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# On characteristic $p$ multizeta values

By

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**Abstract**

In this article, we consider the characteristic $p$ multizeta values introduced by Thakur. We report some recent progress on the analogue of Goncharov's conjecture and a criterion of Eulerian multizeta values. Methods and key ingredients of the proofs are also discussed.

## § 1. Introduction

### § 1.1. Multiple zeta values

Classical multiple zeta values (abbreviated as MZVs) are generalizations of the special values of the Riemann zeta function at positive integers. Precisely, they are defined by

$$\zeta(s_1,\ldots,s_r) := \sum_{n_1>\cdots>n_r\geq 1} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}},$$

where $s_1,\ldots,s_r$ are positive integers with $s_1 \geq 2$. Here $r$ is called the depth and $\sum_{i=1}^{r} s_i$ is called the weight of the MZV $\zeta(s_1,\ldots,s_r)$. These values can be expressed as Chen integrals. Studying the algebraic relations among the MZVs is one of the main themes in this topic. To explain the motivation for the surveyed contents, we first mention two famous conjectures concerning the MZVs.

Let $\mathcal{Z}_w$ be the $\mathbb{Q}$-vector space spanned by all the weight $w$ MZVs, and let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space spanned by 1 and all MZVs. One can see from the defining series

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that the product of two MZVs can be expressed as a $\mathbb{Z}$-linear combination of MZVs and hence $\mathcal{Z}$ has a $\mathbb{Q}$-algebra structure. More precisely, one has

$$\mathcal{Z}_w \mathcal{Z}_{w'} \subseteq \mathcal{Z}_{w+w'}.$$

There are two famous conjectures on MZVs: one is Zagier's dimension conjecture, and the other is Goncharov's direct sum conjecture.

**Conjecture 1.1.** (Zagier) *Set $d_0 = 1, d_1 = 0, d_2 = 1$ and let $d_w$ be defined by the recursive relation*

$$d_w = d_{w-2} + d_{w-3} \text{ for } w \geq 3.$$

*Then*

$$d_w = \dim_{\mathbb{Q}} \mathcal{Z}_w \text{ for } w \geq 2.$$

Note that Terasoma [27] and Goncharov [22] showed that $\dim_{\mathbb{Q}} \mathcal{Z}_w \leq d_w$ for each $w \geq 2$.

**Conjecture 1.2.** (Goncharov [21]) *$\mathcal{Z}$ is a graded algebra over $\mathbb{Q}$, i.e.,*

$$\mathcal{Z} = \mathbb{Q} \oplus \bigoplus_{w \geq 2} \mathcal{Z}_w.$$

In other words, the $\mathbb{Q}$-linear relations among the MZVs are coming from the $\mathbb{Q}$-linear relations among the same weight MZVs. The following conjecture (folklore) is a stronger form of Goncharov's conjecture.

**Conjecture 1.3.** *Let $\overline{\mathcal{Z}}$ be the $\overline{\mathbb{Q}}$-vector space spanned by 1 and all MZVs, and $\overline{\mathcal{Z}}_w$ be the $\overline{\mathbb{Q}}$-vector space spanned by all the weight $w$ MZVs. Then we have that*

$$\overline{\mathcal{Z}} = \overline{\mathbb{Q}} \oplus_{w \geq 2} \overline{\mathcal{Z}}_w$$

*and $\overline{\mathcal{Z}}$ is defined over $\mathbb{Q}$ in the sense that the canonical map $\mathcal{Z} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \overline{\mathcal{Z}}$ is bijective.*

That is, conjecturally all $\overline{\mathbb{Q}}$-polynomial relations among MZVs are $\mathbb{Q}$-homogeneous.

There is one more interesting open problem concerning MZVs. Euler showed that for any $n \in \mathbb{N}$,

$$\zeta(2n)/(2\pi\sqrt{-1})^{2n} \in \mathbb{Q}.$$

Therefore, we shall call a MZV $Z$ *Eulerian* if we have

$$Z/(2\pi\sqrt{-1})^w \in \mathbb{Q},$$

where the $w$ is the weight of $Z$. Since $Z$ is a real number, it follows that if $Z/(2\pi\sqrt{-1})^w \in \mathbb{Q}$, then $w$ must be even. Therefore, in the depth one case we have that for an integer $s \geq 2$, $\zeta(s)/(2\pi\sqrt{-1})^s \in \mathbb{Q}$ if and only if $s$ is even. However, according to the present literatures there is no expected criterion to determine when a given MZV of depth $\geq 2$ is Eulerian.

**Question 1.4.** *Does one have a criterion which can determine when a given MZV of depth $\geq 2$ is Eulerian?*

Recently, the author of the present article proved a function field analogue of Conjecture 1.3 (cf. [12]). Moreover, the authors of [19] establish a criterion of characteristic $p$ Eulerian MZVs. The present paper is a survey on the results mentioned above as well as the strategy of proofs.

### § 1.2. Overview

In § 2, we first review the known results for the depth one MZVs in positive characteristic, the so-called Carlitz zeta values. We then introduce the characteristic $p$ MZVs defined by Thakur and state the result on the analogous question of Conjecture 1.3 (cf. Theorem 2.4).

The aim of § 3 is to state the criterion of Eulerian MZVs. We first introduce the category of $\mathcal{F}$-modules (Frobenius modules), and then introduce the works of Anderson-Thakur [3, 4] in order to set up the necessary foundation. The criterion of characteristic $p$ Eulerian MZVs is stated as Theorem 3.5.

In § 4 we introduce the recent developments and tools in the transcendence theory in positive characteristic. We also review some classical theories for comparison. The materials contain

- Wüstholz's analytic subgroup theorem [33, 34].
- Yu's sub-$t$-module theorem [36].
- Classical Siegel-Shidlovskii theory [6].
- ABP criterion [2].
- Papanikolas' difference Galois theory [24].

In the final section, we sketch the ideas how to prove Theorems 2.4 and 3.5.

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## § 2. Linear independence of multizeta values

This is the theory of multizeta values in characteristic $p$. The arithmetic here comes from the polynomial ring $A = \mathbb{F}_q[\theta]$, where $\mathbb{F}_q$ is the finite field of $q$ elements with characteristic $p$ and $\theta$ is a variable. Here $A$ plays the role of the ring of integers $\mathbb{Z}$ in the classical case. It sits discretely inside the Laurent series field $\mathbb{F}_q((\frac{1}{\theta}))$ with respect to the $\frac{1}{\theta}$-adic topology just as $\mathbb{Z}$ is discretely inside the real line $\mathbb{R}$. The fraction field of $A$ is denoted by $k$, which plays the role of rational number field $\mathbb{Q}$. Throughout this paper, we fix an algebraic closure $\overline{k_\infty}$ of $k_\infty$, and denote by $\bar{k}$ a fixed algebraic closure of $k$ embedded in $\overline{k_\infty}$. Finally, we let $\mathbb{C}_\infty$ be the completion of $\overline{k_\infty}$ with respect to the absolute value extending a given one on $k_\infty$ associated to the $\frac{1}{\theta}$-adic topology. Note that $\mathbb{C}_\infty$ is an algebraically closed field that plays the role of the complex numbers $\mathbb{C}$. See [23, 28].

In what follows, without confusion we still use the symbol $\zeta$ for zeta in the characteristic $p$ setting.

