simplify $\left(r\frac{\partial}{\partial r}\right)^2 u$ as

$$\left(r\frac{\partial}{\partial r}\right)^2U=\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial U}{\partial r}\right)=r^2\frac{\partial^2 U}{\partial r^2}+r\frac{\partial U}{\partial r} \tag{61}$$

and conclude the identity:

Lemma 1.21 *(**Laplace operator in polar coordinates** (r,θ) **of** $\mathbb{R}^2$.) For any C^2 function $u(x,y) = u(r\cos\theta, r\sin\theta) = U(r,\theta)$ defined on $\mathbb{R}^2$, then on the two open sets (48), we have the identity*

$$(x^2+y^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = \left(r\frac{\partial}{\partial r}\right)^2 U + \left(\frac{\partial}{\partial \theta}\right)^2 U, \tag{62}$$

which is the same as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \underbrace{\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}}, \quad r>0, \quad 0<\theta<2\pi. \tag{63}$$

Remark 1.22 *In particular, if $u(x,y) = U(r)$ is a radial function, (63) becomes*

$$U''(r) + \frac{1}{r}U'(r) = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}, \quad r>0, \tag{64}$$

which matches with (41).

Example 1.23 *(**Radial harmonic function on** $\mathbb{R}^2\setminus\{(0,0)\}$.) If a function $u(x,y) = U(r)$, $r = \sqrt{x^2+y^2}$, is **radial**, then by (62) we have*

$$\left(r\frac{\partial}{\partial r}\right)^2 U = (x^2+y^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{65}$$

*Thus a **radial harmonic function** $u(r)$ (defined on $\mathbb{R}^2\setminus\{(0,0)\}$) satisfies*

$$\left(r\frac{\partial}{\partial r}\right)^2 U = r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) = 0,$$

i.e.,

$$r\frac{\partial U}{\partial r} = const.$$

Hence

$$u(x,y) = U(r) = a\ln r + b = a\log\sqrt{x^2+y^2} + b, \quad (x,y)\neq(0,0). \tag{66}$$

for some constants a, b. Note that $u(x,y)$ is defined only on $\mathbb{R}^2\setminus\{(0,0)\}$ and

$$\lim_{(x,y)\to(0,0)} u(x,y) = \infty \quad (\text{if } a>0).$$

*Its **gradient vector is pointing in the radial direction**, given by*

$$\nabla u(x,y) = \left(\frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right) = a\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), \quad (x,y)\neq(0,0). \tag{67}$$

Example 1.24 *(**Angular harmonic function on** $\mathbb{R}^2\setminus\{(x,y): x\geq 0\}$.) If a function $u(x,y) = U(\theta)$ depends only on angle $\theta\in(0,2\pi)$, then by (62) we have*

$$\left(\frac{\partial}{\partial \theta}\right)^2 U = (x^2+y^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{68}$$

*Thus an **angular harmonic function** $U(\theta)$ (defined on $\tilde{\mathbb{R}}^2 = \mathbb{R}^2\setminus\{(x,y): x\geq 0\}$) satisfies $U''(\theta) = 0$, i.e.*

$$u(x,y) = U(\theta) = c\theta + d = c\tan^{-1}\frac{y}{x} + d \ (\text{if } x\neq 0 \text{ and } (x,y) \text{ is in the first quadrant}) \tag{69}$$

for some constants c, d. Its ***gradient vector is perpendicular to the radial direction****, given by*

$$\nabla u(x,y) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = c\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad (x,y) \neq (0,0). \tag{70}$$

Since the derivative of a harmonic function is still harmonic, the functions

$$\frac{x}{x^2+y^2}, \qquad \frac{y}{x^2+y^2}$$

are both harmonic in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Example 1.25 *(**Two important harmonic functions.**) By (63), one can check that* ***for any*** $n \in \mathbb{Z}$ *the two functions* $r^n \cos n\theta$, $r^n \sin n\theta$ *are both* ***harmonic functions*** *defined on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *(not just on* $\mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$*). For* $n \in \mathbb{N}$, *the functions are actually defined on the whole plane* $\mathbb{R}^2$. *The corresponding functions* $u(x,y)$ *are polynomials in the variables* x *and* y *with degree* n, *and are defined on the whole* $\mathbb{R}^2$. *They are called* ***harmonic polynomials*** *on* $\mathbb{R}^2$. *For example, when* $n=1$, *we get* $r\cos\theta = x$, $r\sin\theta = y$ *and for* $n=2$, *we get*

$$r^2\cos 2\theta = x^2 - y^2, \qquad r^2 \sin 2\theta = 2xy,$$

, etc. For $n=-1$, *we get the familiar ones:*

$$r^{-1}\cos(-\theta) = \frac{r\cos\theta}{r^2} = \frac{x}{x^2+y^2}, \qquad r^{-1}\sin(-\theta) = -\frac{r\sin\theta}{r^2} = \frac{-y}{x^2+y^2}.$$

1.4.1 Using a different method to derive Laplace equation in polar coordinates (r,θ).

This method is based on the following important observation (which is a problem in HW 9) and can be easily generalized to the case $u(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$.

Lemma 1.26 *Let* $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^2 *function and let* $\{v,w\} \subset \mathbb{R}^2$ *be* ***any*** *orthonormal basis. Then at any fixed point* $p \in \mathbb{R}^2$ *we have the identity*

$$(\triangle u)(p) = \frac{\partial^2 u}{\partial x^2}(p) + \frac{\partial^2 u}{\partial y^2}(p) = \left.\frac{d^2}{dt^2}\right|_{t=0} u(p+tv) + \left.\frac{d^2}{dt^2}\right|_{t=0} u(p+tw), \quad t \in (-\varepsilon, \varepsilon). \tag{71}$$

Remark 1.27 *The same result holds for a* C^2 *function* $u : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$.

We can use Lemma 1.26 to express the Laplace operator $\triangle$ in terms of the polar coordinates (r,θ). Let $p = (x,y) \in \mathbb{R}^2$ be a fixed point with polar coordinates (r,θ). We can take the orthonormal basis $\{v,w\}$ at p as

$$v = (\cos\theta, \sin\theta), \qquad w = (-\sin\theta, \cos\theta),$$

where we see that the vector v is pointing in the radial direction and the vector w is pointing in the perpendicular direction. Recall that we denote $u(r\cos\theta, r\sin\theta)$ by $U(r,\theta)$. Now we have

$$u(p+tv) = u(r\cos\theta + t\cos\theta,\ r\sin\theta + t\sin\theta) = u((r+t)\cos\theta, (r+t)\sin\theta) = U(r+t,\theta), \tag{72}$$

which gives

$$\left.\frac{d^2}{dt^2}\right|_{t=0} u(p+tv) = \left.\frac{d^2}{dt^2}\right|_{t=0} U(r+t,\theta) = \underbrace{\frac{\partial^2 U}{\partial r^2}(r,\theta)}. \tag{73}$$

Next, we have

$$u(p+tw) = u(r\cos\theta - t\sin\theta,\ r\sin\theta + t\cos\theta),$$

where the vector $p+tw$ has length $\sqrt{r^2+t^2}$ with

$$\begin{aligned}
&(r\cos\theta - t\sin\theta,\ r\sin\theta + t\cos\theta)\\
&= \sqrt{r^2+t^2}\left(\frac{r}{\sqrt{r^2+t^2}}\cos\theta - \frac{t}{\sqrt{r^2+t^2}}\sin\theta,\ \frac{r}{\sqrt{r^2+t^2}}\sin\theta + \frac{t}{\sqrt{r^2+t^2}}\cos\theta\right)\\
&= \sqrt{r^2+t^2}\left(\cos\left(\theta+\theta(t)\right),\ \sin\left(\theta+\theta(t)\right)\right),
\end{aligned} \tag{74}$$

for some angle $\theta(t)$ satisfying

$$\cos\theta(t) = \frac{r}{\sqrt{r^2+t^2}}, \quad \sin\theta(t) = \frac{t}{\sqrt{r^2+t^2}}, \quad \theta(0)=0. \tag{75}$$

In particular, we get

$$\theta'(t) = \frac{r}{r^2+t^2}, \quad \theta''(t) = -\frac{2rt}{(r^2+t^2)^2}, \quad \theta'(0) = \frac{1}{r}, \quad \theta''(0)=0. \tag{76}$$

Now by (74), we have

$$u(r\cos\theta - t\sin\theta,\ r\sin\theta + t\cos\theta) = U\left(\sqrt{r^2+t^2}, \theta+\theta(t)\right)$$

and then by (76) we conclude

$$\begin{aligned}
&\left.\frac{d^2}{dt^2}\right|_{t=0} u(p+tw) = \left.\frac{d^2}{dt^2}\right|_{t=0} U\left(\sqrt{r^2+t^2}, \theta+\theta(t)\right)\\
&= \frac{d}{dt}\left[\frac{\partial U}{\partial r}\left(\sqrt{r^2+t^2}, \theta+\theta(t)\right)\cdot\frac{t}{\sqrt{r^2+t^2}} + \frac{\partial U}{\partial \theta}\left(\sqrt{r^2+t^2}, \theta+\theta(t)\right)\cdot\theta'(t)\right]\Bigg|_{t=0}\\
&= \frac{\partial U}{\partial r}(r,\theta)\cdot\frac{1}{r} + \frac{\partial^2 U}{\partial \theta^2}(r,\theta)\cdot\left(\theta'(0)\right)^2 = \underbrace{\frac{1}{r}\frac{\partial U}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}(r,\theta)}.
\end{aligned} \tag{77}$$

The identity (63) follows due to (73) and (77).

Remark 1.28 ***(Important.)*** *In summary, we have (here $p=(x,y)$ has polar coordinates (r,θ))*

$$\begin{cases} u(p+tv) = U(r+t,\theta), & v = (\cos\theta,\sin\theta)\ \textit{(radial direction)},\\ u(p+tw) = U\left(\sqrt{r^2+t^2}, \theta+\theta(t)\right), & \theta(0)=0, \quad w=(-\sin\theta,\cos\theta)\ \textit{(angular direction)}, \end{cases}$$

and then differentiate each with respect to t twice and apply the chain rule.

1.5 Laplace equation is invariant under radial inversion in $\mathbb{R}^2$.

Lemma 1.29 ***(Laplace equation is invariant under radial inversion on*** $\mathbb{R}^2$.*) Assume $u(x,y)$ is* ***harmonic*** *on $\mathbb{R}^2$ and let $U(r,\theta) = u(r\cos\theta, r\sin\theta)$, $(r,\theta)\in(0,\infty)\times(0,2\pi)$. Then the* ***radial inversion*** *function (i.e. $r\to\frac{1}{r}$)*

$$\tilde{U}(r,\theta) = U\left(\frac{1}{r},\theta\right), \quad (r,\theta)\in(0,\infty)\times(0,2\pi) \tag{78}$$

also satisfies the equation

$$\frac{\partial^2\tilde{U}}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial\tilde{U}}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2\tilde{U}}{\partial\theta^2}(r,\theta) = 0 \tag{79}$$

on $(r,\theta)\in(0,\infty)\times(0,2\pi)$. *Therefore, if* $u(x,y)$ *is* ***harmonic*** *on* $\mathbb{R}^2$, *then the function*

$$\tilde{u}(x,y)=u\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right) \tag{80}$$

is also harmonic on $\mathbb{R}^2\backslash\{(0,0)\}$. *Note that the length of the vector* $\left(x/\left(x^2+y^2\right),y/\left(x^2+y^2\right)\right)$ *is equal to* $1/\sqrt{x^2+y^2}$.

Remark 1.30 *Lemma 1.29 is a special case of the* ***Kelvin transformation*** *in* $\mathbb{R}^n$.

Proof. Assume (x,y) is **harmonic** on $\mathbb{R}^2$ and let $U(r,\theta)=u(r\cos\theta,r\sin\theta)$. It satisfies

$$\frac{\partial^2U}{\partial r^2}(r,\theta)+\frac{1}{r}\frac{\partial U}{\partial r}(r,\theta)+\frac{1}{r^2}\frac{\partial^2U}{\partial\theta^2}(r,\theta)=0 \tag{81}$$

on the domain $(r,\theta)\in(0,\infty)\times(0,2\pi)$. We compute (write $U(r,\theta)$ as $U(s,\theta)$ to avoid confusion)

$$\begin{aligned}
\tilde{U}(r,\theta)&=U(s,\theta),\quad s=\frac{1}{r}\\
\tilde{U}_r(r,\theta)&=U_s(s,\theta)\left(\frac{-1}{r^2}\right)=-s^2U_s(s,\theta)\\
\tilde{U}_{rr}(r,\theta)&=U_{ss}(s,\theta)\left(\frac{1}{r^4}\right)+U_s(s,\theta)\left(\frac{2}{r^3}\right)=s^4U_{ss}(s,\theta)+2s^3U_s(s,\theta)\\
\tilde{U}_{\theta\theta}(r,\theta)&=U_{\theta\theta}(s,\theta)
\end{aligned}$$

and get

$$\begin{aligned}
&\frac{\partial^2\tilde{U}}{\partial r^2}(r,\theta)+\frac{1}{r}\frac{\partial\tilde{U}}{\partial r}(r,\theta)+\frac{1}{r^2}\frac{\partial^2\tilde{U}}{\partial\theta^2}(r,\theta)\\
&=\left[s^4U_{ss}(s,\theta)+2s^3U_s(s,\theta)\right]+s\left(-s^2U_s(s,\theta)\right)+s^2U_{\theta\theta}(s,\theta)\\
&=s^4\left\{\underbrace{U_{ss}(s,\theta)+\frac{1}{s}U_s(s,\theta)+\frac{1}{s^2}U_{\theta\theta}(s,\theta)}\right\}=0.
\end{aligned} \tag{82}$$

The proof is done. □

Example 1.31 *We know* $x^2-y^2=r^2\cos2\theta$ *and* $2xy=r^2\sin2\theta$ *are harmonic on* $\mathbb{R}^2$. *By (80) in Lemma 1.29, the functions*

$$\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=\frac{1}{r^2}\cos2\theta,\qquad\frac{2xy}{\left(x^2+y^2\right)^2}=\frac{1}{r^2}\sin2\theta$$

are also harmonic on $\mathbb{R}^2\backslash\{(0,0)\}$. *Similarly for the harmonic functions* $r^n\cos n\theta,\ r^n\sin n\theta$, *for any* $n\in\mathbb{Z}$.