## § 2.1. Carlitz theory

Let $A_+$ be the set of all monic polynomials in $A$, which plays the role of the set of positive integers $\mathbb{N}$. In [9], Carlitz considered the following series: for $n \in \mathbb{N}$,

$$\zeta(n) := \sum_{a \in A_+} \frac{1}{a^n} \in k_\infty.$$

Notice that since we are in the setting of non-archimedean analysis, the series above also converges at $n = 1$ and one observes that $\zeta(n)$ is non-vanishing. In the classical case, by the work of Euler, the special value of the Riemann zeta function at an even positive integer $2m$ can be expressed in terms of $(2\pi\sqrt{-1})^{2m}$ and the Bernoulli number $B_{2m}$. Note that $2\pi\sqrt{-1}$ is the period of the exponential function for the multiplicative group $\mathbb{G}_m$:

$$0 \to \mathbb{Z} \cdot 2\pi\sqrt{-1} \to \mathbb{C} \twoheadrightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \to 1.$$

Let $\mathbb{G}_a$ be the additive group over $A$. The Carlitz module $C$ is defined to be the group scheme $\mathbb{G}_a$ equipped with a nontrivial $A$-module structure given by

$$\theta *_C x = \theta x + x^q \text{ and } \xi *_C x = \xi x \text{ for } x \in \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty, \xi \in \mathbb{F}_q.$$

One has the Carlitz exponential function

$$\exp_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i},$$

where $D_0 := 1$ and $D_i := \prod_{j=0}^{i-1} \left(\theta^{q^i} - \theta^{q^j}\right)$. From non-archimedean analysis, it is not hard to see that $\exp_C$ converges on whole $\mathbb{C}_\infty$ and therefore is surjective onto $\mathbb{C}_\infty$. Carlitz showed that

$$\exp_C(az) = a *_C \exp_C(z) \text{ for all } a \in A.$$

In other words, the Carlitz exponential

$$\exp_C : \mathbb{C}_\infty \twoheadrightarrow C(\mathbb{C}_\infty) = \mathbb{C}_\infty$$

is an analytic $A$-module homomorphism. The kernel $\Lambda_C := \operatorname{Ker}(\exp_C)$ is shown to be a discrete $A$-module of rank one in $\mathbb{C}_\infty$, and its generator (unique up to scalar multiple by $\mathbb{F}_q^\times$) can be expressed as

$$\tilde{\pi} = (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1}, \tag{2.1}$$

where $(-\theta)^{\frac{1}{q-1}}$ is a fixed $(q-1)$-st root of $(-\theta)$ throughout this paper. In the function field situation, $C$ plays the role of $\mathbb{G}_m$ and $\tilde{\pi}$ plays the role of $2\pi\sqrt{-1}$.

Denote by $A^\times$ the unit group of $A$ and note that the cardinality of $A^\times$ is $q-1$. For positive integers $n$ divisible by $q-1$, we shall call them "*even*" in this function field setting. Carlitz established an analogue of the classical Euler's formula on the special values of the Riemann zeta function at even positive integers: for $n \in \mathbb{N}$ "*even*", one has

$$\zeta(n) = \frac{B_n}{\Gamma_{n+1}} \tilde{\pi}^n, \tag{2.2}$$

where

$$\Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i} \in A \tag{2.3}$$

for $n = \sum n_i q^i$ with $0 \le i \le q-1$ and $B_n \in k$ is defined by

$$\frac{z}{\exp_C(z)} = \sum_{n=0}^{\infty} \frac{B_n}{\Gamma_{n+1}} z^n.$$

## § 2.2. Transcendence theory for Carlitz zeta values

As an analogue of the transcendence of $2\pi\sqrt{-1}$, Wade [32] showed that $\tilde{\pi}$ is transcendental over $k$ and hence by (2.2) each $\zeta(n)$ for $n \in \mathbb{N}$ "*even*" is also transcendental over $k$. The breakthrough on the transcendence of all Carlitz zeta values, particularly the $\zeta(n)$ for $n$ "*odd*" (i.e. $(q-1) \nmid n$), was due to Jing Yu.

**Theorem 2.1.** (Yu [35]) *For each $n \in \mathbb{N}$, $\zeta(n)$ is transcendental over $k$.*

Later on, Yu used his far-reaching result which is the so-called sub-$t$-module theorem to obtain the following $\bar{k}$-linear independence result.

**Theorem 2.2.** (Yu [36]) *Given any positive integers $m$ and $n$, we have*

$$\dim_{\bar{k}} \bar{k}\text{-Span}\,\{1,\tilde{\pi},\cdots,\tilde{\pi}^m,\zeta(1),\cdots,\zeta(n)\} = 1+m+n-\lfloor\frac{\min\{m,n\}}{q-1}\rfloor.$$

*In other words, the $\bar{k}$-linear relations among the set $\{1,\tilde{\pi},\cdots,\tilde{\pi}^m,\zeta(1),\cdots,\zeta(n)\}$ are coming from the Euler-Carlitz relations (2.2).*

Since our field is of characteristic $p$, there are also natural relations valid there:

$$\zeta(pn)=\zeta(n)^p \text{ for any } n\in\mathbb{N}. \tag{2.4}$$

The algebraic relations (2.4) among the Carlitz zeta values are called Frobenius $p$-th power relations. Using the powerful tools developed by Papanikolas [24], Chang and Yu completely determined all the algebraic relations among the Carlitz zeta values.

**Theorem 2.3.** (Chang-Yu [20]) *All the algebraic relations among the Carlitz zeta values are those coming from the Euler-Carlitz relations (2.2) and Frobenius $p$-th power relations (2.4). In particular, for $n\in\mathbb{N}$ we have*

$$\operatorname{tr.deg}_{\overline{k}}\overline{k}\,(\tilde{\pi},\zeta(1),\cdots,\zeta(n)) = 1+n-\lfloor\frac{n}{q-1}\rfloor-\lfloor\frac{n}{p}\rfloor+\lfloor\frac{n}{p(q-1)}\rfloor.$$

*Remark.* In the classical case, conjecturally the Euler relations account for all algebraic relations among the special values of the Riemann zeta function at positive integers ($\geq 2$). In other words, for an integer $n\geq 2$ conjecturally one has

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}\left(2\pi\sqrt{-1},\zeta(2),\ldots,\zeta(n)\right)=n-\lfloor\frac{n}{2}\rfloor.$$

### § 2.3. Transcendence theory for multizeta values

Let $A_+$ be the set of all monic polynomials in $A$. It plays the role of the set of positive integers. In [28], Thakur defined the characteristic $p$ multizeta values: for any $r$-tuple of positive integers $(s_1,\cdots,s_r)\in\mathbb{N}^r$,

$$\zeta(s_1,\ldots,s_r):=\sum\frac{1}{a_1^{s_1}\cdots a_r^{s_r}}\in k_\infty,$$

where the sum is over $(a_1,\ldots,a_r)\in A_+^r$ with $\deg a_1>\cdots>\deg a_r$. (Here deg means the degree of a given polynomial in the variable $\theta$). We call this MZV having *depth* $r$ and *weight* $\sum_{i=1}^r s_i$. In the case of $r=1$, the values above are the Carlitz zeta values at positive integers.

Note that there are no natural orders on $A_+$ and thus the following two results due to Thakur [29, 31] are nontrivial although the classical counterparts are immediate consequence of the defining series.

(a) Each MZV $\zeta(s_1, \cdots, s_r)$ is nonzero.

(b) The product of a weight $w_1$ MZV and a weight $w_2$ MZV can be expressed as an $\mathbb{F}_p$-linear combination of weight $w_1 + w_2$ MZVs.

Let $Z_i$ be a MZV of weight $w_i$ for $i = 1, \cdots, n$. For positive integers $m_1, \cdots, m_n$, we define the total weight of the monomial $Z_1^{m_1} \cdots Z_n^{m_n}$ to be $\sum_{i=1}^n m_i w_i$. For a positive integer $w$, we let $\mathcal{Z}_w$ (resp. $\overline{\mathcal{Z}}_w$) be the $k$-vector space (resp. $\bar{k}$-vector space) spanned by the weight $w$ MZVs. Let $\mathcal{Z}$ (resp. $\overline{\mathcal{Z}}$) be the $k$-vector space (resp. $\bar{k}$-vector space) spanned by 1 and all MZVs. The property (b) implies that $\mathcal{Z}$ (resp. $\overline{\mathcal{Z}}$) is a $k$-algebra (resp. $\bar{k}$-algebra). In [12], the author of the present paper proved a characteristic $p$ analogue of Conjecture 1.3.