1.6 Laplace operator in spherical coordinates (r,θ,φ) of $\mathbb{R}^3$.

Remark 1.32 *There are several different methods to derive the Laplace operator in spherical coordinates* (r,θ,φ) *of* $\mathbb{R}^3$. *Here we only provide the most straightforward method. For more discussions and details, see the file "****Laplace-equation-in-polar-and-spherical-coordinates-2019-4-28.tex****".*

The sphere coordinates in $\mathbb{R}^3$ is given by (r,θ,φ) and its relation with respect to the Euclidean coordinates is

$$x=r\sin\varphi\cos\theta,\quad y=r\sin\varphi\sin\theta,\quad z=r\cos\varphi,\quad x^2+y^2=r^2\sin^2\varphi, \tag{83}$$

where $r>0,\ \theta\in(0,2\pi),\ \varphi\in(0,\pi)$. By (83), we get the inverse identities

$$r=\sqrt{x^2+y^2+z^2},\qquad \theta=\tan^{-1}\frac{y}{x},\qquad \varphi=\cos^{-1}\frac{z}{r}=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right). \tag{84}$$

We have

$$U(r,\theta,\varphi)=u(r\sin\varphi\cos\theta,r\sin\varphi\sin\theta,r\cos\varphi) \tag{85}$$

and

$$\begin{cases} r=\sqrt{x^2+y^2+z^2}, \\ \frac{\partial r}{\partial x}=\frac{x}{r}=\sin\varphi\cos\theta, \\ \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\varphi\sin\theta \\ \frac{\partial r}{\partial z}=\frac{z}{r}=\cos\varphi, \end{cases} \tag{86}$$

and

$$\begin{cases} \theta=\tan^{-1}\frac{y}{x}, \\ \frac{\partial\theta}{\partial x}=\frac{-y}{x^2+y^2}=-\frac{r\sin\varphi\sin\theta}{r^2\sin^2\varphi}=-\frac{\sin\theta}{r\sin\varphi}, \\ \frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{r\sin\varphi\cos\theta}{r^2\sin^2\varphi}=\frac{\cos\theta}{r\sin\varphi}, \\ \frac{\partial\theta}{\partial z}=0, \end{cases} \tag{87}$$

and

$$\begin{cases} \varphi=\cos^{-1}\frac{z}{r}, \\ \frac{\partial\varphi}{\partial x}=\frac{-1}{\sqrt{1-\left(\frac{z}{r}\right)^2}}\frac{-z}{r^2}\frac{\partial r}{\partial x}=\frac{z}{r\sqrt{r^2-z^2}}\frac{\partial r}{\partial x}=\frac{r\cos\varphi}{r\sqrt{r^2\sin^2\varphi}}\sin\varphi\cos\theta=\frac{\cos\varphi\cos\theta}{r} \\ \frac{\partial\varphi}{\partial y}=\frac{r\cos\varphi}{r\sqrt{r^2\sin^2\varphi}}\sin\varphi\sin\theta=\frac{\cos\varphi\sin\theta}{r} \\ \frac{\partial\varphi}{\partial z}=\frac{-1}{\sqrt{1-\left(\frac{z}{r}\right)^2}}\left(\frac{1}{r}-\frac{z}{r^2}\frac{\partial r}{\partial z}\right)=\frac{-1}{\sqrt{r^2-z^2}}\left(1-\cos^2\varphi\right)=-\frac{\sin^2\varphi}{\sqrt{r^2\sin^2\varphi}}=-\frac{\sin\varphi}{r}. \end{cases} \tag{88}$$

With the above, we conclude the **first order operator relation:**

$$\begin{cases} \frac{\partial}{\partial x}=(\sin\varphi\cos\theta)\frac{\partial}{\partial r}-\frac{\sin\theta}{r\sin\varphi}\frac{\partial}{\partial\theta}+\frac{\cos\varphi\cos\theta}{r}\frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial y}=(\sin\varphi\sin\theta)\frac{\partial}{\partial r}+\frac{\cos\theta}{r\sin\varphi}\frac{\partial}{\partial\theta}+\frac{\cos\varphi\sin\theta}{r}\frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial z}=(\cos\varphi)\frac{\partial}{\partial r}-\frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}. \end{cases} \tag{89}$$

By (89), we have the following gradient vector relation similar to (57):

$$\begin{aligned} \nabla u(x,y,z)&=\underbrace{\frac{\partial u}{\partial x}(x,y,z)\,\mathbf{e}_1+\frac{\partial u}{\partial y}(x,y,z)\,\mathbf{e}_2+\frac{\partial u}{\partial z}(x,y,z)\,\mathbf{e}_3} \\ &=\frac{\partial U}{\partial r}\begin{pmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{pmatrix}+\frac{1}{r\sin\varphi}\frac{\partial U}{\partial\theta}\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}+\frac{1}{r}\frac{\partial U}{\partial\varphi}\begin{pmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{pmatrix} \\ &=\underbrace{\frac{\partial U}{\partial r}\mathbf{e}_r+\frac{1}{r\sin\varphi}\frac{\partial U}{\partial\theta}\mathbf{e}_\theta+\frac{1}{r}\frac{\partial U}{\partial\varphi}\mathbf{e}_\varphi}, \end{aligned} \tag{90}$$

where $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\varphi\}$ is an **orthonormal basis** given by

$$\begin{cases} \mathbf{e}_r=(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi) \\ \mathbf{e}_\theta=(-\sin\theta,\cos\theta,0) \\ \mathbf{e}_\varphi=(\cos\varphi\cos\theta,\cos\varphi\sin\theta,-\sin\varphi). \end{cases} \tag{91}$$

Theorem 1.33 *(Laplace operator in spherical coordinates* (r,θ,φ) ***of*** $\mathbb{R}^3$*.) Under the* ***spherical coordinates*** (r,θ,φ) *in* $\mathbb{R}^3$*, we have the identity:*

$$\left[\left(r\frac{\partial}{\partial r}\right)^2 U+\left(r\frac{\partial}{\partial r}\right)U\right]+\frac{1}{\sin^2\varphi}\left[\left(\frac{\partial}{\partial\theta}\right)^2 U+\left((\sin\varphi)\frac{\partial}{\partial\varphi}\right)^2 U\right]$$
$$= \left(x^2+y^2+z^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=r^2\triangle u\left(x,y,z\right), \tag{92}$$

which is the same as

$$\triangle u\left(x,y,z\right)=\left(\frac{\partial^2 U}{\partial r^2}+\frac{2}{r}\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2 U}{\partial\theta^2}+\frac{1}{r^2}\left(\frac{\partial^2 U}{\partial\varphi^2}+\frac{\cos\varphi}{\sin\varphi}\frac{\partial U}{\partial\varphi}\right). \tag{93}$$

Proof. We omit its proof. It is a tedious computation. □

If $u\left(x,y,z\right)=U\left(r\right)$ is a radial function only, then **a radial harmonic function** $U\left(r\right)$ (defined on $\mathbb{R}^3\backslash\left\{(0,0,0)\right\}$) satisfies

$$U''\left(r\right)+\frac{2}{r}U'\left(r\right)=0\quad\text{on}\quad(0,\infty). \tag{94}$$

Its solution is given by

$$U\left(r\right)=\frac{a}{r}+b,\quad r\in(0,\infty),\quad a,\ b\text{ are constants} \tag{95}$$

or

$$u\left(x,y,z\right)=\frac{a}{\sqrt{x^2+y^2+z^2}}+b,\quad (x,y,z)\in\mathbb{R}^3\backslash\left\{(0,0,0)\right\}, \tag{96}$$

which has a singularity at the origin $O=(0,0,0)\in\mathbb{R}^3$ with

$$\lim_{(x,y,z)\to(0,0,0)}u\left(x,y,z\right)=\infty\quad(\text{if } a>0). \tag{97}$$

On the other hand, if $u\left(x,y,z\right)=U\left(\theta\right)$, i.e. it depends only on the angle θ, we have $U''\left(\theta\right)=0$, $U\left(\theta\right)=c\theta+d$ for some constants $c,\ d$. This gives the **θ-angular harmonic function:** (here we use the representation $\theta=\tan^{-1}\frac{y}{x}$)

$$u\left(x,y,z\right)=c\tan^{-1}\frac{y}{x}+d\quad(\text{if } (x,y,z) \text{ lies in the first octant of } \mathbb{R}^3).$$

The domain of $u\left(x,y,z\right)=U\left(\theta\right)$ is $\mathbb{R}^3\backslash\left\{(x,0,z):x\geq 0,\ z\in\mathbb{R}\right\}$ (same as $\theta\in(0,2\pi)$). Note that the set $\left\{(x,0,z):x\geq 0,\ z\in\mathbb{R}\right\}$ has **measure zero** in $\mathbb{R}^3$. Any point p on this measure zero set has angle $\theta=0$ or $\theta=2\pi$.

Finally, if $u\left(x,y,z\right)=U\left(\varphi\right)$, i.e. it depends only on the angle φ, by solving the ODE

$$\left((\sin\varphi)\frac{\partial}{\partial\varphi}\right)^2 U\left(\varphi\right)=0,$$

we have $(\sin\varphi)U'\left(\varphi\right)=c$ for some constant c and then

$$U\left(\varphi\right)=c\int\frac{1}{\sin\varphi}d\varphi+d=c\int\csc\varphi d\varphi+d=c\log\left|\csc\varphi-\cot\varphi\right|+d, \tag{98}$$

which gives the **φ-angular harmonic function:** (here we use the representation $\varphi=\cos^{-1}\frac{z}{r}$)

$$u\left(x,y,z\right)=c\log\left|\csc\varphi-\cot\varphi\right|+d$$
$$=c\log\left|\csc\left(\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}\right)-\cot\left(\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right|+d, \tag{99}$$

where $c,\ d$ are constants. The domain of $u\left(x,y,z\right)=U\left(\varphi\right)$ is $\mathbb{R}^3$ minus the z-axis (same as $\varphi\in(0,\pi)$). Any point $p=(0,0,z)$ on z-axis has angle $\varphi=0$ or $\varphi=\pi$ and $\log\left|\csc\varphi-\cot\varphi\right|$ is undefined at $\varphi=0$ and $\varphi=\pi$.

1.7 The application of Green identities to Dirichlet problem for Poisson equation on bounded domains.

When $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, the most important question related to the Laplace operator is the Dirichlet problem for Poisson equation on Ω. Let $\Omega \subseteq \mathbb{R}^2$ ($\Omega \subseteq \mathbb{R}^n$ is also OK ..) be a bounded C^1 domain and let $f(x)$, $h(x)$ be continuous functions on Ω and $\partial\Omega$ respectively. The **Dirichlet problem for Poisson equation** has the form

$$\begin{cases} \triangle u(x) = f(x) \quad \text{in} \quad \Omega \subseteq \mathbb{R}^2 \\ u(x) = h(x) \quad \text{on} \quad \partial\Omega. \end{cases} \tag{100}$$

One can use (35) to show that (100) has a **unique** solution (we will not discuss the **existence** of a solution here). In PDE theory, the boundary condition $u(x) = h(x)$ on $\partial\Omega$ is also called **Dirichlet condition**.

Lemma 1.34 *(**Uniqueness of solution for Dirichlet problem of Poisson equation.**) Let $u,\ v \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$ be two solutions of (100) on Ω, where $f \in C^0(\Omega)$ and $h \in C^1(\partial\Omega)$ are given. Then we must have $u \equiv v$ on $\bar{\Omega}$.*

Proof. Set $w = u - v \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$. It satisfies

$$\begin{cases} \triangle w(x) = 0 \quad \text{in} \quad \Omega \subseteq \mathbb{R}^2 \\ w(x) = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{101}$$

By (35), we have the identity

$$\int_\Omega w \triangle w dx + \int_\Omega |\nabla w|^2 dx = \int_{\partial\Omega} w \frac{\partial w}{\partial \mathbf{N}} d\sigma \ (d\sigma = ds \text{ here since we are in } \mathbb{R}^2), \tag{102}$$

which, together with (101), gives the identity

$$\int_\Omega |\nabla w|^2 dx = 0, \tag{103}$$

where we also know that $|\nabla w|^2$ is a continuous function on Ω with $|\nabla w|^2 \geq 0$ everywhere. Hence we conclude $|\nabla w|^2 \equiv 0$ on Ω and $w(x)$ must be a constant function on Ω. As $w(x) = 0$ on $\partial\Omega$, we must have $w(x) \equiv 0$ on $\bar{\Omega}$. The proof is done. □

Remark 1.35 *(**Important.**) The above uniqueness result does not hold on **unbounded domains**. This is because the divergence theorem and the Green identities **are valid only on bounded domains** $\Omega \subset \mathbb{R}^n$. See Remark 1.6. For example, the two functions*

$$w(x,y) \equiv 0, \quad v(x,y) = y, \quad (x,y) \in \mathbb{R}^2_{y+} = \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\}$$

all satisfy the equation $\triangle u = 0$ on $\mathbb{R}^2_{y+}$ with $u \equiv 0$ on $\partial\mathbb{R}^2_{y+}$.

1.8 The weak maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.

From now on, Ω will denote a **bounded domain** in $\mathbb{R}^2$ or $\mathbb{R}^3$. However, the maximum/minimum principle for harmonic functions on any bounded domain $\Omega \subset \mathbb{R}^n$ is also valid.

Lemma 1.36 *(**Weak maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.**) Let Ω be a **bounded domain** in $\mathbb{R}^2$ or $\mathbb{R}^3$. Assume $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$. If $\triangle u \geq 0$ (≤ 0) everywhere in Ω, then*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u \quad \left(\min_{\bar{\Omega}} u = \min_{\partial\Omega} u\right). \tag{104}$$

*Consequently for a **harmonic** function $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$, we have*

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u, \quad \forall\, x \in \bar{\Omega}. \tag{105}$$

*In particular, for a **harmonic** function $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$, we have*

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{106}$$

Remark 1.37 *The condition that Ω is a **bounded** domain is **essential** in the above lemma. Lemma 1.36 is **false** on unbounded domains. For example, take $u(x,y) = y$ on $\mathbb{R}^2_{y+}$. It does not satisfy the maximum principle. Another interesting example is the **nonzero harmonic function** $u(x,y) = \sinh x \cos y$ on $\Omega \subset \mathbb{R}^2$ with $u \equiv 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^2$ is the **unbounded semi-infinite strip***

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 : x > 0,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}. \tag{107}$$

Remark 1.38 *By the above lemma, we conclude that, if $\triangle u \geq 0$ in Ω, it satisfies the **maximum principle**; and if $\triangle u \leq 0$ in Ω, it satisfies the **minimum principle.** One can also write the maximum principle as*

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u \tag{108}$$

and write the minimum principle as

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u. \tag{109}$$

Remark 1.39 *(**An interesting simple example.**) Here is a simple example in the 1-dimensional case. Let $f(x) = x^2,\ x \in (-1,1)$. It satisfies $f''(x) = 2 > 0$ everywhere in $\Omega = (-1,1) \subset \mathbb{R}$. The maximum value of $f(x)$ on $\bar{\Omega}$ is 1, attained at $x = \pm 1 \in \partial\Omega$. Similarly, the function $g(x) = -x^2,\ x \in (-1,1)$ satisfies $g''(x) = -2 < 0$ everywhere in $\Omega = (-1,1) \subset \mathbb{R}$. The minimum value of $g(x)$ on $\bar{\Omega}$ is -1, attained at $x = \pm 1 \in \partial\Omega$. Finally, the function $h(x) = ax + b$ (a, b are any two numbers) is **harmonic** on $\Omega = (-1,1)$. Its maximum value and minimum value attained at $x = \pm 1 \in \partial\Omega$ respectively. Unless $h(x)$ is a **constant** function, otherwise, it is impossible for the harmonic function $h(x)$ to attain its maximum value (or minimum value) at some $x_0 \in \Omega = (-1,1)$ (this is the **strong maximum/minimum principle**, to be proved later on).*

Proof. Assume first that $\triangle u > 0$ everywhere in Ω. Then since u is continuous on $\bar{\Omega}$ (compact set), there is some point $p \in \bar{\Omega}$ such that $u(p) = \max_{\bar{\Omega}} u$. If $p \in \partial\Omega$, the result follows. If $p \in \Omega$ (interior point), we get

$$\triangle u(p) = \frac{\partial^2 u}{\partial x^2}(p) + \frac{\partial^2 u}{\partial y^2}(p) \leq 0, \tag{110}$$

a contradiction. Hence p must lie on the boundary of Ω and so $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$ (note that we always have $\max_{\bar{\Omega}} u \geq \max_{\partial\Omega} u$).

Next, assume that $\triangle u \geq 0$ everywhere in Ω. We can use a small perturbation argument. Let

$$v(x,y) = u(x,y) + \varepsilon\left(x^2 + y^2\right), \quad (x,y) \in \bar{\Omega},$$

where $\varepsilon > 0$ is a constant. We have

$$\triangle v(x,y) = \triangle u(x,y) + 4\varepsilon \geq 0 + 4\varepsilon > 0 \quad \text{everywhere in} \quad \Omega.$$

Hence we have

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} v = \max_{\partial\Omega} v \leq \left(\max_{\partial\Omega} u\right) + \varepsilon \max_{\partial\Omega} \left(x^2 + y^2\right). \tag{111}$$

As $\varepsilon > 0$ is arbitrary and Ω is a **bounded** domain (hence $\max_{\partial\Omega} (x^2 + y^2)$ is finite), letting $\varepsilon \to 0^+$ in (111), we conclude

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u.$$

On the other hand, we also have $\max_{\bar{\Omega}} u \geq \max_{\partial\Omega} u$ and so (104) is verified. The proof of the minimum case is similar.

Finally, it is easy to see that for a **harmonic** function $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$, we have (106). Note that we have the inequality

$$-|u(x)| \leq u(x) \leq |u(x)|, \quad \forall\, x \in \bar{\Omega}$$

and so

$$\begin{cases} \max_{\bar{\Omega}} u = \max_{\partial\Omega} u \leq \max_{\partial\Omega} |u| \\ \min_{\bar{\Omega}} u = \min_{\partial\Omega} u \geq \min_{\partial\Omega} (-|u|) = -\max_{\partial\Omega} |u|. \end{cases}$$

The above implies

$$\underbrace{-\max_{\partial\Omega} |u|} \leq \min_{\bar{\Omega}} u \leq u(x) \leq \max_{\bar{\Omega}} u \leq \underbrace{\max_{\partial\Omega} |u|}, \quad \forall\, x \in \bar{\Omega},$$

which implies

$$|u(x)| \leq \underbrace{\max_{\partial\Omega} |u|}, \quad \forall\, x \in \bar{\Omega}. \tag{112}$$

By (112), we clearly have (106).

1.8.1 Application of the weak maximum/minimum principle.

Let $f \in C^0(\Omega)$ and $h \in C^0(\partial\Omega)$. We can consider the following problem on bounded domain Ω:

$$\begin{cases} \triangle u(x) = f(x) & \text{in} \quad \Omega \\ u(x) = h(x) & \text{on} \quad \partial\Omega. \end{cases} \tag{113}$$

This problem is **well-posed** and we have the following **uniqueness** property due to the maximum principle:

Lemma 1.40 ***(Uniqueness of solution for Dirichlet problem of Poisson equation.)*** *The problem (113) has at most one solution $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$.*

Remark 1.41 *Recall that we have used Green identity to prove Lemma 1.40 before for the case $u \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$. See Lemma 1.34.*

Proof. Assume there are two solutions $u,\ v \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$. Then the function $w = u - v \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$ satisfies

$$\begin{cases} \triangle w(x) = 0 & \text{in} \quad \Omega \\ w(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{114}$$

By the weak maximum/minimum principle, we have $w \equiv 0$ in Ω. Hence $u \equiv v$ in Ω (and so on $\bar{\Omega}$). □

1.9 Poisson integral formula in the plane; Dirichlet problem on a disc.

Let $B_a(0) \subset \mathbb{R}^2$ be the open disc centered at the origin $(0,0)$ with radius $a > 0$. We want to solve the **Dirichlet problem** (this is a **well-posed problem**):

$$\begin{cases} \triangle u(x,y) = 0 \quad \text{in} \quad (x,y) \in B_a(0) \subset \mathbb{R}^2 \\ u(x,y) = h(x,y) \quad \text{on} \quad (x,y) \in \partial B_a(0) \ \text{(the boundary of } B_a(0)\text{)}, \end{cases} \tag{115}$$

where $h(x,y)$ is a given **continuous** function defined on $\partial B_a(0)$.

Due to the **symmetry** of the domain and the **symmetry** of the Laplace operator, there is a **solution formula** for this problem. The solution lies in the space $C^2(B_a(0)) \bigcap C^0\left(\overline{B_a(0)}\right)$. Moreover, by the maximum principle, the solution in the function space $C^2(B_a(0)) \bigcap C^0\left(\overline{B_a(0)}\right)$ is **unique**.

As the domain $B_a(0)$ is a disc, to solve the problem (115), it is natural to use **polar coordinates** (r,θ) instead of the Euclidean coordinates (x,y). The Laplace equation under polar coordinates (r,θ) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad u = u(r,\theta), \quad 0 < r < a, \quad 0 < \theta < 2\pi. \tag{116}$$

and we we want to solve it. Since in the problem (115) the solution $u(x,y)$ is to be defined at $(x,y) = (0,0)$, we hope equation (116) can also be defined at $r = 0$ and the solution $u(r,\theta)$ **will not have a singularity at** $r = 0$ (i.e. $u(r,\theta)$ is **well-defined at** $r = 0$). In view of this, we multiply the equation by r^2 to get

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad u = u(r,\theta), \quad 0 < r < a, \quad 0 < \theta < 2\pi \tag{117}$$

and focus on equation (117). Moreover, we also want the solution $u(r,\theta)$ to be 2π**-periodic in** $\theta \in (-\infty,\infty)$ so that the corresponding solution $u(x,y)$ can be defined on the whole open disc $B_a(0)$. Using polar coordinates (r,θ), the problem (115) can be expressed as (note that since the radius of $B_a(0)$ is fixed, the continuous boundary function $h(x,y)$ on $\partial B_a(0)$ is a 2π**-periodic** continuous function depending only on $\theta \in [0,2\pi]$)

$$\begin{cases} (1).\ r^2\frac{\partial^2 u}{\partial r^2}(r,\theta) + r\frac{\partial u}{\partial r}(r,\theta) + \frac{\partial^2 u}{\partial \theta^2}(r,\theta) = 0, \quad (r,\theta) \in [0,a) \times [0,2\pi] \\ (2).\ u(a,\theta) = h(\theta), \quad \forall\, \theta \in [0,2\pi]. \end{cases} \tag{118}$$

Recall that there is a family of **separable solutions** $\{u_n(r,\theta)\}_{n=0}^{\infty}$ for equation (1) in (118), which are defined on $(r,\theta) \in [0,a) \times [0,2\pi]$, given by

$$u_n(r,\theta) = \begin{cases} r^n(A_n \cos n\theta + B_n \sin n\theta), \quad n \in \mathbb{N}, \quad (r,\theta) \in [0,a) \times [0,2\pi] \\ A_0, \quad n = 0, \end{cases} \tag{119}$$

where $A_0,\ A_n,\ B_n$ are arbitrary constants. They satisfy the boundary condition

$$u_n(a,\theta) = a^n(A_n \cos n\theta + B_n \sin n\theta), \quad \forall\, \theta \in [0,2\pi]. \tag{120}$$

Therefore, if $h(\theta)$ is a finite linear combination of $\cos n\theta$ and $\sin n\theta$ for $n \in \mathbb{N}$, then (118) can be easily solved.

Remark 1.42 *Note that each corresponding function $u_n(x,y)$ is a* ***harmonic polynomial*** *on the whole disc $B_a(0)$ (or on $\mathbb{R}^2$).*

Example 1.43 *Solve the problem on* $B_a(0)$ *:*

$$\begin{cases} (1).\ r^2\frac{\partial^2 u}{\partial r^2}(r,\theta)+r\frac{\partial u}{\partial r}(r,\theta)+\frac{\partial^2 u}{\partial \theta^2}(r,\theta)=0, & (r,\theta)\in[0,a)\times[0,2\pi] \\ (2).\ u(a,\theta)=3\sin\theta-5\cos\theta+7\cos(2\theta), & \forall\,\theta\in[0,2\pi]. \end{cases} \tag{121}$$

Solution:

The function

$$u(r,\theta)=3\left(\frac{r}{a}\right)\sin\theta-5\left(\frac{r}{a}\right)\cos\theta+7\left(\frac{r}{a}\right)^2\cos(2\theta),\quad (r,\theta)\in[0,a)\times[0,2\pi]$$

is clearly a solution of (121). In terms of (x,y)-coordinates, the corresponding $u(x,y)$ is

$$u(x,y)=3\left(\frac{y}{a}\right)-5\left(\frac{x}{a}\right)+7\left(\frac{x^2-y^2}{a^2}\right),\quad (x,y)\in B_a(0)$$

and the corresponding function $h(x,y)$ on $\partial B_a(0)$ is given by (note that $\theta=\tan^{-1}\frac{y}{x}$)

$$h(x,y)=3\frac{y}{\sqrt{x^2+y^2}}-5\frac{x}{\sqrt{x^2+y^2}}+7\left(\frac{x^2-y^2}{x^2+y^2}\right),\quad (x,y)\in\partial B_a(0).$$

□

We can use the above observation as a **motivation** to solve the general Dirichlet problem (118) for **arbitrary continuous function** $h(\theta)$ "**if one can express** $h(\theta)$ **as a linear combination of all possible** $\cos n\theta$ **and** $\sin n\theta$ **for all possible** $n\in\mathbb{N}$". This is indeed possible if we assume $h(\theta)$ is a 2π**-periodic** C^1 **function** on $\theta\in[0,2\pi]$. More precisely, we have the following result from **Fourier series theory**:

Lemma 1.44 *(**Fourier series result.**) Assume $h(\theta)$ is a 2π-**periodic** C^1 **function** defined on $\theta\in[0,2\pi]$ (or on $\theta\in\mathbb{R}$). Then the following series*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\theta)+b_n\sin(n\theta)\right],\quad \theta\in[0,2\pi] \tag{122}$$

*converges **absolutely** and **uniformly** to $h(\theta)$ on $[0,2\pi]$. That is*

$$h(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\theta)+b_n\sin(n\theta)\right],\quad \forall\,\theta\in[0,2\pi], \tag{123}$$

where

$$\begin{cases} a_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}h(\varphi)\cos(n\varphi)\,d\varphi, & n=0,\ 1,\ 2,\ 3,\ ..., \\ b_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}h(\varphi)\sin(n\varphi)\,d\varphi, & n=1,\ 2,\ 3,\ .... \end{cases} \tag{124}$$

*The series (122) is called the **Fourier series** of the function h on $\theta\in[0,2\pi]$. We can also rewrite (123) as*

$$h(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{1}{\pi}\int_0^{2\pi}\underbrace{h(\varphi)\cos(n\varphi-n\theta)}\,d\varphi,\quad \theta\in[0,2\pi]. \tag{125}$$

Remark 1.45 *The above lemma* ***fails*** *if we only assume $h(\theta)$ is a 2π-periodic* ***continuous*** *function.*

Proof. Omit. $\square$

We now assume that $h(\theta)$ is a 2π-periodic C^1 **function** and see what we can do (in the problem (118) we only assume h to be a continuous function). Motivated by the Fourier series, we now consider the **sum** of $u_n(r,\theta)$ from (119) for all $n = 0,\ 1,\ 2,\ 3,\ ...$, and get a series of the form

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta\right), \quad (r,\theta) \in [0,a) \times [0,2\pi], \tag{126}$$

where the constants $A_0,\ A_n,\ B_n$ will be chosen so that when $r = a$ it reduces to the Fourier series of $h(\theta)$ on $\theta \in [0,2\pi]$.

Remark 1.46 *Since $A_0/2$ and each $r^n(A_n \cos n\theta + B_n \sin n\theta)$ are harmonic on $(r,\theta) \in [0,a) \times [0,2\pi]$, we "**expect**" that the series (126) to converge to a* ***harmonic function*** *on $[0,a) \times [0,2\pi]$. Moreover, if we choose the coefficients $A_0,\ A_n$ and B_n suitably, then the sum of the series (i.e. the function $u(r,\theta)$ in (126)) will tend to $h(\theta)$ as $r \to a$ and the Dirichlet problem (118) can be solved. See below for details.*

To satisfy the boundary condition at $r = a$, we need to require

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} a^n \left(A_n \cos n\theta + B_n \sin n\theta\right) = h(\theta), \quad \forall\ \theta \in [0,2\pi]. \tag{127}$$

Hence, by (124), we must require

$$\begin{cases} A_0 = \dfrac{1}{\pi} \int_0^{2\pi} h(\varphi)\, d\varphi \\ A_n = \dfrac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \cos(n\varphi)\, d\varphi, \qquad B_n = \dfrac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \sin(n\varphi)\, d\varphi, \quad n \in \mathbb{N}. \end{cases} \tag{128}$$

Remark 1.47 *It is* ***impossible*** *to have $u(a,\theta) = h(\theta)$ for* ***arbitrary*** *$h(\theta)$ if we only consider* ***finite sum*** *in (127). This is because if we want to use $\cos n\theta$ and $\sin n\theta$ to express $h(\theta)$, we need* ***infinitely many*** *trigonometric functions.*

By the above, we arrive at a series (at least formally for now) of the form

$$\begin{cases} \dfrac{1}{2\pi} \int_0^{2\pi} h(\varphi)\, d\varphi \\ + \sum_{n=1}^{\infty} r^n \left[\left(\frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \cos(n\varphi)\, d\varphi \right) \cos n\theta + \left(\frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \sin(n\varphi)\, d\varphi \right) \sin n\theta \right] \end{cases} \tag{129}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} h(\varphi)\, d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{2\pi} \underbrace{\left(\frac{r}{a}\right)^n h(\varphi) \left[\cos n\varphi \cos n\theta + \sin n\varphi \sin n\theta\right]} d\varphi \right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} h(\varphi)\, d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{2\pi} \underbrace{\left(\frac{r}{a}\right)^n \cos(n(\theta - \varphi))} \cdot h(\varphi)\, d\varphi \right). \tag{130}$$

Now we note the following:

Lemma 1.48 *The series in (130) does converge on $(r,\theta) \in [0,a] \times [0,2\pi]$ and when $r = a,\ \theta \in [0,2\pi]$, its sum in (130) is equal to $h(\theta)$ (here we assume $h(\theta)$ is a 2π***-periodic*** *C^1* ***function*** *on $\theta \in [0,2\pi]$).*

Proof. We have the estimate

$$\left|\int_0^{2\pi}\left(\frac{r}{a}\right)^n \cos\left(n\left(\theta-\varphi\right)\right)\cdot h\left(\varphi\right)d\varphi\right|\leq\left(2\pi\max_{\varphi\in[0,2\pi]}\left|h\left(\varphi\right)\right|\right)\left(\frac{r}{a}\right)^n,\quad \forall\,\theta,\ \varphi\in[0,2\pi].$$

By comparison theory, if $r\in[0,a)$, the series in (130) will converge absolutely (since $\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n$ converges on $r\in[0,a)$). Finally, at $r=a,\ \theta\in[0,2\pi]$, the sum in (130) is equal to the **Fourier series** of $h(\theta)$ on $[0,2\pi]$ (see (129)) and so equal to $h(\theta)$. The proof is done. □

By Lemma 1.48, we can denote the sum of the series as $u(r,\theta)$, where $(r,\theta)\in[0,a]\times[0,2\pi]$ and we still need to know what $u(r,\theta)$ is when $(r,\theta)\in[0,a)\times[0,2\pi]$ and also need to know if $u(r,\theta)$ is a **harmonic function** or not on the domain $(r,\theta)\in[0,a)\times[0,2\pi]$ (same as on the domain $(x,y)\in B_a(0)\subset\mathbb{R}^2$).