**Theorem 2.4.** *Let $w_1, \ldots, w_\ell$ be $\ell$ distinct positive integers. Let $V_i$ be a finite set consisting of some monomials of multizeta values of total weight $w_i$ for $i = 1, \ldots, \ell$. If $V_i$ is a linearly independent set over $k$ for $i = 1, \ldots, \ell$, then the set*

$$\{1\} \cup \bigcup_{i=1}^{\ell} V_i$$

*is linearly independent over $\bar{k}$. In particular, we have that*

$$\overline{\mathcal{Z}} = \bar{k} \oplus \bigoplus_{w \in \mathbb{N}} \overline{\mathcal{Z}}_w$$

*and $\overline{\mathcal{Z}}$ is defined over $k$ in the sense that the canonical map*

$$\mathcal{Z} \otimes_k \bar{k} \to \overline{\mathcal{Z}} \text{ is bijective.}$$

### § 2.4. Remark on algebraic relations among MZVs

Let $\mathfrak{Z}_w$ be the $k$-vector space spanned by the monomials of MZVs of total weight $w$. By the property (b) in the previous subsection we see that $\mathfrak{Z}_w = \mathcal{Z}_w$. The main goal of transcendence theory for MZVs is to understand and determine all the $\bar{k}$-algebraic relations among the MZVs. Note that $\bar{k}$-algebraic relations among MZVs can be regarded as $\bar{k}$-linear relations among the monomials of MZVs. So Theorem 2.4 implies that all the $\bar{k}$-algebraic relations among the MZVs are coming from the $k$-linear relations among the same weight MZVs. That is, the whole program of transcendence theory for MZVs boils down to the following question, which is still open.

**Question 2.5.** *What is the dimension* $\dim_k \mathcal{Z}_w$ *for each $w \in \mathbb{N}$?*

*Remark.* Unlike the classical case such as Conjecture 1.1, one does not know what the expected answer of Question 2.5 should be.

The following result is a consequence of Theorem 2.4.

**Corollary 2.6.** *Let $Z_1$ and $Z_2$ be two multizeta values of the same weight. Then either the ratio $Z_1/Z_2$ is in $k$ or $Z_1$ and $Z_2$ are algebraically independent over $k$.*

*Proof.* Suppose that $Z_1/Z_2 \notin k$. By Theorem 2.4 the ratio $Z_1/Z_2$ is transcendental over $k$. Suppose on the contrary that $Z_1$ and $Z_2$ are algebraically dependent over $k$. Then by Theorem 2.4 there exists a homogenous polynomial $F(X,Y) \in k[X,Y]$ of positive degree so that $F(Z_1,Z_2) = 0$. Let $d$ be the total degree of $F$. Then dividing the equation $F(Z_1,Z_2) = 0$ by $Z_2^d$ we see that the ratio $Z_1/Z_2$ satisfies a nontrivial polynomial over $k$, whence a contradiction. □

For a MZV $Z$ of weight $w$, we call $Z$ "Eulerian" if $Z/\tilde{\pi}^w \in k$. Note that because of (2.1) we have that $\tilde{\pi}^w \notin k_\infty$ if and only if $w$ is "*odd*". It follows that if $Z$ is Eulerian, then $w$ must be "*even*". The following result is an interesting phenomenon called *Euler dichotomy* ([12, Cor. 2.3.3]).

**Theorem 2.7.** *Every multizeta value is either Eulerian or is algebraically independent from $\tilde{\pi}$. In particular, every multizeta value of "odd" weight is algebraically independent from $\tilde{\pi}$.*

**Question 2.8.** *One can ask if the classical MZVs have the analogous Euler dichotomy as above.*

## § 3. Criterion of Eulerian multizeta values

In what follows, we will state an algebraic criterion that determines when a given multizeta value is Eulerian (cf. [19]).

### § 3.1. Rationality and algebraicity of $Z/\tilde{\pi}^w$

Concerning the Eulerian MZVs, Carlitz gave a clear description in the depth one case (see (2.2)):

**Theorem 3.1.** (Carlitz [9]) *Let $s$ be a positive integer. Then $\zeta(s)$ is Eulerian if and only if $s$ is "even".*

In [35], Yu proved that the algebraicity of $\zeta(s)/\tilde{\pi}^s$ implies the rationality.

**Theorem 3.2.** (Yu [35]) *Let $s$ be a positive integer. Then we have that*

$$\zeta(s)/\tilde{\pi}^s \in k \text{ if and only if } \zeta(s)/\tilde{\pi}^s \in \bar{k}.$$

The following result, which is a consequence of Theorem 2.7, is a generalization of Yu's theorem above for any MZV of arbitrary depth.

**Theorem 3.3.** *Let $Z$ be a MZV of weight $w$. Then we have that*

$$Z/\tilde{\pi}^w \in k \text{ if and only if } Z/\tilde{\pi}^w \in \bar{k}.$$

*Remark.* For the classical MZVs, conjecturally one expects that

$$\zeta(s_1, \cdots, s_r)/(2\pi\sqrt{-1})^{s_1+\cdots+s_r} \in \mathbb{Q} \text{ if and only if } \zeta(s_1, \cdots, s_r)/(2\pi\sqrt{-1})^{s_1+\cdots+s_r} \in \overline{\mathbb{Q}}.$$

## § 3.2. Frobenius modules

Let $t$ be a new variable independent from $\theta$. We consider the Laurent series field $\mathbb{C}_\infty((t))$ and equip it with a Frobenius twisting automorphism:

$$\begin{aligned} \sigma : \mathbb{C}_\infty((t)) &\to \mathbb{C}_\infty((t)) \\ f := \textstyle\sum_i a_i t^i &\mapsto \sigma(f) := \textstyle\sum_i a_i^{\frac{1}{q}} t^i. \end{aligned}$$

For convenience we write $f^{(-n)} := \sigma^n(f)$ for an integer $n$. We then extend such twisting to the matrices with entries in $\mathbb{C}_\infty((t))$ by entrywise twisting. The Frobenius twisting operation stabilizes several subrings and subfields such as $\overline{k}[t]$, the Tate algebra $\mathbb{T}$ of power series over $\mathbb{C}_\infty$ convergent on the closed unit disk, $\overline{k}(t)$ and the fraction field of $\mathbb{T}$ denoted by $\mathbb{L}$. For their invariants under $\sigma$, we note:

$$\overline{k}[t]^\sigma = \mathbb{F}_q[t],\ \mathbb{T}^\sigma = \mathbb{F}_q[t],\ \overline{k}(t)^\sigma = \mathbb{F}_q(t),\ \mathbb{L}^\sigma = \mathbb{F}_q(t).$$

**Definition 3.4.** An $\digamma$-module is a pair $(M, \digamma)$ equipped with the following two properties:

- $M$ is a free left $\overline{k}[t]$-module of finite rank;
- $\digamma : M \to M$ is a $\sigma$-semilinear map, i.e., $\digamma$ is additive and satisfies $\digamma(am) = a^{(-1)}\digamma(m)$ for $a \in \overline{k}[t]$, $m \in M$.

A morphism of $\digamma$-modules is a left $\overline{k}[t]$–module homomorphism that is compatible with the $\digamma s$. We denote by $\mathfrak{F}$ the category consisting of all $\digamma$-modules.

We mention that our $\digamma$-modules here are slightly different from the terminology in [26], but their concepts are the same. The notion of such $\digamma$-modules originated from the theory of $t$-motives initiated by Anderson [1]. The simplest example of an $\digamma$-module is the trivial object denoted by $\mathbf{1}$, where the underlying space of $\mathbf{1}$ is $\overline{k}[t]$, and the action of $\digamma$ on $\mathbf{1}$ is given as

$$\digamma(f) := f^{(-1)} \text{ for } f \in \overline{k}[t].$$

Another example is the $n$-th tensor power of the Carlitz motive denoted by $C^{\otimes n}$. Here $n$ is a positive integer, and the underlying space of $C^{\otimes n}$ is $\overline{k}[t]$ equipped with the $\sigma$-action given by

$$\sigma(f) := (t-\theta)^n f^{(-1)} \text{ for } f \in C^{\otimes n}.$$

Let $M$ be an $\sigma$-module. We fix a $\overline{k}[t]$-basis $\{m_1, \ldots, m_r\}$ of $M$. Then the action of $\sigma$ on this basis is represented as

$$\sigma \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = \Phi \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix}$$

for some matrix $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$. Conversely, a matrix $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ determines an object $M$ in $\mathfrak{F}$, where $M$ is free of rank $r$ over $\overline{k}[t]$ and the action of $\sigma$ on certain $\overline{k}[t]$-basis of $M$ is represented by the matrix $\Phi$. In this case, we shall say that $M$ is defined by $\Phi$.

## § 3.3. The criterion

**3.3.1. Anderson-Thakur polynomials** $H_n$ In what follows, we briefly review the theory of Anderson-Thakur [3, 4]. For $n = 0, 1, 2, \ldots$, we recall that $\Gamma_{n+1}$ is defined in (2.3) and now we define the sequence of Anderson-Thakur polynomials $H_n \in A[t]$ by the generating function identity

$$\sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}|_{\theta=t}} x^n := \left(1 - \sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{(t^{q^i} - \theta^{q^j})}{(t^{q^i} - t^{q^{j-1}})} x^{q^i}\right)^{-1}.$$

Note that for $0 \leq n \leq q-1$ we have $H_n = 1$. We shall mention that we make change of notation by $t \leftarrow T$ in [4, (2)] in order to match the notation in [12].

Put

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{C}_\infty[\![t]\!]$$

and note that it is entire on $\mathbb{C}_\infty$ and satisfies the functional equation

$$\Omega^{(-1)} = (t-\theta)\Omega.$$

We further note that $\tilde{\pi} = 1/\Omega(\theta)$.

The important identity developed in [3] is that for each positive integer $n$, the polynomial $H_n(t) \in A[t]$ satisfies

$$(H_{s-1}\Omega^s)^{(d)}(\theta) = \frac{\Gamma_s S_d(s)}{\tilde{\pi}^s} \text{ for any } s, d \in \mathbb{N}, \tag{3.1}$$

where

$$S_d(s) := \sum_{\substack{a \in A_+ \\ \deg a = d}} \frac{1}{a^s} \in k.$$

It follows that the specialization of the following series

$$\sum_{i_1 > \cdots > i_r \geq 0} (\Omega^{s_r} H_{s_r - 1})^{(i_r)} \cdots (\Omega^{s_1} H_{s_1 - 1})^{(i_1)}$$

at $t = \theta$ gives

$$\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta(s_1, \ldots, s_r) / \tilde{\pi}^{s_1 + \cdots + s_r}.$$

**3.3.2. The $\mathrm{Ext}^1$-group** Fix an $r$-tuple $\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{N}^r$. We define the following matrix:

(3.2)

$$\Phi_{\mathbf{s}} := \begin{pmatrix} (t-\theta)^{s_1 + \cdots + s_r} & 0 & 0 & \cdots & 0 \\ H_{s_1 - 1}^{(-1)} (t-\theta)^{s_1 + \cdots + s_r} & (t-\theta)^{s_2 + \cdots + s_r} & 0 & \cdots & 0 \\ 0 & H_{s_2 - 1}^{(-1)} (t-\theta)^{s_2 + \cdots + s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \cdots & 0 & H_{s_r - 1}^{(-1)} (t-\theta)^{s_r} & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{k}[t]).$$

Let $\Phi_{\mathbf{s}'}$ be the square matrix of size $r$ cut from the upper left square of $\Phi_{\mathbf{s}}$:

(3.3)

$$\Phi_{\mathbf{s}'} := \begin{pmatrix} (t-\theta)^{s_1 + \cdots + s_r} & & & \\ H_{s_1 - 1}^{(-1)} (t-\theta)^{s_1 + \cdots + s_r} & (t-\theta)^{s_2 + \cdots + s_r} & & \\ & \ddots & \ddots & \\ & & H_{s_{r-1} - 1}^{(-1)} (t-\theta)^{s_{r-1} + s_r} & (t-\theta)^{s_r} \end{pmatrix} \in \mathrm{Mat}_r(\overline{k}[t]).$$

Denote by $M_{\mathbf{s}}$ and $M_{\mathbf{s}'}$ the objects in $\mathfrak{F}$ defined by the matrices $\Phi_{\mathbf{s}}$ and $\Phi_{\mathbf{s}'}$ respectively. Note that the $M_{\mathbf{s}}$ fits into the short exact sequence

$$0 \to M_{\mathbf{s}'} \to M_{\mathbf{s}} \twoheadrightarrow \mathbf{1} \to 0.$$

and so $M_{\mathbf{s}}$ belongs to $\mathrm{Ext}^1_{\mathfrak{F}}(\mathbf{1}, M_{\mathbf{s}})$. We note that the group $\mathrm{Ext}^1_{\mathfrak{F}}(\mathbf{1}, M_{\mathbf{s}})$ has a natural $\mathbb{F}_q[t]$-module structure coming from the Baer sum and pushout of morphisms of $M_{\mathbf{s}'}$. More precisely, if $M_1$ and $M_2$ are two objects in $\mathrm{Ext}^1_{\mathfrak{F}}(\mathbf{1}, M_{\mathbf{s}'})$ defined by the following two matrices respectively:

$$\Phi_1 := \begin{pmatrix} \Phi_{\mathbf{s}'} & 0 \\ \mathbf{v}_1 & 1 \end{pmatrix}, \quad \Phi_2 := \begin{pmatrix} \Phi_{\mathbf{s}'} & 0 \\ \mathbf{v}_2 & 1 \end{pmatrix}.$$

Then the Baer sum $M_1 +_B M_2$ is the object in $\mathfrak{F}$ defined by the matrix:

$$\begin{pmatrix} \Phi_{\mathbf{s}'} & 0 \\ \mathbf{v}_1 + \mathbf{v}_2 & 1 \end{pmatrix}.$$

Furthermore, for any $a \in \mathbb{F}_q[t]$ the action of $a$ on $M_1$ is the object $a * M_1 \in \mathfrak{F}$ defined by the matrix:

$$\begin{pmatrix} \Phi_{\mathbf{s}'} & 0 \\ a\mathbf{v}_1 & 1 \end{pmatrix}.$$

In [19], a criterion of characteristic $p$ Eulerian multizeta values is established. The result is stated as follows.

**Theorem 3.5.** *$\zeta(s_1, \cdots, s_r)$ is Eulerian if and only if $M_{\mathbf{s}}$ is a torsion element in the $\mathbb{F}_q[t]$-module $\mathrm{Ext}^1_{\mathfrak{F}}(\mathbf{1}, M_{\mathbf{s}'})$.*

**Corollary 3.6.** *If $\zeta(s_1, \ldots, s_r)$ is Eulerian, then*

$$\zeta(s_2, \ldots, s_r), \zeta(s_3, \ldots, s_r), \cdots, \zeta(s_r)$$

*are simultaneously Eulerian and so $s_i$ is "even"for all $i = 1, \ldots, r$.*

*Proof.* The proof can be seen in the proof of Theorem 3.5. □

## § 4. Current developments on methods of transcendence theory

In this section, we will introduce the current methods of transcendence theory over function fields in positive characteristic. The first systematic development is Yu's theorem, the so-called sub-$t$-module theorem. It is a function field analogue of Wüstholz's analytic subgroup theorem. So we first introduce Wüstholz's theory in order to motivate Yu's theorem. We list the order of the introductions of this section as follows.

- Wüstholz's analytic subgroup theorem [33, 34].
- Yu's sub-$t$-module theorem [36].
- Siegel-Shidlovskii theory [6].
- ABP criterion [2].
- Papanikolas' difference Galois theory [24].