To answer the above question, we note that the identity (130) gives us a **motivation** to look at the series

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos\left(n\left(\theta-\varphi\right)\right),\tag{131}$$

which is **convergent** when $(r,\theta,\varphi)\in[0,a)\times[0,2\pi]\times[0,2\pi]$ and we can find its sum explicitly. Before we go on, we note the following comparison:

1. If we assume $h(\theta)$ is a 2π-periodic C^1 **function,** then the series (130) (which has integral sign in it) **converges** at $r=a$ and it is equal to the Fourier series of $h(\theta)$.

2. On the other hand, the series (131) (which does not have integral sign in it) **diverges** at $r=a$ for any $\theta,\ \varphi$. At $r=a$, it has the form

$$\sum_{n=1}^{\infty}\cos n\left(\theta-\varphi\right)$$

   and note that $\lim_{n\to\infty}\cos n\left(\theta-\varphi\right)$ **does not converge to zero for any values of** $\theta,\ \varphi$ (recall that in Calculus, if a series $\sum_{n=1}^{\infty}a_n$ converges, we must have $\lim_{n\to\infty}a_n=0$).

To go on, motivated by (130), we study the following series properties:

Lemma 1.49 *The series*

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos\left(n\left(\theta-\varphi\right)\right)\tag{132}$$

converges ***absolutely*** *on* $(r,\theta,\varphi)\in[0,a)\times[0,2\pi]\times[0,2\pi]$ *and* ***uniformly*** *on* $(r,\theta,\varphi)\in[0,a-\varepsilon]\times[0,2\pi]\times[0,2\pi]$ *for any small* $\varepsilon>0$. *We also have*

$$\frac{\partial^k}{\partial r^k}\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos\left(n\left(\theta-\varphi\right)\right)\right)=\sum_{n=1}^{\infty}\frac{\partial^k}{\partial r^k}\left[\left(\frac{r}{a}\right)^n\cos\left(n\left(\theta-\varphi\right)\right)\right]\tag{133}$$

and

$$\frac{\partial^k}{\partial \theta^k}\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos\left(n\left(\theta-\varphi\right)\right)\right)=\sum_{n=1}^{\infty}\frac{\partial^k}{\partial \theta^k}\left[\left(\frac{r}{a}\right)^n\cos\left(n\left(\theta-\varphi\right)\right)\right]\tag{134}$$

on $(r,\theta,\varphi)\in[0,a)\times[0,2\pi]\times[0,2\pi]$ *for all* $k\in\mathbb{N}$, *which is due to the property that for each* $k\in\mathbb{N}$, *the series on the* ***right hand side*** *of (133) and (134) also converge* ***absolutely*** *on* $[0,a)\times[0,2\pi]\times[0,2\pi]$ *and* ***uniformly*** *on* $[0,a-\varepsilon]\times[0,2\pi]\times[0,2\pi]$.

Proof. The convergence result is a consequence of standard Series Theory. We omit its proof. □

By Lemma 1.49, we can **move the summation** $\sum_{n=1}^{\infty}$ **into the integral sign** and also **differentiate under the integral sign**, as long as we confine $(r,\theta) \in [0,a) \times [0,2\pi]$, i.e.

Corollary 1.50 ***(Commute the summation and the integral; differentiate under the integral; differentiate under the summation.)*** *We have*

$$u(r,\theta) = \frac{1}{2\pi}\left[\int_0^{2\pi} h(\varphi)\,d\varphi + 2\sum_{n=1}^{\infty}\int_0^{2\pi} \underbrace{\left(\frac{r}{a}\right)^n \cos(n(\theta-\varphi))}\cdot h(\varphi)\,d\varphi\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\underbrace{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\varphi))}\right)\cdot h(\varphi)\,d\varphi, \quad \forall\ (r,\theta) \in [0,a)\times[0,2\pi]. \tag{135}$$

Moreover, on the domain $(r,\theta) \in [0,a)\times[0,2\pi]$ *and for all* $k \in \mathbb{N}$, *we have the identities:*

$$\frac{\partial^k}{\partial r^k}u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left(\underbrace{\frac{\partial^k}{\partial r^k}\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\varphi))\right)}\right)h(\varphi)\,d\varphi \tag{136}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\underbrace{2\sum_{n=1}^{\infty}\frac{\partial^k}{\partial r^k}\left[\left(\frac{r}{a}\right)^n \cos(n(\theta-\varphi))\right]}\right)h(\varphi)\,d\varphi \tag{137}$$

and

$$\frac{\partial^k}{\partial \theta^k}u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left(\underbrace{\frac{\partial^k}{\partial \theta^k}\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\varphi))\right)}\right)h(\varphi)\,d\varphi \tag{138}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\underbrace{2\sum_{n=1}^{\infty}\frac{\partial^k}{\partial \theta^k}\left[\left(\frac{r}{a}\right)^n \cos(n(\theta-\varphi))\right]}\right)h(\varphi)\,d\varphi. \tag{139}$$

The lemma below says that we can simplify the series (132) and **find its sum explicitly**.

Lemma 1.51 ***(Evaluating the series.)*** *We have the identity*

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\varphi) = \frac{a^2-r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}, \quad \forall\ (r,\theta,\varphi) \in [0,a)\times[0,2\pi]\times[0,2\pi]. \tag{140}$$

In particular, by Lemma 1.49 (note that each $\left(\frac{r}{a}\right)^n \cos n(\theta-\varphi)$ *is* ***harmonic*** *on* $(r,\theta) \in [0,a) \times [0,2\pi]$ *and one can* ***differentiate under the summation sign****), we know that for any fixed* $a > 0$ *and* $\varphi \in [0,2\pi]$, *the function*

$$\frac{a^2-r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2} \tag{141}$$

is ***harmonic*** *on the domain* $(r,\theta) \in [0,a)\times[0,2\pi]$.

Proof. To prove (140), you may have to use **Euler's formula for complex number** $z = re^{i\theta}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$, which is

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta), \quad \forall\, n \in \mathbb{N}.$$

Then look at the series

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n [\cos n(\theta-\varphi) + i\sin n(\theta-\varphi)]$$

and use the identity

$$1 + 2\sum_{n=1}^{\infty} z^n = \frac{2}{1-z} - 1, \quad \forall\, z \in \mathbb{C} \text{ with } |z| < 1$$

to get

$$\begin{aligned}
&1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\varphi) = \mathrm{Re}\left[1 + 2\sum_{n=1}^{\infty} z^n\right], \quad z = \frac{r}{a}e^{i(\theta-\varphi)}, \quad r \in [0,a), \quad |z| < 1\\
&= \mathrm{Re}\left[\frac{2}{1-z} - 1\right] = \mathrm{Re}\left[\frac{2}{1-\frac{r}{a}e^{i(\theta-\varphi)}} - 1\right]\\
&= \mathrm{Re}\left[\frac{2a[a - r\cos(\theta-\varphi) + ir\sin(\theta-\varphi)]}{[a - r\cos(\theta-\varphi) - ir\sin(\theta-\varphi)][a - r\cos(\theta-\varphi) + ir\sin(\theta-\varphi)]} - 1\right]\\
&= \frac{2a[a - r\cos(\theta-\varphi)] - [a^2 - 2ar\cos(\theta-\varphi) + r^2]}{a^2 - 2ar\cos(\theta-\varphi) + r^2} = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}.
\end{aligned}$$

The proof of (140) is done. □

By Lemma 1.51 we conclude the important formula:

$$\begin{aligned}
&u(r,\theta)\\
&= \tfrac{1}{2\pi}\left[\int_0^{2\pi} h(\varphi)\,d\varphi + 2\sum_{n=1}^{\infty}\int_0^{2\pi}\left(\underbrace{\left(\tfrac{r}{a}\right)^n \cos n(\theta-\varphi)}\right) h(\varphi)\,d\varphi\right] \text{ (def. on } (r,\theta) \in [0,a] \times [0,2\pi]\text{)} \qquad (142)\\
&= \tfrac{1}{2\pi}\int_0^{2\pi}\left(\underbrace{1 + 2\sum_{n=1}^{\infty}\left(\tfrac{r}{a}\right)^n \cos n(\theta-\varphi)}\right) h(\varphi)\,d\varphi \text{ (def. on } (r,\theta) \in [0,a) \times [0,2\pi]\text{)} \qquad (143)\\
&= \tfrac{1}{2\pi}\int_0^{2\pi} \underbrace{\tfrac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}} \cdot h(\varphi)\,d\varphi \text{ (def. on } (r,\theta) \in [0,a) \times [0,2\pi]\text{)}. \qquad (144)
\end{aligned}$$

Definition 1.52 *The identity in (144) is called the* ***Poisson Integral Formula on the disc*** $B_a(0) \subset \mathbb{R}^2$.

Remark 1.53 ***(Important.)*** *Since the function* $\underbrace{\cdot}$ *in (144) is* ***harmonic*** *on* $(r,\theta) \in [0,a) \times [0,2\pi]$, *the function* $u(r,\theta)$ *in (144) will also be* ***harmonic*** *on* $(r,\theta) \in [0,a) \times [0,2\pi]$ *(one can differentiate under the integral sign).*

Note that the interchange of the summation $\sum_{n=1}^{\infty}$ and the integral $\int_0^{2\pi}$ in (143) is valid only for $r \in [0,a)$, **not** including $r = a$. In case $h(\theta)$ is a C^1 **function** on $\partial B_a(0)$, then the series in (142) **converges** at $r = a$ and the total sum in (142) is equal to $h(\theta)$ for all $\theta \in [0,2\pi]$ (see Lemma 1.48). Also, we cannot let $r = a$ in the identity (143) since the series $\sum_{n=1}^{\infty}\cos n(\theta-\varphi)$ diverges for any θ, φ.

Finally, we note that the integral in (144) is defined only for $(r,\theta)\in[0,a)\times[0,2\pi]$, and as $(r,\theta)\to(a,\theta_0)$ we have the limit

$$\lim_{(r,\theta)\to(a,\theta_0)}\left(\frac{1}{2\pi}\int_0^{2\pi}\underbrace{\frac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}}\cdot h(\varphi)\,d\varphi\right) \tag{145}$$

$$=\lim_{(r,\theta)\to(a,\theta_0)}\left(\frac{a^2-r^2}{2\pi}\cdot\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi\right)\ (0\cdot\infty\text{ form})=h(\theta_0) \tag{146}$$

as long as $h(\theta)$ is a **continuous function** on $\partial B_a(0)$. We will prove the limit in (146) in Theorem 1.56 below.

Lemma 1.54 *(**Geometric way to express the function in (140)**.) We have*

$$\frac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}=\frac{a^2-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{z}|^2},\quad (r,\theta,\varphi)\in[0,a)\times[0,2\pi]\times[0,2\pi]. \tag{147}$$

where $\mathbf{x}=(r\cos\theta,r\sin\theta)\in B_a(0)$, $\mathbf{z}=(a\cos\varphi,a\sin\varphi)\in\partial B_a(0)$. *For each fixed* $\mathbf{z}\in\partial B_a(0)$, *the function*

$$\frac{a^2-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{z}|^2},\quad \mathbf{x}\in B_a(0) \tag{148}$$

*is **harmonic** in* $\mathbf{x}\in B_a(0)$.

Remark 1.55 *For each fixed* $\mathbf{z}=(x_0,y_0)\in\partial B_a(0)$, *one can give a direct proof that the function in (148), given by*

$$\frac{a^2-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{z}|^2}=\frac{a^2-(x^2+y^2)}{(x-x_0)^2+(y-y_0)^2},\quad \mathbf{x}=(x,y)\in B_a(0), \tag{149}$$

*is **harmonic** in* $\mathbf{x}\in B_a(0)$. *This will be a homework problem for you.*

Proof. This is a simple verification. We have $|\mathbf{x}|^2=r^2$ and

$$|\mathbf{x}-\mathbf{z}|^2=(r\cos\theta-a\cos\varphi)^2+(r\sin\theta-a\sin\varphi)^2=a^2-2ar\cos(\theta-\varphi)+r^2.$$

The proof is done. □

By (147), one can also write $u(r,\theta)$ in a more **geometric way** as

$$u(\mathbf{x})=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_0^{2\pi}\frac{h(\varphi)}{|\mathbf{x}-\mathbf{z}|^2}a\,d\varphi=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds,\quad \mathbf{x}\in B_a(0), \tag{150}$$

where

$$\mathbf{x}=(r\cos\theta,r\sin\theta)\in B_a(0),\qquad \mathbf{z}=(a\cos\varphi,a\sin\varphi)\in\partial B_a(0), \tag{151}$$

and the integral on the right hand side of (150) is the **line integral with respect to arc length parameter** s **on** $\partial B_a(0)$, where we know that $ds=a\,d\varphi$.

The Poisson Integral Formula (144) is motivated by the fact that $h\in C^1$ (so that we can apply the Fourier series theory). However, to solve the Dirichlet problem (115), it suffices to assume that $h\in C^0$ (i.e. h is a continuous function).

Our main result is the following:

Theorem 1.56 *Let h be a **continuous** function on $\partial B_a(0)$ and let*

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, \quad \mathbf{x} = (x,y) \in B_a(0), \quad |\mathbf{x}| < a. \tag{152}$$

*Then $u(\mathbf{x}) = u(x,y)$ is **harmonic** in $B_a(0)$ and for each fixed $\mathbf{p} \in \partial B_a(0)$ we have the limit*

$$\lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)} u(\mathbf{x}) = h(\mathbf{p}). \tag{153}$$

To Be Continued

Remark 1.57 *(**Important.**) In terms of (r,θ), the limit (153) is the same as (assume $\mathbf{p} \in \partial B_a(0)$ has angle $\theta_0 \in [0, 2\pi]$)*

$$\begin{aligned} &\lim_{(r,\theta)\to(a,\theta_0)} u(r,\theta) \\ &= \lim_{(r,\theta)\to(a,\theta_0)} \left(\frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\varphi)}{a^2 - 2ar\cos(\theta-\varphi) + r^2} d\varphi \right) = h(\theta_0). \end{aligned} \tag{154}$$

Remark 1.58 *(**Important.**) Note that the integral*

$$\int_{|\mathbf{z}|=a} \frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds$$

diverges if $\mathbf{x} = (r\cos\theta, r\sin\theta) \in \partial B_a(0)$ (i.e. at $r = a$). In such a case we have

$$\int_{|\mathbf{z}|=a} \frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds = \int_0^{2\pi} \frac{h(\varphi)}{2a^2 - 2a^2\cos(\theta-\varphi)} a\, d\varphi = \frac{1}{2a} \int_0^{2\pi} \frac{h(\varphi)}{1 - \cos(\theta-\varphi)} d\varphi$$

and we know that the improper integral $\int_0^{2\pi} \frac{1}{1-\cos(\theta-\varphi)} d\varphi$ diverges (when φ is near θ, $1 - \cos(\theta-\varphi)$ is like $(\varphi-\theta)^2/2$). However, as $\mathbf{x} \to \mathbf{p} \in \partial B_a(0)$, the term $\left(a^2 - |\mathbf{x}|^2\right)/2\pi a$ will tend to zero. As a result of balance, we will get the limit (153).

Proof. In the integral (152), we have $|\mathbf{x}-\mathbf{z}| \neq 0$ for each $\mathbf{x} \in B_a(0)$ and $\mathbf{z} \in \partial B_a(0)$. Thus the integral is a **regular** integral (not an improper integral) and one can differentiate (with respect to x or y) under the integral sign. Thus $u(\mathbf{x}) = u(x,y)$ is **harmonic** in $B_a(0)$.

Next we note that

$$\begin{aligned} &\frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{1}{|\mathbf{x}-\mathbf{z}|^2} ds \\ &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{1}{a^2 - 2ar\cos(\theta-\varphi) + r^2} d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta-\varphi) \right) d\varphi = 1, \quad \forall\, \mathbf{x} \in B_a(0), \end{aligned} \tag{155}$$

where in the above we have used the identity

$$\int_0^{2\pi} \left(2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta-\varphi) \right) d\varphi = 2\sum_{n=1}^{\infty} \left(\left(\frac{r}{a}\right)^n \int_0^{2\pi} \cos n(\theta-\varphi)\, d\varphi \right) = 2\sum_{n=1}^{\infty} 0 = 0.$$

Hence

$$\begin{aligned} &|u(\mathbf{x}) - h(\mathbf{p})| \\ &= \left| \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{h(\mathbf{z}) - h(\mathbf{p})}{|\mathbf{x}-\mathbf{z}|^2} ds \right| \le \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{|h(\mathbf{z}) - h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2} ds. \end{aligned}$$

Since h is continuous at $\mathbf{p}$, for any $\varepsilon > 0$ there exists a small arc $C(\mathbf{p}) \subset \partial B_a(0)$ centered at $\mathbf{p}$ with length 2δ such that

$$|h(\mathbf{z}) - h(\mathbf{p})| < \varepsilon \quad \text{if} \quad \mathbf{z} \in C(\mathbf{p}).$$

Now

$$\begin{aligned} |u(\mathbf{x}) - h(\mathbf{p})| &\le \begin{cases} \frac{a^2-|\mathbf{x}|^2}{2\pi a} \int_{\mathbf{z}\in C(\mathbf{p})} \frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2} ds \\ + \frac{a^2-|\mathbf{x}|^2}{2\pi a} \int_{\mathbf{z}\in \partial B_a(0)\setminus C(\mathbf{p})} \frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2} ds \end{cases} \\ &\le \varepsilon + \underbrace{\frac{a^2-|\mathbf{x}|^2}{2\pi a} \int_{\mathbf{z}\in \partial B_a(0)\setminus C(\mathbf{p})} \frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2} ds}. \end{aligned}$$

Now if $\mathbf{x} \in B_a(0)$ and $\mathbf{x}$ is close to $\mathbf{p} \in \partial B_a(0)$ with $|\mathbf{x}-\mathbf{p}| < \delta/2$, then by the triangle inequality

$$|\mathbf{z}-\mathbf{p}| \le |\mathbf{x}-\mathbf{z}| + |\mathbf{x}-\mathbf{p}|,$$

we will have for $\mathbf{z} \in \partial B_a(0) \setminus C(\mathbf{p})$ the estimate

$$|\mathbf{x}-\mathbf{z}| \ge |\mathbf{z}-\mathbf{p}| - |\mathbf{x}-\mathbf{p}| \ge \delta - \frac{\delta}{2} = \frac{\delta}{2}. \tag{156}$$

Hence for $\mathbf{x}$ close to $\mathbf{p} \in \partial B_a(0)$ with $|\mathbf{x}-\mathbf{p}| < \delta/2$, we have the estimate

$$\int_{\mathbf{z}\in \partial B_a(0)\setminus C(\mathbf{p})} \frac{|h(\mathbf{z})-h(\mathbf{p})|}{|\mathbf{x}-\mathbf{z}|^2} ds \le \int_{\mathbf{z}\in \partial B_a(0)\setminus C(\mathbf{p})} \frac{|h(\mathbf{x})-h(\mathbf{p})|}{\left(\frac{\delta}{2}\right)^2} ds \le \frac{2M}{\left(\frac{\delta}{2}\right)^2} \cdot 2\pi a$$

where $M = \sup_{\mathbf{z}\in\partial B_a(0)} |h(\mathbf{z})|$. Hence

$$\lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)} |u(\mathbf{x}) - h(\mathbf{p})| \le \varepsilon + \lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)} \left(\frac{a^2-|\mathbf{x}|^2}{2\pi a} \cdot \frac{2M}{\left(\frac{\delta}{2}\right)^2} \cdot 2\pi a \right) = \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, we obtain $\lim_{\mathbf{x}\to\mathbf{p},\ \mathbf{x}\in B_a(0)} |u(\mathbf{x}) - h(\mathbf{p})| = 0$. The proof is done. □

We can summarize the following:

Theorem 1.59 ***(Solution of the Dirichlet problem (115) when*** h ***is continuous on*** $\partial B_a(0)$***.)*** *Consider the Dirichlet problem for the Laplace equation on* $B_a(0)$*:*

$$\begin{cases} \triangle u(x,y) = 0 & \text{in} \quad (x,y) \in B_a(0) \\ u(x,y) = h(x,y) & \text{on} \quad (x,y) \in \partial B_a(0) \ \ (\text{the boundary of } B_a(0)\,), \end{cases} \tag{157}$$

where $h(x,y)$ *is a given* ***continuous*** *function defined on* $\partial B_a(0)$*. The solution in the space*

$$C^2(B_a(0)) \bigcap C^0\left(\bar{B}_a(0)\right)$$

is ***unique*** *and is given by*

$$u(\mathbf{x}) = \begin{cases} \dfrac{a^2-|\mathbf{x}|^2}{2\pi a} \displaystyle\int_{|\mathbf{z}|=a} \frac{h(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds, & \mathbf{x} \in B_a(0) \\ h(\mathbf{x}), & \mathbf{x} \in \partial B_a(0). \end{cases} \tag{158}$$

Remark 1.60 ***(Important observation.)*** *The representation formula (158) says that the values of* u *at interior points* $\mathbf{x} \in B_a(0)$ *is completely determined by its boundary data* h*. This matches with the maximum/minimum principle.*

Remark 1.61 *In terms of the polar coordinates* (r,θ), *the function* $u(\mathbf{x})$ *in (158) can be written as (note that* $ds = a d\varphi$*)*

$$u(r,\theta) = \begin{cases} \dfrac{a^2 - r^2}{2\pi} \displaystyle\int_0^{2\pi} \frac{h(\varphi)}{a^2 - 2ar\cos(\theta - \varphi) + r^2} d\varphi, & (r,\theta) \in [0,a) \times [0,2\pi] \\ h(\theta), \quad r = a, \quad \theta \in [0,2\pi]. \end{cases} \tag{159}$$

1.10 The case when h is a C^1 function on $\partial B_a(0)$.

In case h is a C^1 function on $\partial B_a(0)$ in the Dirichlet problem (115), then $h(\theta)$ is a 2π**-periodic** C^1 **function** defined on $\theta \in [0,2\pi]$ with Fourier series expansion

$$h(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \quad \forall\, \theta \in [0,2\pi], \tag{160}$$

where the coefficients $a_n,\ b_n$ are given by (124). In this situation, we have:

Theorem 1.62 *(**Solution of the Dirichlet problem (115) when** h **is** C^1 **on** $\partial B_a(0)$.) Assume* $h(\theta)$ *is a* 2π***-periodic*** C^1 ***function*** *defined on* $\theta \in [0,2\pi]$. *The solution of the Dirichlet problem (115) in the function space* $C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$ *can also be expressed as*

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cdot \underbrace{\left(\frac{r}{a}\right)^n \cos n\theta} + b_n \cdot \underbrace{\left(\frac{r}{a}\right)^n \sin n\theta} \right], \quad (r,\theta) \in [0,a] \times [0,2\pi] \tag{161}$$

where $a_n,\ b_n$ *are the* ***Fourier series coefficients*** *of* $h(\theta)$ *on* $\theta \in [0,2\pi]$.

Remark 1.63 *(**Important.**) Note that the series in (161) is defined at* $r = a$ *with sum equal to* $h(\theta)$ *for all* $\theta \in [0,2\pi]$. *Since we have*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0,$$

which, together with the ***uniform convergence*** *of the series (132) on* $(r,\theta,\varphi) \in [0, a-\varepsilon] \times \cdots$ *for any small* $\varepsilon > 0$, *we see that the function* $u(r,\theta)$ *defined by the series (161) is continuous on* $(r,\theta) \in [0,a) \times [0,2\pi]$, *and* $u(a,\theta) = h(\theta)$ *for all* $\theta \in [0,2\pi]$. *However,* ***it is not clear whether*** $u(r,\theta)$ ***is continuous up to*** $r = a$ ***or not.*** *Therefore, we know that* $u(r,\theta)$ *is harmonic on* $[0,a) \times [0,2\pi]$ *and* $u(a,\theta) = h(\theta)$ *for all* $\theta \in [0,2\pi]$, *but* ***we do not know*** *if we have*

$$\lim_{(r,\theta)\to(a,\theta_0)} u(r,\theta) = u(a,\theta_0).$$

Proof. We already know that the solution $u(r,\theta)$ is given by

$$\begin{aligned} u(r,\theta) &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\varphi)}{a^2 - 2ar\cos(\theta-\varphi) + r^2} d\varphi, \quad (r,\theta) \in [0,a) \times [0,2\pi] \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \varphi) \right) h(\varphi)\, d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (\cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi) \right) h(\varphi)\, d\varphi \end{aligned} \tag{162}$$

and we can change the order of $\int_0^{2\pi}(*)\,d\varphi$ and $\sum_{n=1}^{\infty}(*)$ on $(r,\theta)\in[0,a)\times[0,2\pi]$ to get

$$\begin{aligned} &u(r,\theta)\\ &=\frac{1}{2\pi}\int_0^{2\pi}h(\varphi)\,d\varphi+\sum_{n=1}^{\infty}\left[\begin{array}{c}\left(\frac{1}{\pi}\int_0^{2\pi}h(\varphi)(\cos n\varphi)\,d\varphi\right)\underbrace{\left(\frac{r}{a}\right)^n\cos n\theta}\\ +\left(\frac{1}{\pi}\int_0^{2\pi}h(\varphi)(\sin n\varphi)\,d\varphi\right)\underbrace{\left(\frac{r}{a}\right)^n\sin n\theta}\end{array}\right]\\ &=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cdot\underbrace{\left(\frac{r}{a}\right)^n\cos n\theta}+b_n\cdot\underbrace{\left(\frac{r}{a}\right)^n\sin n\theta}\right],\quad(r,\theta)\in[0,a)\times[0,2\pi]. \end{aligned}$$

Since the function

$$\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi$$

is continuous up to $\partial B_a(0)$ with $\lim_{(r,\theta)\to(a,\theta_0)}(*)=h(\theta_0)$ for all $\theta_0\in[0,2\pi]$, the series (161) must be also continuous up to $\partial B_a(0)$, i.e. it lies in the function space $C^2(B_a(0))\bigcap C^0\left(\bar{B}_a(0)\right)$. The proof is done. □

1.11 Representation formula for harmonic functions on a disc.

An important consequence of Theorem 1.59 is the following:

Theorem 1.64 *(**Representation formula for harmonic functions on the disc** $B_a(0)$.) Assume*

$$u(\mathbf{x})\in C^2(B_a(0))\bigcap C^0\left(\bar{B}_a(0)\right)$$

*is a **harmonic function** on $B_a(0)$ and is **continuous up to the boundary** $\partial B_a(0)$. Then on $B_a(0)$ we have the identity*

$$u(\mathbf{x})=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds,\quad\forall\,\mathbf{x}\in B_a(0),\tag{163}$$

which is the same as

$$u(r,\theta)=\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{u(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi,\quad(r,\theta)\in[0,a)\times[0,2\pi],\tag{164}$$

where $u(\varphi)=u(a\cos\varphi,a\sin\varphi)$, $\varphi\in[0,2\pi]$.

Remark 1.65 *See Theorem 1.72 below for a similar result when $u(\mathbf{x})$ satisfying $\triangle u(\mathbf{x})\geq 0$ on $B_a(0)$ (or $\triangle u(\mathbf{x})\leq 0$ on $B_a(0)$).*

Remark 1.66 *(**Important.**) On the disc $B_a(\mathbf{x}_0)$ centered at some $\mathbf{x}_0\in\mathbb{R}^2$, the identity (163) becomes*

$$u(\mathbf{x})=\frac{a^2-|\mathbf{x}-\mathbf{x}_0|^2}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds,\quad\forall\,\mathbf{x}\in B_a(\mathbf{x}_0).\tag{165}$$

To see this, let $v(\mathbf{x})=u(\mathbf{x}+\mathbf{x}_0)$, then v is harmonic on $B_a(0)$ if and only if u is harmonic on $B_a(\mathbf{x}_0)$. For v, we have

$$v(\mathbf{x})=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{v(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds,\quad\forall\,\mathbf{x}\in B_a(0),\tag{166}$$

and back to u we have

$$u(\mathbf{x}+\mathbf{x}_0)=\frac{a^2-|(\mathbf{x}+\mathbf{x}_0)-\mathbf{x}_0|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{u(\mathbf{z}+\mathbf{x}_0)}{|(\mathbf{x}+\mathbf{x}_0)-(\mathbf{z}+\mathbf{x}_0)|^2}ds,\quad \forall\ \mathbf{x}\in B_a(0). \tag{167}$$

If we let $\mathbf{y}=\mathbf{x}+\mathbf{x_0}\in B_a(\mathbf{x}_0)$ *and* $\tilde{\mathbf{z}}=\mathbf{z}+\mathbf{x}_0\in\partial B_a(\mathbf{x}_0)$, *then the above is the same as*

$$u(\mathbf{y})=\frac{a^2-|\mathbf{y}-\mathbf{x}_0|^2}{2\pi a}\int_{|\tilde{\mathbf{z}}-\mathbf{x}_0|=a}\frac{u(\tilde{\mathbf{z}})}{|\mathbf{y}-\tilde{\mathbf{z}}|^2}ds,\quad \forall\ \mathbf{y}\in B_a(\mathbf{x}_0), \tag{168}$$

which gives (165).