In [2], Anderson, Brownawell and Papanikolas developed a linear independence criterion, the so-called ABP criterion. The authors of [2] mentioned that they came up with the criterion in the process of searching for a $t$-motivic translation of Yu's sub-$t$-module theorem, and they are inclined to believe that at the end of the day the ABP criterion and Yu's sub-$t$-module theorem differ insignificantly in terms of ability to detect $\bar{k}$-linear independence. We mention that ABP criterion can be regarded as a special case (with restricted conditions) of the function field analogue of the Siegel-Shidlovskii criterion on $E$-functions. A refined version of the ABP criterion which relaxes the conditions is given in [10].

In the final part of this section, we will introduce Papanikolas' theory, which can be regarded as a function field analogue of Grothendieck's periods conjecture. Some applications on algebraic independence results will be also mentioned.

## § 4.1. Wüstholz's analytic subgroup theorem

Let $G$ be a commutative algebraic group defined over a number field $K$. The set of $\mathbb{C}$-valued points of $G$, denoted by $G(\mathbb{C})$, can be viewed as a complex manifold for a given embedding of $K$ into $\mathbb{C}$. It follows that one can view $G(\mathbb{C})$ as a complex Lie group as the "multiplication"and inverse maps are holomorphic. Denote by $\mathfrak{g}$ the Lie algebra of the group variety $G$. Then the Lie algebra of the complex Lie group $G(\mathbb{C})$ is the complex vector space $\mathfrak{g} \otimes_K \mathbb{C}$. Both the Lie group and its Lie algebra are related by the exponential map

$$\exp_G : \mathfrak{g} \otimes_K \mathbb{C} \to G(\mathbb{C}),$$

which is defined using one-parameter subgroups.

The celebrated analytic subgroup theorem of Wüstholz is stated as follows.

**Theorem 4.1.** (Wüstholz [33, 34]) *Let* $\mathbf{u} \in \mathfrak{g} \otimes_K \mathbb{C}$ *satisfy* $\exp_G(\mathbf{u}) \in G(\overline{\mathbb{Q}})$*. Let* $T_{\mathbf{u}} \subset \mathfrak{g} \otimes_K \mathbb{C}$ *be the smallest vector subspace of* $\mathfrak{g} \otimes_K \mathbb{C}$ *defined over* $\overline{\mathbb{Q}}$ *and containing* $\mathbf{u}$*. Then* $T_{\mathbf{u}}$ *is the tangent space at the identity of an algebraic subgroup of* $G \times_K \overline{\mathbb{Q}}$ *that is defined over* $\overline{\mathbb{Q}}$*.*

Wüstholz's theorem above has many important applications. For example, one can give proofs of the following $\overline{\mathbb{Q}}$-linear independence results:

- Baker's theorem on linear forms of logarithms of algebraic numbers.
- $\overline{\mathbb{Q}}$-linear independence of elliptic logarithms at algebraic points.
- $\overline{\mathbb{Q}}$-linear independence of the periods of the first, second and third kinds for elliptic curves over $\overline{\mathbb{Q}}$.

For more details, we refer the reader to [5].

### § 4.2. Yu's sub-$t$-module theorem

We first review the theory of $t$-modules introduced by Anderson [1]. Let $\mathbb{G}_a$ be the additive group over $k$, and $K$ be a field extension of $k$. By a $t$-module of dimension $n$ over $K$, we mean a $t$-action given by a ring homomorphism

$$\phi : \mathbb{F}_q[t] \to \mathrm{End}(\mathbb{G}^n_{a/K})$$

satisfying the following conditions:

- For constants $\xi \in \mathbb{F}_q$, $\phi_\xi$ is the scalar multiplication by $\xi$.
- $(d\phi_t - \theta Id)^N \,\mathrm{Lie}(\mathbb{G}^n_a) = 0$ for some integer $N > 0$.

These $t$-modules $G = (\mathbb{G}^n_{a/K}, \phi)$ have exponential maps

$$\exp_G : \mathrm{Lie}\, G(\mathbb{C}_\infty) = \mathbb{C}^n_\infty \to \mathbb{C}^n_\infty = G(\mathbb{C}_\infty).$$

These are entire $\mathbb{F}_q$-linear maps defined on $\mathbb{C}^n_\infty$ and satisfying

$$\exp_G(d\phi_a(\mathbf{z})) = \phi_a\left(\exp_G(\mathbf{z})\right) \text{ for all } a \in \mathbb{F}_q[t], \mathbf{z} \in \mathbb{C}^n_\infty.$$

To each $t$-module $G$, there is a unique exponential map $\exp_G$ which depends functorially on $G$. We shall note that in general $\exp_G$ is not surjective. If it is surjective, then the $t$-module in question is called uniformizable.

A $t$-module $G = (\mathbb{G}^n_{a/\bar{k}}, \phi)$ defined over $\bar{k}$ is called *regular* if there is an integer $r$ such that for all $a \in \mathbb{F}_q[t]$ the kernel of $\phi_a$ in $G(\bar{k})$ is a free $A/(a)$-module of rank $r$. We shall mention that most $t$-modules interesting to us are regular. For instance, the Drinfeld $\mathbb{F}_q[t]$-modules (of generic characteristic), which are one-dimensional nontrivial $t$-modules defined over $\bar{k}$, are regular.

A connected algebraic subgroup of $\mathbb{G}^n_a$ invariant under $\phi_a$ for all $a \in A$, will be called a sub-$t$-module of $(\mathbb{G}^n_a, \phi)$. Yu's sub-$t$-module theorem is stated as follows:

**Theorem 4.2.** (Yu [36]) *Let $G = (\mathbb{G}^n_a, \phi)$ be a regular $t$-module of dimension $n$ defined over $\bar{k}$. Let $\mathbf{u}$ be a point in $\mathrm{Lie}\, G(\mathbb{C}_\infty)$ such that $\exp_G(\mathbf{u}) \in G(\bar{k})$ . Then the smallest vector subspace in $\mathrm{Lie}\, G(\mathbb{C}_\infty)$ defined over $\bar{k}$ which is invariant under $d\phi_t$ and contains $\mathbf{u}$ is the tangent space at the origin of a sub-$t$-module of $G$ that is defined over $\bar{k}$.*

Yu's theorem has many applications. For example, one can use it to show:

- $\bar{k}$-linear independence of Drinfeld logarithms at algebraic points [36].

- $\overline{k}$-linear independence of Carlitz zeta values [36].
- $\overline{k}$-linear independence of periods and quasi-periods of Drinfeld modules [7].
- $\overline{k}$-linear independence of special gamma values [8].

## § 4.3. Siegel-Shidlovskii theory

For an algebraic number $\alpha$ we denote by $\widehat{\alpha}$ the maximal complex absolute value of the conjugates of $\alpha$. An entire function $f(z)$ on $\mathbb{C}$ given by the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

with $a_n \in \overline{\mathbb{Q}}$ for all $n$, is called an $E$-function if

- $f$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;
- both $\widehat{a_n}$ and the common denominator $\mathrm{den}(a_0, a_1, \ldots, a_n)$ are bounded by $C^n$, where $C$ is a positive number depending only on $f$.

The typical example is the exponential function $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. The following celebrated theorem is a refined version of the classical Sigel-Shidlovskii criterion.

**Theorem 4.3.** (Beukers [6]) *Let $f_1, \ldots, f_n$ be a set of $E$-functions which satisfy the system of first-order differential equations*

$$\frac{d}{dz} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = B \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix},$$

*where $B$ is an $n \times n$ matrix with entries in $\overline{\mathbb{Q}}(z)$. Denote by $T(z)$ the common denominator of the entries of $B$. Then for any $\xi \in \overline{\mathbb{Q}}^{\times}$ such that $T(\xi) \neq 0$, any $\overline{\mathbb{Q}}$-linear relation among the values $f_1(\xi), \ldots, f_n(\xi)$ is the specialization of a linear relation among the functions $f_1, \ldots, f_n$ over $\overline{\mathbb{Q}}(z)$.*

## § 4.4. ABP-criterion

The first instance of a function field analogue of the Siegel-Shidlovskii criterion was invented by Anderson-Brownawell-Papanikolas [2] and so we call it the ABP criterion. Later on, the author of the present article followed the methods of [2] to give a refined version which relaxes the conditions of the ABP-criterion in [2]. Notice that the original version in [2] is to deal with the case of uniformizable $t$-motives and hence the restrictions of conditions come up naturally from $t$-motives.