1.12 Mean value formula for harmonic functions.

Theorem 1.67 *(**Mean value formula; line integral version.**) Assume* $u\in C^2(\Omega)$ *is **harmonic** on some open set* $\Omega\subset\mathbb{R}^2$, *then for any open disc* $B_a(\mathbf{x}_0)\subset\subset\Omega$, $a>0$, *we have the identity*

$$u(\mathbf{x}_0)=\frac{1}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}u(\mathbf{z})\,ds\ \ (\textbf{line integral} \text{ on the circle } \partial B_a(\mathbf{x}_0)\ ), \tag{169}$$

*i.e. the value of u at the **center*** $\mathbf{x}_0$ *of the disc* $B_a(\mathbf{x}_0)$ *is equal to its average on the circumference* $|\mathbf{z}-\mathbf{x}_0|=a$. *The integral in (169) is the line integral with respect to arc length parameter s on* $\partial B_a(\mathbf{x}_0)$, $ds=ad\varphi,\ \varphi\in[0,2\pi]$.

Remark 1.68 *For harmonic functions* $u\in C^2(B_a(0))\bigcap C^0(\bar{B}_a(0))$ *defined on an open disc* $B_a(0)\subset\mathbb{R}^n$ *for arbitrary* $n\in\mathbb{N}$, *there is also a mean value formula.*

Proof. On the disc $B_a(\mathbf{x}_0)$, by the representation formula (165), we have

$$u(\mathbf{x})=\frac{a^2-|\mathbf{x}-\mathbf{x}_0|^2}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds,\quad \forall\ \mathbf{x}\in B_a(\mathbf{x}_0)$$

and so

$$\begin{aligned}u(\mathbf{x}_0)&=\frac{a^2}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}\frac{u(\mathbf{z})}{|\mathbf{x}_0-\mathbf{z}|^2}ds\\&=\frac{a^2}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}\frac{u(\mathbf{z})}{a^2}ds=\frac{1}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}u(\mathbf{z})\,ds.\end{aligned}$$

The proof is done. □

Theorem 1.69 *(**Mean value formula; double integral version.**) Assume* $u\in C^2(\Omega)$ *is harmonic on some open set* $\Omega\subset\mathbb{R}^2$, *then for any open disc* $B_a(\mathbf{x}_0)\subset\subset\Omega$, $a>0$, *we have the identity*

$$u(\mathbf{x}_0)=\frac{1}{\pi a^2}\iint_{B_a(\mathbf{x}_0)}u(\mathbf{x})\,d\mathbf{x}\ \ (\textbf{double integral} \text{ on the disc } B_a(\mathbf{x}_0)\ ). \tag{170}$$

Proof. By (169), we have

$$ru(\mathbf{x}_0)=\frac{1}{2\pi}\int_{|\mathbf{z}-\mathbf{x}_0|=r}u(\mathbf{z})\,ds=\frac{1}{2\pi}\int_0^{2\pi}u(\mathbf{x}_0+r(\cos\theta,\sin\theta))\cdot rd\theta$$

for any $B_r(\mathbf{x}_0)\subset B_a(\mathbf{x}_0)\subset\subset\Omega$, $r\in[0,a]$, and if we integrate with respect to the radius r from 0 to a, we get

$$\int_0^a ru(\mathbf{x}_0)\,dr=\frac{1}{2\pi}\int_0^a\left[\int_0^{2\pi}u(\mathbf{x}_0+r(\cos\theta,\sin\theta))\cdot rd\theta\right]dr$$

and so

$$\frac{a^2}{2}u(\mathbf{x}_0) = \frac{1}{2\pi}\underbrace{\int_0^{2\pi}\int_0^a u(\mathbf{x}_0 + r(\cos\theta, \sin\theta))\cdot r dr d\theta}, $$

which, by the change of variables formula for double integral in the plane, we have

$$u(\mathbf{x}_0) = \frac{1}{\pi a^2}\underbrace{\iint_{B_a(\mathbf{x}_0)} u(\mathbf{x})\, d\mathbf{x}}.$$

The proof is done. □

1.13 Mean value inequality for subharmonic and superharmonic functions.

We first note that the "**name**" for subharmonic and superharmonic functions are due to the following properties:

Lemma 1.70 *Assume Ω is a **bounded domain** and assume $u,\ v \in C^2(\Omega)\bigcap C^0(\bar{\Omega})$ with*

$$(1).\ \triangle u \geq 0 \quad in\ \ \Omega. \qquad (2).\ \triangle v = 0 \quad in\ \ \Omega. \qquad (3).\ u \leq v \quad on\ \ \partial\Omega, \tag{171}$$

*then $u \leq v$ in Ω (this is why we call u a **subharmonic** function). Similarly, if we have*

$$(1).\ \triangle u \leq 0 \quad in\ \ \Omega. \qquad (2).\ \triangle v = 0 \quad in\ \ \Omega. \qquad (3).\ u \geq v \quad on\ \ \partial\Omega, \tag{172}$$

*then $u \geq v$ in Ω (this is why we call u a **superharmonic** function).*

Remark 1.71 *Draw an one-dimensional picture for this.*

Proof. This is a consequence of the weak maximum/minimum principle. We prove the first case. Let $w = u - v$. It satisfies

$$\triangle w \geq 0 \ \text{ in } \ \Omega \qquad \text{and} \qquad w \leq 0 \ \text{ on } \ \partial\Omega.$$

Therefore, by the weak maximum principle, we have $\max_{\bar{\Omega}} w = \max_{\partial\Omega} w \leq 0$, which implies that $w = u - v \leq 0$ in Ω. For the second case, let $w = u - v$ and apply the weak minimum principle. □

Theorem 1.72 *(**Poisson integral inequality for subharmonic and superharmonic functions.**) Let $B_a(0)$ be the open disc in $\mathbb{R}^2$ centered at the point $0 = (0,0)$ with radius $a > 0$. Assume $u \in C^2(B_a(0))\bigcap C^0(\bar{B}_a(0))$ and satisfies $\triangle u(\mathbf{x}) \geq 0$ on $B_a(0)$ (i.e. it is subharmonic on $B_a(0)$). Then we have*

$$u(\mathbf{x}) \leq \frac{a^2 - |\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds, \quad \forall\ \mathbf{x} \in B_a(0). \tag{173}$$

Similarly, if $u(\mathbf{x})$ satisfies $\triangle u(\mathbf{x}) \leq 0$ on $B_a(0)$, then we have

$$u(\mathbf{x}) \geq \frac{a^2 - |\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{z}|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds, \quad \forall\ \mathbf{x} \in B_a(0). \tag{174}$$

Remark 1.73 *In case the disc is $B_a(\mathbf{x}_0)$, then (173) and (174) will become*

$$u(\mathbf{x}) \leq \frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds, \quad \forall\ \mathbf{x} \in B_a(\mathbf{x}_0) \tag{175}$$

and

$$u(\mathbf{x}) \geq \frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds, \quad \forall\ \mathbf{x} \in B_a(\mathbf{x}_0). \tag{176}$$

Proof. This is a consequence of Lemma 1.70. Let

$$v(\mathbf{x})=\begin{cases}\dfrac{a^2-|\mathbf{x}|^2}{2\pi a}\displaystyle\int_{|\mathbf{z}|=a}\frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}ds, & \mathbf{x}\in B_a(0)\\ u(\mathbf{x}), \quad \mathbf{x}\in\partial B_a(0).\end{cases}$$

We know that $v(\mathbf{x})\in C^2(B_a(0))\bigcap C^0(\bar{B}_a(0))$ is **harmonic** in $B_a(0)$ with $v(\mathbf{x})=u(\mathbf{x})$ on $\partial B_a(0)$. If we have $\triangle u(\mathbf{x})\geq 0$ on $B_a(0)$, then Lemma 1.70 implies $u(\mathbf{x})\leq v(\mathbf{x})$ in Ω, which gives (173). The proof of (174) is similar. □

Theorem 1.74 ***(Mean value inequality for subharmonic and superharmonic functions.)*** *For any open set $\Omega\subset\mathbb{R}^2$, if $u\in C^2(\Omega)$ is* ***subharmonic*** *on Ω (i.e. $\triangle u\geq 0$ on Ω), then for any open disc $B_a(\mathbf{x}_0)\subset\subset\Omega$, $a>0$, we have the following* ***mean value inequality***

$$u(\mathbf{x}_0)\leq\frac{1}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}u(\mathbf{z})\,ds \text{ (line integral on the circle } \partial B_a(\mathbf{x}_0)\text{)} \tag{177}$$

and

$$u(\mathbf{x}_0)\leq\frac{1}{\pi a^2}\iint_{B_a(\mathbf{x}_0)}u(\mathbf{x})\,d\mathbf{x} \text{ (double integral on the disc } B_a(\mathbf{x}_0)\text{)}. \tag{178}$$

Similarly, if $u\in C^2(\Omega)$ is ***superharmonic*** *on Ω (i.e. $\triangle u\leq 0$ on Ω), then we have (177) and (178) with " $\leq$ " replaced by " $\geq$ ".*

Proof. Assume u is **subharmonic**. For any open disc $B_a(\mathbf{x}_0)\subset\subset\Omega$, by (175) (evaluated at $\mathbf{x}=\mathbf{x}_0$), we have

$$u(\mathbf{x}_0)\leq\frac{a^2-|\mathbf{x}_0-\mathbf{x}_0|^2}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}\frac{u(\mathbf{z})}{|\mathbf{x}_0-\mathbf{z}|^2}ds=\frac{1}{2\pi a}\int_{|\mathbf{z}-\mathbf{x}_0|=a}u(\mathbf{z})\,ds,$$

which proves (177).

For (178), we note that the inequality (177) is valid for any radius $r>0$ as long as $B_r(\mathbf{x}_0)\subset B_a(\mathbf{x}_0)\subset\subset\Omega$. Hence we have

$$ru(\mathbf{x}_0)\leq\frac{1}{2\pi}\int_{|\mathbf{z}-\mathbf{x}_0|=r}u(\mathbf{z})\,ds=\frac{1}{2\pi}\int_0^{2\pi}u(\mathbf{x}_0+r(\cos\theta,\sin\theta))\cdot r d\theta,\quad\forall\, r\in[0,a]$$

and if we integrate with respect to the radius r from 0 to a, we get

$$\int_0^a ru(\mathbf{x}_0)\,dr\leq\frac{1}{2\pi}\int_0^a\left[\int_0^{2\pi}u(\mathbf{x}_0+r(\cos\theta,\sin\theta))\cdot r d\theta\right]dr$$

and so

$$\frac{a^2}{2}u(\mathbf{x}_0)\leq\frac{1}{2\pi}\int_0^{2\pi}\int_0^a u(\mathbf{x}_0+r(\cos\theta,\sin\theta))\cdot r dr d\theta=\frac{1}{2\pi}\iint_{B_a(\mathbf{x}_0)}u(\mathbf{x})\,d\mathbf{x},$$

where the last identity in the above is due to **the change of variables formula for double integrals in the plane**. The proof is done for the subharmonic case. The proof for the superharmonic case is similar. □

1.14 The strong maximum/minimum principle for harmonic functions on the disc $B_a(0)$.

Theorem 1.75 *(**Strong maximum/minimum principle for harmonic functions on the disc** $B_a(0)$.) Assume that $u(\mathbf{x}) \in C^2(B_a(0)) \bigcap C^0(\bar{B}_a(0))$, $\mathbf{x} = (x, y) \in \mathbb{R}^2$, is **harmonic** on $B_a(0)$. Let $M = \max_{\mathbf{x}\in\bar{B}_a(0)} u(\mathbf{x})$ and $m = \min_{\mathbf{x}\in\bar{B}_a(0)} u(\mathbf{x})$. If there exists $\mathbf{x}_0 \in B_a(0)$ ($\mathbf{x}_0$ is an interior point of $\bar{B}_a(0)$) such that $u(\mathbf{x}_0) = M$ (or $u(\mathbf{x}_0) = m$), then u must be a **constant function** on $\bar{B}_a(0)$.*

Remark 1.76 *Note that the strong maximum/minimum principle will imply the weak maximum/minimum principle.*

Proof. By the **Poisson *Integral Formula,*** we have

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z})}{|\mathbf{x} - \mathbf{z}|^2} ds, \quad \forall\ \mathbf{x} = (x, y) \in B_a(0)$$

where $u(\mathbf{z}) \le M$ for all $\mathbf{z} \in \partial B_a(0)$. By the identity (see (155) also)

$$0 = u(\mathbf{x}_0) - M = \frac{a^2 - |\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z}) - M}{|\mathbf{x}_0 - \mathbf{z}|^2} ds, \quad \text{where } a^2 - |\mathbf{x}_0|^2 > 0, \tag{179}$$

we must have

$$\int_{|\mathbf{z}|=a} \frac{u(\mathbf{z}) - M}{|\mathbf{x}_0 - \mathbf{z}|^2} ds = 0, \tag{180}$$

where we also note that $u(\mathbf{z}) - M \le 0$ for all $\mathbf{z} \in \partial B_a(0)$ due to $M = \max_{\mathbf{x}\in\bar{B}_a(0)} u(\mathbf{x})$. Therefore we must have $u(\mathbf{z}) \equiv M$ on $\partial B_a(0)$. Hence $u(\mathbf{x})$ is harmonic in $B_a(0)$ and has constant value on $\partial B_a(0)$. By the **weak maximum principle,** $u(\mathbf{x})$ must be a constant function on $\bar{B}_a(0)$ with $u(\mathbf{x}) \equiv M$. Similar result holds for the case $u(\mathbf{x}_0) = m$. □

Remark 1.77 *Draw a one-dimensional picture for the above theorem.*

1.15 Gradient estimate and Liouville theorem for harmonic functions.

Theorem 1.78 *(**Derivatives estimate of harmonic functions.**) Assume $u(x, y) \in C^2(\Omega)$ is harmonic on $\Omega \subset \mathbb{R}^2$. Then we have*

$$\left|\frac{\partial u}{\partial x}(x_0, y_0)\right| \left(or \left|\frac{\partial u}{\partial y}(x_0, y_0)\right|\right) \le \frac{2}{a} \max_{\partial B_a(x_0,y_0)} |u| \tag{181}$$

as long as $B_a(x_0, y_0) \subset\subset \Omega$.

Proof. First note that $u \in C^\infty(\Omega)$ (note that a C^2 harmonic function is automatically a C^∞ function; we can see this from Theorem 1.85 below). In particular, the function $\frac{\partial u}{\partial x}\left(\text{or } \frac{\partial u}{\partial y}\right)$ is also a **harmonic function** on Ω. Hence by the mean value formula we have

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{1}{\pi a^2} \underbrace{\iint_{B_a(x_0,y_0)} \frac{\partial u}{\partial x}(x, y)\, dxdy,}_{} \tag{182}$$

where by the **divergence theorem**, we know

$$\underbrace{\iint_{B_a(x_0,y_0)} \frac{\partial u}{\partial x}(x, y)\, dxdy}_{} = \iint_{B_a(x_0,y_0)} \operatorname{div}(u(x, y), 0)\, dxdy$$

$$= \int_{\partial B_a(x_0,y_0)} \langle (u(x, y), 0),\ \mathbf{N}_{out}(x, y)\rangle\, ds. \tag{183}$$

Therefore

$$\begin{aligned}\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| &\le \left|\frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)}\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle\, ds\right| \\ &\le \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)}\underbrace{|\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle|}\, ds \\ &\le \frac{2\pi a}{\pi a^2}\underbrace{\max_{\partial B_a(x_0,y_0)}|(u(x,y),0)|} = \frac{2}{a}\max_{\partial B_a(x_0,y_0)}|u|. \end{aligned} \tag{184}$$

The proof is done. □

We also have:

Corollary 1.79 *(**Derivatives estimate of nonnegative harmonic functions.**) Assume $u(x,y) \in C^2(\Omega)$ is harmonic on $\Omega \subset \mathbb{R}^2$ with $u \ge 0$ **everywhere** in Ω. Then we have*

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \left(or \left|\frac{\partial u}{\partial y}(x_0,y_0)\right|\right) \le \frac{2}{a}u(x_0,y_0) \tag{185}$$

as long as $B_a(x_0,y_0) \subset\subset \Omega$.

Remark 1.80 *Note that estimate (185) is better than (181) when $u \ge 0$ in Ω.*

Proof. By (182) and (183), we have

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)}\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle\, ds$$

and so

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \le \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)}|\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle|\, ds.$$

Since $u \ge 0$ everywhere in Ω, we have

$$\underbrace{|\langle (u(x,y),0),\ \mathbf{N}_{out}(x,y)\rangle| \le |(u(x,y),0)| = u(x,y)}, \quad \forall\ (x,y) \in \partial B_a(x_0,y_0).$$

By this, we get (we use mean value property again)

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \le \frac{1}{\pi a^2}\int_{\partial B_a(x_0,y_0)} u(x,y)\, ds = \frac{2\pi a}{\pi a^2}\left(\frac{1}{2\pi a}\int_{\partial B_a(x_0,y_0)} u(x,y)\, ds\right) = \frac{2}{a}u(x_0,y_0).$$

The proof is done. □

Example 1.81 *Let $\Omega \subset \mathbb{R}^2$ be a domain. Assume $u(x,y) \in C^2(\Omega)$ is harmonic on $\Omega \subset \mathbb{R}^2$ with $u \ge -13$ everywhere in Ω. Assume $B_a(x_0,y_0) \subset\subset \Omega$ and $u(x_0,y_0) = -5$ (here $B_a(x_0,y_0)$ is an open disc in $\mathbb{R}^2$ centered at (x_0,y_0) with radius $a > 0$). Then we have the following derivatives estimate:*

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| \left(or \left|\frac{\partial u}{\partial y}(x_0,y_0)\right|\right) \le \_\_\_\_\_\_\_\_\_\_\_\_.$$

<u>**Solution**</u>:

The function $v(x,y) = u(x,y) + 13$ is harmonic on $\Omega \subset \mathbb{R}^2$ with $v \ge 0$ everywhere in Ω. Hence we have

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right| = \left|\frac{\partial v}{\partial x}(x_0,y_0)\right| \le \frac{2}{a}v(x_0,y_0) = \frac{2}{a}(u(x_0,y_0)+13) = \frac{2}{a}(-5+13) = \frac{16}{a}.$$

□

Theorem 1.82 *(Liouville theorem of harmonic functions on entire space.) If u is harmonic on $\mathbb{R}^2$ and is* ***bounded either above or below*** *on $\mathbb{R}^2$, then u must be a constant function.*

Remark 1.83 *Note that here $u(x,y)$ is defined on the* ***whole space*** *$\mathbb{R}^2$. This is essential.*

Remark 1.84 *The same property holds if u is harmonic on $\mathbb{R}^n$, $n \geq 3$, and is* ***bounded either above or below*** *on $\mathbb{R}^n$.*

Proof. Without loss of generality, assume u is bounded below (if u is bounded above, we can look at $-u$). By adding a large constant if necessary, we may assume $u \geq 0$ on $\mathbb{R}^2$. Apply (185) to $u(x,y)$ with $\Omega = \mathbb{R}^2$, $a \to \infty$, to get

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial u}{\partial y}(x_0,y_0) \equiv 0, \quad \forall\ (x_0,y_0) \in \mathbb{R}^2.$$

As the point (x_0,y_0) is arbitrary, we are done. □

1.16 C^2 harmonic functions are automatically C^∞ functions.

Theorem 1.85 *Let $\Omega \subset \mathbb{R}^2$ be any open set (may not be bounded) and $u \in C^2(\Omega)$ is harmonic on Ω. Then we must have $u \in C^\infty(\Omega)$.*

Proof. For any $\mathbf{x}_0 \in \Omega$ one can find a small open disc $B_a(\mathbf{x}_0) \subset\subset \Omega$ and we have the identity

$$\begin{aligned} u(\mathbf{x}) &= \frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \frac{u(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2} ds \\ &= \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \underbrace{\frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{z}|^2}} u(\mathbf{z})\, ds, \quad \forall\ \mathbf{x} \in B_a(\mathbf{x}_0) \end{aligned} \tag{186}$$

Note that, for $\mathbf{z} \in \partial B_a(\mathbf{x}_0)$, the function $\left(a^2 - |\mathbf{x}-\mathbf{x}_0|^2\right)/|\mathbf{x}-\mathbf{z}|^2$ is a C^∞ function of $\mathbf{x} \in B_a(\mathbf{x}_0)$ and at any fixed $\mathbf{x}_* \in B_a(\mathbf{x}_0)$ the line integral

$$\frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \left(\underbrace{\left.\frac{\partial^{m+n}}{\partial x^m \partial y^n}\right|_{\mathbf{x}=\mathbf{x}_*} \left(\frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{z}|^2} \right)} \right) u(\mathbf{z})\, ds, \quad \forall\ m,\ n \in \mathbb{N} \bigcup \{0\}$$

still converges, by standard theorem in Advanced Calculus (see any textbook), the function $u(\mathbf{x})$ is a C^∞ function of $\mathbf{x} \in B_a(\mathbf{x}_0)$ and one can differentiate under the integral sign, i.e.

$$\begin{aligned} &\left.\frac{\partial^{m+n}}{\partial x^m \partial y^n}\right|_{\mathbf{x}=\mathbf{x}_*} u(\mathbf{x}) \\ &= \frac{1}{2\pi a} \int_{|\mathbf{z}-\mathbf{x}_0|=a} \left(\underbrace{\left.\frac{\partial^{m+n}}{\partial x^m \partial y^n}\right|_{\mathbf{x}=\mathbf{x}_*} \left(\frac{a^2 - |\mathbf{x}-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{z}|^2} \right)} \right) u(\mathbf{z})\, ds, \quad \forall\ m,\ n \in \mathbb{N} \bigcup \{0\}. \end{aligned}$$

The proof is done. □

1.17 The strong maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.

Recall that we already have the weak maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains; see Lemma 1.36. Now we can use **mean value inequality** to prove the following **strong** maximum/minimum principle :

Theorem 1.86 *(**Strong maximum/minimum principle for harmonic, subharmonic, and superharmonic functions on bounded domains.**) Let $\Omega \subset \mathbb{R}^2$ be a **bounded domain** (**open and connected**) in $\mathbb{R}^2$ and $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$. Assume u is **subharmonic** on Ω (i.e. $\triangle u \geq 0$ on Ω) and there exists a point $\mathbf{p} \in \Omega$ such that*

$$u(\mathbf{p}) = \max_{\bar{\Omega}} u, \tag{187}$$

*then u must be a **constant function** on Ω. Similarly, if u is **superharmonic** on Ω (i.e. $\triangle u \leq 0$ on Ω) and there exists a point $\mathbf{p} \in \Omega$ such that*

$$u(\mathbf{p}) = \min_{\bar{\Omega}} u, \tag{188}$$

*then u must be a **constant function** on Ω. In particular, if u is **harmonic** on Ω and there exists a point $\mathbf{p} \in \Omega$ such that $u(\mathbf{p}) = \max_{\bar{\Omega}} u$ or $u(\mathbf{p}) = \min_{\bar{\Omega}} u$, then u must be a **constant function** on Ω.*

Proof. Assume u is **subharmonic** on Ω and we have $u(\mathbf{p}) = \max_{\bar{\Omega}} u$ (call this value M) for some $\mathbf{p} \in \Omega$. Since Ω is an open set, one can find an open disc $B_a(\mathbf{p}) \subset\subset \Omega$ for some $a > 0$. By the mean value inequality, we have

$$0 = u(\mathbf{p}) - M \leq \frac{1}{\pi a^2} \iint_{B_a(\mathbf{p})} (u(\mathbf{x}) - M)\, d\mathbf{x} \leq 0, \quad \text{where } u(\mathbf{x}) - M \leq 0 \text{ on } B_a(\mathbf{p}), \tag{189}$$

which implies $u(\mathbf{x}) \equiv M$ on $B_a(\mathbf{p})$. Therefore, the nonempty set

$$D = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = M\} \subset \Omega, \quad \mathbf{p} \in D$$

is **open** in Ω. We claim that $D \subset \Omega$ is also **closed** in Ω (in topology, this means that any limit point of the set D in Ω must lie in D). Let $\mathbf{q}_n \in D$ be a sequence in D with $\lim_{n\to\infty} \mathbf{q}_n = \mathbf{q}_* \in \Omega$ (i.e. the sequence $q_n \in D$ has a limit point $\mathbf{q}_* \in \Omega$). We claim that $\mathbf{q}_* \in D$. To see this, since $u \in C^0(\bar{\Omega})$, we have

$$M = \lim_{n\to\infty} u(\mathbf{q}_n) = u\left(\lim_{n\to\infty} \mathbf{q}_n\right) = u(\mathbf{q}_*), \quad \mathbf{q}_* \in \Omega,$$

which implies that $\mathbf{q}_* \in D$. Therefore, by definition, this means that $D \subset \Omega$ is also **closed** in Ω. As Ω is **connected**, the only set which is **both open and closed in** Ω is either the empty set $\varnothing$ or the whole set Ω. Since D is not empty (because $\mathbf{p} \in D$), we must have $D = \Omega$. That is, $u \equiv M$ on all of Ω.

The proof for the **superharmonic** case is similar since one can apply the above argument to the subharmonic function $-u$. The proof is done. □

Remark 1.87 *(**Important.**) (**can put this as homework problem.**) In some cases, Theorem 1.86 is also valid on unbounded domains. For example, assume $u \in C^2(\mathbb{R}^2)$ is a **subharmonic function** on the whole plane $\mathbb{R}^2$ (unbounded domain) and there exists $\mathbf{x}_0 \in \mathbb{R}^2$ such that*

$$u(\mathbf{x}_0) \geq u(\mathbf{x}) \quad \textit{for all} \quad \mathbf{x} \in \mathbb{R}^2.$$

Then if we look at the function $u(\mathbf{x})$ on the open disc $B_R(\mathbf{x}_0)$ and let $R \to \infty$, we will have $u(\mathbf{x}) \equiv u(\mathbf{x}_0)$ on $\mathbb{R}^2$.

1.18 Application of the weak maximum/minimum principle.

The following result is important and easy to prove using the weak maximum/minimum principle.

Lemma 1.88 *Let $\Omega \subset \mathbb{R}^n$ be a* ***bounded domain*** *and $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$ satisfy*

$$\triangle u = f \quad in \quad \Omega, \qquad u = g \quad on \quad \partial\Omega, \tag{190}$$

where f and g are continuous function on Ω and $\partial\Omega$ respectively. Let $B_R(x_0)$ be a ball centered at ***some*** *$x_0 \in \Omega$ such that $\Omega \subset B_R(x_0)$. Then we have the estimate*

$$\max_{\bar{\Omega}} |u| \le \max_{\partial\Omega} |g| + \frac{R^2}{2n} \sup_{\Omega} |f|. \tag{191}$$

Remark 1.89 *We will use the weak maximum/minimum principle to prove the above lemma. The weak maximum/minimum principle is actually valid on a* ***bounded domain*** *Ω in $\mathbb{R}^n$ for* ***any*** *$n \in \mathbb{N}$.*

Remark 1.90 *Since we assume $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$, the function $f \in C^0(\Omega)$ is defined only on Ω and, in general, may not be continuous up to $\bar{\Omega}$. Therefore, $\max_{\bar{\Omega}} |f|$ may not exist in general. In case f is continuous on $\bar{\Omega}$, we replace $\sup_{\Omega} |f|$ in (191) by $\max_{\bar{\Omega}} |f|$.*

Remark 1.91 *When $f \equiv 0$ in (190), then u is harmonic on Ω and we get $\max_{\bar{\Omega}} |u| \le \max_{\partial\Omega} |g|$ (same as $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |g|$), which coincides with previous result.*

Remark 1.92 *To make the estimate (191) as best as possible, one may choose $x_0 \in \Omega$ with $R > 0$ as smallest as possible.*

Remark 1.93 *(**A good example**.) From (191) we see that the diameter R of Ω comes into play. For example, take $\Omega = B_a(0) \subset \mathbb{R}^2$ to be an open disc, and solve*

$$\triangle u(x, y) = 4 \quad in \quad B_a(0), \qquad u = 0 \quad on \quad \partial B_a(0).$$

The unique solution is $u(x, y) = x^2 + y^2 - a^2$, $(x, y) \in B_a(0)$, with

$$\max_{\bar{\Omega}} |u| = |u(0,0)| = a^2, \qquad \max_{\partial\Omega} |g| + \frac{a^2}{2n} \sup_{\Omega} |f| = \frac{a^2}{4} \cdot 4 = a^2, \qquad n = 2.$$

We see that ***the diameter comes into play****. In this example, we actually have* ***equality*** *in (191). This is because we choose $x_0 = 0$ to be the center of the disc $B_a(0)$ and the radius for the ball $B_R(x_0)$ is the smallest, i.e. $R = a$.*