**Theorem 4.4.** (Refined version of ABP criterion [2, 10]) *Fix a matrix* $\Phi = \Phi(t) \in \mathrm{Mat}_\ell(\bar{k}[t])$ *such that* $\det \Phi$ *is a polynomial in* $t$ *satisfying* $\det \Phi(0) \neq 0$. *Fix a vector* $\psi = [\psi_1(t), \cdots, \psi_\ell(t)]^{\mathrm{tr}} \in \mathrm{Mat}_{\ell\times 1}(\mathbb{T})$ *satisfying the functional equation* $\psi^{(-1)} = \Phi\psi$. *Let* $\xi \in \overline{k}^{\times} \setminus \overline{\mathbb{F}_q}^{\times}$ *satisfy*

$$\det \Phi(\xi^{\frac{1}{q^i}}) \neq 0 \text{ for all } i = 1, 2, 3, \cdots .$$

*Then for every vector* $\rho \in \mathrm{Mat}_{1\times\ell}(\bar{k})$ *such that* $\rho\psi(\xi) = 0$ *there exists a vector* $P = P(t) \in \mathrm{Mat}_{1\times\ell}(\bar{k}[t])$ *such that*

$$P(\xi) = \rho \text{ and } P\psi = 0.$$

*Remark.* Let $\Phi$ and $\psi$ be given as in the theorem above. Then all the entries of $\psi$ are convergent on whole $\mathbb{C}_\infty$ by [2, Prop. 3.1.3].

ABP criterion is a powerful tool when working on transcendence problems related to $t$-motives. For instance, it is the key ingredient which enables Papanikolas to establish the analogue of Grothendieck's periods conjecture (see the next subsection). Moreover, one can use it to show:

- Function field analogue of Rohrlich-Lang's conjecture [2].
- Function field analogue of Conjecture 1.3 [12].
- A criterion of Eulerian multizeta values [19].

### § 4.5. Papanikolas' theory

In what follows, we fix a matrix $\Phi \in \mathrm{Mat}_r(\bar{k}[t]) \cap \mathrm{GL}_r(\bar{k}(t))$ and suppose that there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfying the systems of Frobenius difference equations

$$\sigma\Psi = \Psi^{(-1)} = \Phi\Psi.$$

We let $Z_\Psi$ be the smallest closed subscheme of $\mathrm{GL}_r / \bar{k}(t)$ so that $\Psi$ is an $\mathbb{L}$-points of $Z_\Psi$. Equivalently, the defining ideal of $Z_\Psi$ is the kernel of the following $\bar{k}(t)$-algebra homomorphism

$$X_{ij} \mapsto \Psi_{ij} : \bar{k}(t)[X, 1/\det X] \to \mathbb{L},$$

where $\bar{k}(t)[X, 1/\det X]$ is the affine coordinate ring of $\mathrm{GL}_r / \bar{k}(t)$ and $X = (X_{ij})$. Fix an algebraic closure of $\mathbb{L}$, which is denoted by $\overline{\mathbb{L}}$. Then we set $\Gamma_\Psi$ to be the smallest closed subscheme of $\mathrm{GL}_r / \mathbb{F}_q(t)$ so that the $\overline{\mathbb{L}}$-valued points of $\Gamma_\Psi$ contain $\Psi^{-1} Z_\Psi(\overline{\mathbb{L}})$. Let $\bar{k}(t)(\Psi)$ be the field over $\bar{k}(t)$ generated by all entries of $\Psi$. Papanikolas developed a function field analogue of classical Galois theory of linear differential equations [25].

**Theorem 4.5.** (Papanikolas [24]) *The scheme $\Gamma_\Psi$ is a closed $\mathbb{F}_q(t)$-subgroup scheme of $\mathrm{GL}_r/\mathbb{F}_q(t)$, and the closed $\overline{k}(t)$-subscheme $Z_\Psi$ of $\mathrm{GL}_r/\overline{k}(t)$ is stable under right-multiplication of $\overline{k}(t)\times_{\mathbb{F}_q(t)}\Gamma_\Psi$ and is a $\left(\overline{k}(t)\times_{\mathbb{F}_q(t)}\Gamma_\Psi\right)$-torsor. Moreover, we have:*

1. *The $\overline{k}(t)$-scheme $Z_\Psi$ is smooth and geometrically connected.*
2. *The $\mathbb{F}_q(t)$-scheme $\Gamma_\Psi$ is smooth and geometrically connected.*
3. *The dimension of $\Gamma_\Psi$ over $\mathbb{F}_q(t)$ is equal to the transcendence degree of $\overline{k}(t)(\Psi)$ over $\overline{k}(t)$.*

## § 4.6. Comparison between $t$-motivic methods and classical differential Galois theory

We fix a matrix $\Phi\in\mathrm{Mat}_r(\overline{k}[t])\cap\mathrm{GL}_r(\overline{k}(t))$ and suppose that there exists $\Psi\in\mathrm{Mat}_r(\mathbb{T})\cap\mathrm{GL}_r(\mathbb{L})$ so that $\Psi^{(-1)}=\Phi\Psi$. Fix a $\xi\in\overline{k}^\times\backslash\overline{\mathbb{F}_q}^\times$ such that $(\Phi,\Psi,\xi)$ satisfies the conditions of Theorem 4.4. We consider the Kronecker $n$-th tensor product of $\Psi^{(-1)}=\Phi\Psi$, whence obtaining

$$\left(\Psi^{\otimes n}\right)^{(-1)}=\Phi^{\otimes n}\Psi^{\otimes n}.$$

We further take the direct sum

$$\left(\oplus_n\Psi^{\otimes n}\right)^{(-1)}=\left(\oplus_n\Phi^{\otimes n}\right)\left(\oplus_n\Psi^{\otimes n}\right)$$

and notice that

- $(\oplus_n\Phi^{\otimes n},\oplus_n\Psi^{\otimes n},\xi)$ still satisfies the conditions of the refined ABP-criterion;
- the entries of $\Psi^{\otimes n}$ are degree $n$ monomials of the entries of $\Psi$.

Therefore, according to Theorem 4.4 we see that any $\overline{k}$-polynomial relation among the entries $\Psi$ is a specialization of a $\overline{k}[t]$-polynomial relation among the entries of $\Psi$. By computing Hilbert series (cf. [24]) we have that

$$\mathrm{tr.deg}_{\overline{k}}\,\overline{k}\left(\Psi(\xi)\right)=\mathrm{tr.deg}_{\overline{k}(t)}\,\overline{k}(t)\left(\Psi\right).$$

It follows that combining Theorem 4.5 we have the following important equality, which was invented by Papanikolas.

**Theorem 4.6.** (Papanikolas) *Given a matrix $\Phi\in\mathrm{Mat}_r(\overline{k}[t])\cap\mathrm{GL}_r(\overline{k}(t))$, we suppose that there exists $\Psi\in\mathrm{Mat}_r(\mathbb{T})\cap\mathrm{GL}_r(\mathbb{L})$ so that $\Psi^{(-1)}=\Phi\Psi$. Fix any $\xi\in\overline{k}^\times\backslash\overline{\mathbb{F}_q}^\times$ so that $(\Phi,\Psi,\xi)$ satisfies the conditions of Theorem 4.4. Then we have*

$$\dim\Gamma_\Psi=\mathrm{tr.deg}_{\overline{k}}\,\overline{k}\left(\Psi(\xi)\right). \tag{4.1}$$

The equality above can be thought of as a function field analogue of Grothendieck's periods conjecture. In the case of uniformizable $t$-motives $M$, they are some specific objects in $\mathfrak{F}$ together with nice difference equations $\Psi^{(-1)} = \Phi\Psi$. We mention that in this case the field $\overline{k}\,(\Psi(\theta))$ contains the "periods"of the $t$-motive $M$. Thereby, appealing to the equality above enables one to prove some algebraic independence results concerning certain periods of $t$-motives. For instance, see [15] for Drinfeld modules which naturally correspond to certain uniformizable $t$-motives. The theorem above has many applications, for example we have:

- Algebraic independence of Carlitz zeta values [20, 18];
- Algebraic independence of Carlitz zeta values and special gamma values [16, 17];
- Algebraic independence of periods and logarithms for Drinfeld modules [14, 15];
- Algebraic independence of arithmetic Drinfeld modular forms at CM points [11];
- Algebraic independence of periods of the first, second and third kinds for rank 2 Drinfeld modules [13].