Proof. Let $A = \max_{\partial\Omega} |g| \ge 0$, $B = \sup_{\Omega} |f| \ge 0$. We may assume both A, $B < \infty$, otherwise estimate (191) is valid automatically. Define

$$w(x) = \left[A + \frac{R^2 - |x - x_0|^2}{2n} \cdot B \right] \pm u(x), \quad x \in \bar{\Omega} = \Omega \bigcup \partial\Omega, \tag{192}$$

then (note that $R^2 - |x - x_0|^2 \ge 0$ for all $x \in \bar{\Omega}$ and $\triangle |x - x_0|^2 = 2n$)

$$\begin{aligned} &(1). \ w \ge 0 \quad \text{on} \quad \partial\Omega \\ &(2). \ \triangle w = -B \pm \triangle u \le 0 \quad \text{on} \quad \Omega \\ &(3). \ w \in C^2(\Omega) \bigcap C^0(\bar{\Omega}). \end{aligned} \tag{193}$$

Minimum principle implies that $w \ge 0$ in Ω and so

$$\begin{aligned} |u(x)| &\le A + \frac{B}{2n}\left(R^2 - |x - x_0|^2\right) \\ &\le A + \frac{B}{2n} R^2 = \max_{\partial\Omega} |g| + \frac{R^2}{2n} \sup_{\Omega} |f|, \quad \forall \ x \in \Omega. \end{aligned}$$

The proof is done. □

1.19 Singularity of harmonic functions.

Definition 1.94 *Assume $n \geq 2$. We say u is a harmonic function with a* ***point singularity*** *at $x = x_0 \in \mathbb{R}^n$ if it is defined and* ***harmonic*** *in $B_R(x_0) \setminus \{x_0\}$ for some $R > 0$ and there exists a sequence $x_n \to x_0$ such that $u(x_n) \to \infty$ (or $-\infty$) as $n \to \infty$.*

Example 1.95 *The function*

$$u(x,y) = \frac{x}{x^2+y^2} \quad (\text{or } \frac{y}{x^2+y^2}), \quad (x,y) \neq (0,0) \in \mathbb{R}^2 \tag{194}$$

is harmonic on $\mathbb{R}^2 \setminus \{(0,0)\}$ and has a ***point singularity*** *at $(0,0)$. Note that $u(x,y) \equiv 0$ along the y-axis, but it satisfies the above definition.*

Another application of the weak maximum/minimum principle is the following:

Theorem 1.96 ***(Removable singularity of harmonic functions.)*** *Assume $n \geq 2$. Let $B_R = B_R(0) \subset \mathbb{R}^n$ be the open ball centered at $x = 0$ with radius $R > 0$. Suppose u is* ***harmonic*** *in $B_R \setminus \{0\}$ and satisfies*

$$u(x) = \begin{cases} o(\log|x|), & n = 2 \\ o\left(|x|^{2-n}\right), & n \geq 3 \end{cases} \quad \text{as} \quad |x| \to 0, \tag{195}$$

which means that

$$\begin{cases} \lim_{r\to 0}\left(\dfrac{\max_{\partial B_r}|u|}{\log r}\right) = 0, & n = 2 \\ \lim_{r\to 0}\left(\dfrac{\max_{\partial B_r}|u|}{r^{2-n}}\right) = 0, & n \geq 3, \end{cases} \tag{196}$$

then u can be defined at $x = 0$ such that it is C^2 and harmonic on B_R. That is: there exists a C^2 harmonic function $v(x)$ defined on B_R such that

$$u(x) \equiv v(x) \quad \text{for all} \quad x \in B_R \setminus \{0\}. \tag{197}$$

Remark 1.97 ***(Important.)*** *The above says that if a harmonic function has a "****point singularity****" at $x = 0$ (or at any point $x = p$), then the limit in (196) cannot be satisfied. In most cases, the limits in (196) will be $\pm\infty$. See Example 1.103 below.*

<u>**Solution**</u>:

We first look at the case $n \geq 3$. Without loss of generality, we may assume u is continuous on $0 < |x| \leq R$, i.e. continuous up to ∂B_R (otherwise we can look at u on $0 < |x| \leq R - \varepsilon$ for some small $\varepsilon > 0$).

Let $v \in C^2(B_R) \bigcap C^0(\bar{B}_R)$ be the solution of the Dirichlet problem:

$$\begin{cases} \triangle v = 0 & \text{in} \quad B_R \\ v = u & \text{on} \quad \partial B_R. \end{cases}$$

We will prove $u = v$ on $B_R \setminus \{0\}$. With this, u can be defined at $x = 0$ (just define its value at $x = 0$ as $v(0)$) such that $u \equiv v$ on B_R and it is harmonic on the whole B_R.

Set $w = v - u$ in $B_R \setminus \{0\}$ and let $M_r = \max_{\partial B_r}|w|$, where $0 < r < R$ is fixed. Clearly we have w is harmonic in the region $\Omega_r = B_R \setminus \{0\} - B_r \setminus \{0\}$. **We shall compare w with**

the harmonic function $|x|^{2-n}/r^{2-n}$. Note that $|x|^{2-n}/r^{2-n}$ is also **harmonic** in Ω_r. By the **maximum principle**, we have (note that $w \equiv 0$ on ∂B_R)

$$|w(x)| \leq M_r \frac{|x|^{2-n}}{r^{2-n}} \quad \text{on} \quad \Omega_r. \tag{198}$$

On the other hand, we have (note that v is harmonic in B_R, continuous up to ∂B_R and $\max_{\bar{B}_R}|v| \leq \max_{\partial B_R}|v|$)

$$M_r = \max_{\partial B_r}|v-u| \leq \underbrace{\max_{\partial B_r}|v|} + \max_{\partial B_r}|u| \leq \underbrace{\max_{\partial B_R}|v|} + \max_{\partial B_r}|u| = \max_{\partial B_R}|u| + \max_{\partial B_r}|u|.$$

From (198) we get

$$\begin{aligned} |w(x)| &\leq \left(\max_{\partial B_R}|u| + \max_{\partial B_r}|u|\right)\frac{|x|^{2-n}}{r^{2-n}} \\ &= \left(\frac{\max_{\partial B_R}|u|}{r^{2-n}} + \frac{\max_{\partial B_r}|u|}{r^{2-n}}\right)|x|^{2-n}, \quad \forall\, x \in \Omega_r = B_R\backslash\{0\} - B_r\backslash\{0\}. \end{aligned} \tag{199}$$

By the assumption, we know that $\lim_{r\to 0}(\max_{\partial B_r}|u|)/r^{2-n} = 0$; hence if we let $r \to 0$ in (199) (here $x \in \Omega_r$ is fixed), we get $w(x) = 0$. Since for any $x \in B_R\backslash\{0\}$ we can find a small $r > 0$ such that $x \in \Omega_r$ and can apply the above argument with $r \to 0$, we can conclude

$$w(x) \equiv 0 \quad \text{in} \quad B_R\backslash\{0\}.$$

The proof is done.

For $n = 2$, we may assume $R < 1$ and u is continuous on $0 < |x| \leq R$. Now we replace (198) by

$$|w(x)| \leq M_r \frac{\log|x|}{\log r} \quad \text{on} \quad \Omega_r, \tag{200}$$

where the harmonic function $\log|x|/\log r$ has value 1 on ∂B_r and has **positive** value $\log R/\log r$ on ∂B_R (note that $w \equiv 0$ on ∂B_R). Therefore, we conclude

$$|w(x)| \leq \left(\max_{\partial B_R}|u| + \max_{\partial B_r}|u|\right)\frac{\log|x|}{\log r}, \quad \forall\, x \in \Omega_r \tag{201}$$

and if we let $r \to 0$ in the above, we get

$$w(x) \equiv 0 \quad \text{in} \quad B_R\backslash\{0\}.$$

The proof is done. □

An interesting consequence of Theorem 1.96 is the following:

Corollary 1.98 *Assume $n \geq 2$ and $u(x)$ is **harmonic** in $\mathbb{R}^n\backslash\{0\}$ and is also a **bounded function (bounded above and bounded below)** on $\mathbb{R}^n\backslash\{0\}$, then $u(x)$ must be a constant function.*

Proof. By (196) in Theorem 1.96, $u(x)$ can be defined at $x = 0$ and is **harmonic** on the whole $\mathbb{R}^n$. Therefore it must be a constant function due to Liouville theorem. □

Remark 1.99 *Note that in the above corollary, we **cannot** replace the condition by **either** bounded above **or** bounded below on $\mathbb{R}^n\backslash\{0\}$. For example, the function r^{2-n}, $n \geq 3$, is **harmonic** in $\mathbb{R}^n\backslash\{0\}$ and is **bounded below** since $r^{2-n} > 0$ on $\mathbb{R}^n\backslash\{0\}$, however, it is not a constant function.*

Question 1.100 *An interesting question for me is: can we find a **harmonic** function on $\mathbb{R}^2\setminus\{0\}$ which is **either** bounded above **or** bounded below on $\mathbb{R}^2\setminus\{0\}$, but it is not a constant function ?*

Corollary 1.101 *Assume $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ is an open set and $p_1, \ ..., \ p_k, \ k \in \mathbb{N}$, are finite distinct points in Ω. If $u \in C^2(\Omega\setminus\{p_1, \ ..., \ p_k\})\bigcap C^0(\Omega)$ and is **harmonic** on $\Omega\setminus\{p_1, \ ..., \ p_k\}$, then $u \in C^2(\Omega)$ and is **harmonic** on Ω. The same result holds if $u \in C^2(\Omega\setminus\{p_1, \ ..., \ p_k\})$, **harmonic** on $\Omega\setminus\{p_1, \ ..., \ p_k\}$, and is a bounded function on Ω.*

Remark 1.102 *(**Important.**) The above result is not correct in the case $n = 1$. For example, take $u(x) = |x|, \ x \in \Omega = (-\infty, \infty)$. Note that for $n = 1$, the fundamental solution of the Laplace equation $u''(x) = 0$ is given by the radial function*

$$u(x) = A|x| + B, \quad x \in (-\infty, \infty),$$

where $A, \ B$ are two arbitrary constants .

Example 1.103 *The function*

$$u(x, y) = \frac{x}{x^2 + y^2} \ (or \ \frac{y}{x^2 + y^2}), \quad (x, y) \neq (0, 0) \in \mathbb{R}^2 \tag{202}$$

*is harmonic on $\mathbb{R}^2\setminus\{(0,0)\}$ and has a **point singularity** at $(0,0)$ (note that $u(x,y) = \frac{\partial}{\partial x}(\log r)$). Along each line $y = mx, \ m \in (-\infty, \infty)$, we have*

$$\lim_{x\to 0} u(x, mx) = \lim_{x\to 0} \frac{x}{(1+m^2)x^2} = \frac{1}{1+m^2}\lim_{x\to 0}\frac{1}{x} = \pm\infty.$$

*It is **impossible** to define $u(x,y)$ at $(0,0)$ so that it becomes harmonic on the whole plane $\mathbb{R}^2$. One can also see that the condition (196) cannot be satisfied, i.e.*

$$\lim_{r\to 0}\left(\frac{\max_{\partial B_r}|u|}{\log r}\right) = \lim_{r\to 0}\left(\frac{\max_{\theta\in[0,2\pi]}\left|\frac{r\cos\theta}{r^2}\right|}{\log r}\right) = \lim_{r\to 0}\left(\frac{1}{r\cdot\log r}\right) = -\infty \neq 0. \tag{203}$$

Therefore, the harmonic function $u(x,y)$ has a singularity at $(0,0)$ which blows up faster than the fundamental solution $\log r$ at $(0,0)$.

As a consequence of Theorem 1.96, we have the following result on the **singularity behavior of harmonic functions:**

Theorem 1.104 *(**Singularity behavior of harmonic functions.**) Assume $n \geq 2$. Suppose u is **harmonic** in $B_R(0)\setminus\{0\} \subset \mathbb{R}^n$ and it has a **point singularity** at $x = 0 \in \mathbb{R}^n$. Then, for $n = 2$, the limit*

$$\lim_{r\to 0}\left(\frac{\max_{\partial B_r}|u|}{\log r}\right) \tag{204}$$

either exists with value $L \neq 0$ (for example, $u(x) = 5\log|x|, \ L = 5, \ x \in \mathbb{R}^2$) or does not exists (usually, it is either $+\infty$ or $-\infty$). Similarly, for $n \geq 3$, the limit

$$\lim_{r\to 0}\left(\frac{\max_{\partial B_r}|u|}{r^{2-n}}\right) \tag{205}$$

either exists with value $L \neq 0$ or does not exists (usually, it is either $+\infty$ or $-\infty$).

1.20 Growth behavior of harmonic functions on $\mathbb{R}^2$ as $|x| \to \infty$.

As an application of Theorem 1.96, we have the following:

Theorem 1.105 *Assume $u : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function on $\mathbb{R}^2$ and satisfies*

$$\lim_{r\to+\infty} \left(\frac{\max_{\partial B_r} |u|}{\log r} \right) = 0, \tag{206}$$

*where B_r is the open ball centered at $x = 0$ with radius $r > 0$. Then it must be a **constant function**.*

Remark 1.106 *(**Be caregul.**) Note that the function $u(\mathbf{x})$ in the above must be defined on all $\mathbb{R}^2$, otherwise it is not correct. For example, the function $u(r,\theta) = r^{-n}\cos n\theta,\ n \in \mathbb{N}$, is harmonic and defined on $\mathbb{R}^2 \backslash \{(0,0)\}$ satisfying (206), but it is **not** a constant function.*

Remark 1.107 *For $n \geq 3$, the condition*

$$\lim_{r\to+\infty} \left(\frac{\max_{\partial B_r} |u|}{r^{2-n}} \right) = 0, \qquad n > 2,$$

*is **trivial** because it will imply that u is bounded on $\mathbb{R}^n$ and so it must be a **constant function**.*

Proof. Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\} \backslash \{0\}$ and let

$$v(x) = u\left(\frac{x}{|x|^2}\right), \quad x \in \Omega, \quad \left|\frac{x}{|x|^2}\right| = \frac{1}{|x|} > 1, \quad v(r) = u\left(\frac{1}{r}\right).$$

By Lemma 1.29, the function $v(x)$ is also harmonic on Ω. It satisfies

$$\begin{aligned}
&\lim_{r\to 0} \left(\frac{\max_{\partial B_r} |v|}{\log r} \right) = \lim_{r\to 0} \left(\frac{\max_{\partial B_{1/r}} |u|}{\log r} \right) \\
&= \lim_{r\to 0} \left(\frac{\max_{\partial B_{1/r}} |u|}{-\log 1/r} \right) \left(\text{let } s = \frac{1}{r} \right) = \lim_{s\to+\infty} \left(-\frac{\max_{\partial B_s} |u|}{\log s} \right) = 0.
\end{aligned}$$

By Theorem 1.96, $v(x)$ can be defined at $x = 0$ such that it is C^2 and harmonic in the whole open ball $\{x \in \mathbb{R}^2 : |x| < 1\}$. In particular, $v(x)$ is bounded near $x = 0$ and so $u(x)$ is **bounded** near $|x| = +\infty$. Hence $u : \mathbb{R}^2 \to \mathbb{R}$ is a bounded harmonic function on the entire space $\mathbb{R}^2$ and it must be a constant function. □

<u>This is the end of elliptic equations.</u>