We shall mention that although the theory above provides some hope to work on algebraic independence of special values in question, there are two general difficulties occurring in the procedure:

- How does one give a $t$-motivic interpretation of the special values in question if it is possible? i.e., how does one create suitable $(\Phi, \Psi, \xi)$ which satisfies Theorem 4.4 and the field $\overline{k}\,(\Psi(\xi))$ contains the special values in question?
- Assuming the step above valid, how does one compute the algebraic Galois group $\Gamma_\Psi$?

In the classical case, for some nice systems of linear differential equations with solution functions as $E$-functions, one also has the equality such as (4.1). That is, the dimension of the differential Galois group in question is equal to the transcendence degree over $\overline{\mathbb{Q}}$ of the field generated by the solution functions ($E$-functions) at an algebraic number satisfying the conditions of Theorem 4.3. However, the values of $E$-functions at algebraic numbers are not "periods", and so many classical algebraic independence problems concerning "periods"are still open.

## § 5. Key ingredients of the proofs

In what follows, we sketch the key ingredients of proofs of Theorems 2.4 and 3.5.

### § 5.1. Key ingredients of proof of Theorem 2.4

The primary tool of proving Theorem 2.4 is to use the ABP-criterion (Theorem 4.4). Here we list the key steps in [12]:

(I) Abstraction for the values having the MZ properties.

(II) Generalization of Theorem 2.4 for the values having the MZ properties.

(III) MZVs have the MZ properties.

We explain some details of the steps above in the following subsections.

**5.1.1. Step I.** In [4], Anderson and Thakur gave a $t$-motivic explanation for MZVs and the author of the present paper observes that the difference equations associated the MZV in question have some specific properties, and thereby gives the following abstraction for simplicity.

**Definition 5.1.** Let $\mathcal{E}$ be the ring of power series in $\mathbb{C}_\infty[\![t]\!]$ that converge on whole $\mathbb{C}_\infty$. A nonzero element $Z \in k_\infty^\times$ is said to have the *MZ* (Multizeta) property with weight $w$ if there exists $\Phi \in \mathrm{Mat}_d(\bar{k}[t])$ and $\psi \in \mathrm{Mat}_{d\times 1}(\mathcal{E})$ with $d \geq 2$ so that

(1) $\psi^{(-1)} = \Phi\psi$ and $(\Phi, \psi, \theta)$ satisfies the conditions of the ABP-criterion;

(2) The last column of $\Phi$ is of the form $(0, \ldots, 1)^{\mathrm{tr}}$ (whose entries are zero except the last entry being 1);

(3) $\psi(\theta)$ is of the form (with specific first and last entries):

$$\psi(\theta) = \begin{pmatrix} 1/\tilde{\pi}^w \\ \vdots \\ cZ/\tilde{\pi}^w \end{pmatrix}$$

for some $c \in k^\times$ ;

(4) for any positive integer $N$, $\psi(\theta^{q^N})$ is of the form:

$$\psi(\theta^{q^N}) = \begin{pmatrix} 0 \\ \vdots \\ (cZ/\tilde{\pi}^w)^{q^N} \end{pmatrix}$$

(whose entries are zero except the last entry).

*Remark.* One can see from Theorem 5.3 that any nonzero $Z$ having the *MZ* property has a unique weight.

The abstraction above has some convenience. For example, the *MZ* property is invariant under product.

**Proposition 5.2.** *Let $Z_1, \ldots, Z_n$ be nonzero values having the MZ property with weights $w_1, \ldots, w_n$ respectively. For nonnegative integers $m_1, \ldots, m_n$, not all zero, the monomial*

$$Z_1^{m_1} \cdots Z_n^{m_n}$$

*has the MZ property with weight $\sum_{i=1}^{n} m_i w_i$.*

*Proof.* We consider the Kronecker product:

$$\Phi := \Phi_1^{\otimes m_1} \otimes \cdots \otimes \Phi_n^{\otimes m_n} \text{ and } \psi := \psi_1^{\otimes m_1} \otimes \cdots \otimes \psi_n^{\otimes m_n}.$$

Then one has $\psi^{(-1)} = \Phi\psi$ and the result follows from Definition 5.1. □

**5.1.2. Step II.** The following result is a generalized version of Theorem 2.4.

**Theorem 5.3.** *Let $w_1, \ldots, w_\ell$ be $\ell$ distinct positive integers. Let $V_i$ be a finite set of some nonzero values having the MZ-property with weight $w_i$, and suppose that $V_i$ is a linearly independent set over $k$ for $i = 1, \ldots, \ell$. Then the union*

$$\{1\} \cup \bigcup_{i=1}^{\ell} V_i$$

*is a linearly independent set over $\bar{k}$.*

Here, we will not give the detailed proof of the theorem above. Instead, we outline the key steps of the proof. Let the notation and assumptions be as in Theorem 5.3. Without loss of generality, we may assume that $w_1 > \cdots > w_\ell$. Suppose on the contrary that the set

$$\{1\} \cup \bigcup_{i=1}^{\ell} V_i$$

is linearly dependent over $\bar{k}$. By induction on the weight, we may further assume that there are nontrivial $\bar{k}$-linear relations connecting $V_1$ and $\{1\} \cup \bigcup_{i=2}^{\ell} V_i$. Under such hypotheses, the major two steps of the proof are the following.

- Show that $V_1$ is a linearly dependent set over $\bar{k}$.
- Show that $V_1$ is a linearly dependent set over $k$, whence a contradiction.

The proofs of the two steps above use techniques of Frobenius difference equations. For more details, see [12].

Here we present how to prove the transcendence of a nonzero value $Z$ having the *MZ*-property. Let $Z$ be with weight $w$ and $(\Phi, \psi)$ be associated with $Z$ given in the Definition 5.1. Note that $\psi(\theta)$ is of the form

$$\begin{pmatrix} 1/\tilde{\pi}^w \\ \vdots \\ cZ/\tilde{\pi}^w \end{pmatrix}$$

for some $c \in k^\times$ and $\psi(\theta^{q^N})$ is of the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ (cZ/\tilde{\pi}^w)^{q^N} \end{pmatrix}.$$

If $Z \in \bar{k}^\times$, then by Theorem 4.4 there exists a vector $P = (f_1, \ldots, f_d) \in \mathrm{Mat}_{1\times d}\left(\bar{k}[t]\right)$ so that $P\psi = 0$ and $f_1(\theta) = -cZ$, $f_d(\theta) = 1$ and $f_i(\theta) = 0$ for $i \neq 1, d$. Pick a sufficiently large integer $N$ so that the polynomial $f_d$ is non-vanishing at $t = \theta^{q^N}$. Then using the specific form of $\psi(\theta^{q^N})$ and specializing the equation $P\psi = 0$ at $t = \theta^{q^N}$ give rise to the vanishing of $Z/\tilde{\pi}^w$, whence a contradiction.

**5.1.3. Step III.** It is clear that Theorem 2.4 will be a consequence of Theorem 5.3 if one shows that each MZV has the *MZ* property. We first fix a multizeta value $\zeta(s_1, \ldots, s_r)$. Put $\mathbf{s} = (s_1, \ldots, s_r)$ and let $\Phi_{\mathbf{s}}$ be defined as in (3.2). Define the series:

$$\begin{aligned} &L_2(t) := \textstyle\sum_{i=0}^{\infty} \left(\Omega^{s_1} H_{s_1-1}\right)^{(i)} \\ &L_3(t) := \textstyle\sum_{i_1 > i_2 \geq 0} \left(\Omega^{s_2} H_{s_2-1}\right)^{(i_2)} \left(\Omega^{s_1} H_{s_1-1}\right)^{(i_1)} \\ &\vdots \\ &L_{r+1}(t) := \textstyle\sum_{i_1 > \cdots > i_r \geq 0} \left(\Omega^{s_r} H_{s_r-1}\right)^{(i_r)} \ldots \left(\Omega^{s_1} H_{s_1-1}\right)^{(i_1)} \end{aligned} \tag{5.1}$$

By § 3.3.1 one has that for each $1 \leq j \leq r$,

$$L_{j+1}(\theta) = \Gamma_{s_1} \cdots \Gamma_{s_j} \zeta(s_1, \ldots, s_j)/\tilde{\pi}^{s_1+\cdots+s_j}. \tag{5.2}$$

Moreover, if we put

$$\psi_{\mathbf{s}} := \begin{pmatrix} \Omega^{s_1+\cdots+s_r} \\ \Omega^{s_2+\cdots+s_r} L_2 \\ \vdots \\ \Omega^{s_r} L_r \\ L_{r+1} \end{pmatrix},$$

then we have $\psi_{\mathbf{s}}^{(-1)} = \Phi_{\mathbf{s}}\psi_{\mathbf{s}}$. As is checked immediately, the conditions (1)-(3) of Definition 5.1 are satisfied. We first note that the function $\Omega$ has simple zero at $t = \theta^{q^N}$ for each positive integer $N$. Therefore, to check the condition (4) it suffices to show that for each positive integer $N$ we have

$$L_{j+1}(\theta^{q^N}) = \left(\Gamma_{s_1}\cdots\Gamma_{s_j}\zeta(s_1,\dots,s_j)/\tilde{\pi}^{s_1+\cdots+s_j}\right)^{q^N}$$

for $1 \le j \le r$.

To show the formula above, we express $L_{j+1}$ as $L_{j+1} = L_{j+1}^{<N} + L_{j+1}^{\ge N}$, where

$$\begin{aligned} L_{j+1}^{<N}(t) &:= \textstyle\sum_{\substack{i_1 > \cdots > i_j \ge 0; \\ i_j < N}} \left(\Omega^{s_j} H_{s_j-1}\right)^{(i_j)} \dots \left(\Omega^{s_1} H_{s_1-1}\right)^{(i_1)} \\ L_{j+1}^{\ge N}(t) &:= \textstyle\sum_{i_1 > \cdots > i_j \ge N} \left(\Omega^{s_j} H_{s_j-1}\right)^{(i_j)} \dots \left(\Omega^{s_1} H_{s_1-1}\right)^{(i_1)}. \end{aligned}$$

Then using the functional equation $\Omega^{(-1)} = (t-\theta)\Omega$ we can interpret $L_{r+1}^{<N}(t)$ as

$$L_{j+1}^{<N}(t) = \sum_{\substack{i_1 > \cdots > i_j \ge 0; \\ i_j < N}} \frac{\Omega^{s_1+\cdots+s_j} H_{s_j-1}^{(i_j)} \dots H_{s_1-1}^{(i_1)}}{\Big((t-\theta^q)\dots(t-\theta^{q^{i_j}})\Big)^{s_j} \dots \Big((t-\theta^q)\dots(t-\theta^{q^{i_1}})\Big)^{s_1}}.$$

Then one observes that the vanishing order of general term at $t = \theta^{q^N}$ is positive, whence $L_{j+1}^{<N}(\theta^{q^N}) = 0$. Therefore, the desired formula follows from the equality

$$L_{j+1}^{\ge N}(t) = \left(L_{j+1}(t)\right)^{(N)}.$$

## § 5.2. Key ingredients of proof of Theorem 3.5

**5.2.1. Construction of $\Psi_{\mathbf{s}}$** We fix an $r$-tuple $\mathbf{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ and follow the notation above. To prove Theorem 3.5, we need to work out the solution matrix of the system of difference equations

$$\Psi_{\mathbf{s}}^{(-1)} = \Phi_{\mathbf{s}}\Psi_{\mathbf{s}}.$$

The construction of such $\Psi_{\mathbf{s}}$ is due to Anderson and Thakur [4] and is reviewed as follows. For $1 < \ell < j \le r+1$, we put

$$L_{j\ell}(t) := \sum_{i_\ell > \cdots > i_j \ge 0} \left(\Omega^{s_j} H_{s_j-1}\right)^{(i_j)} \cdots \left(\Omega^{s_\ell} H_{s_\ell-1}\right)^{(i_\ell)} \in \mathbb{T}$$

and note that

$$L_{j\ell}(\theta) = \Gamma_{s_{\ell-1}}\cdots\Gamma_{s_{j-1}}\zeta(s_{\ell-1},\dots,s_{j-1})/\tilde{\pi}^{s_{\ell-1}+\cdots+s_{j-1}}.$$

Put

$$\Psi_{\mathbf{s}} := \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & & & & \\ \Omega^{s_2+\cdots+s_r}L_{21} & \Omega^{s_2+\cdots+s_r} & & & \\ \vdots & \Omega^{s_3+\cdots+s_r}L_{32} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \Omega^{s_r}L_{r1} & \Omega^{s_r}L_{r2} & & \ddots & \Omega^{s_r} & \\ L_{(r+1),1} & L_{(r+1),2} & \cdots & \cdots & L_{(r+1),r} & 1 \end{pmatrix},$$

then we have $\Psi_{\mathbf{s}}^{(-1)} = \Phi_{\mathbf{s}}\Psi_{\mathbf{s}}$.

**5.2.2. Ideas of the proof of Theorem 3.5** In what follows, we sketch the ideas of the proof of Theorem 3.5. We fix a column vector $\mathbf{m} \in \mathrm{Mat}_{(r+1)\times 1}(M_{\mathbf{s}})$ so that the entries of $\mathbf{m}$ comprise a $\bar{k}[t]$-basis of $M_s$ with the property

$$F(\mathbf{m}) = \Phi_s \mathbf{m}.$$

Note that $M_{\mathbf{s}}$ is trivial in $\mathrm{Ext}^1_{\mathfrak{F}}(\mathbf{1}, M_{\mathbf{s}'})$ if and only if there exists $\gamma \in \mathrm{GL}_{r+1}(\bar{k}[t])$ of the form

$$\gamma = \begin{pmatrix} I_r & 0 \\ \gamma_1, \ldots, \gamma_r & 1 \end{pmatrix}$$

so that

$$F(\gamma\mathbf{m}) = \begin{pmatrix} \Phi_{\mathbf{s}'} & \\ & 1 \end{pmatrix} (\gamma\mathbf{m}).$$

It is equivalent to

$$\gamma^{(-1)}\Phi_{\mathbf{s}} = \begin{pmatrix} \Phi_{\mathbf{s}'} & \\ & 1 \end{pmatrix} \gamma.$$

If the equality above holds, we observe that the matrix $\gamma\Psi_{\mathbf{s}}$ satisfies

$$(\gamma\Psi_{\mathbf{s}})^{(-1)} = \begin{pmatrix} \Phi_{\mathbf{s}'} & \\ & 1 \end{pmatrix} (\gamma\Psi_{\mathbf{s}}).$$

Let $\Psi_{\mathbf{s}'}$ be the square matrix of size $r$ cut off from the upper left square of $\Psi_{\mathbf{s}}$. We note that $\Psi_{\mathbf{s}'}$ satisfies $\Psi_{\mathbf{s}'}^{(-1)} = \Phi_{\mathbf{s}'}\Psi_{\mathbf{s}'}$. It follows that

$$\begin{pmatrix} \Psi_{\mathbf{s}'} & \\ & 1 \end{pmatrix}^{(-1)} = \begin{pmatrix} \Phi_{\mathbf{s}'} & \\ & 1 \end{pmatrix} \begin{pmatrix} \Psi_{\mathbf{s}'} & \\ & 1 \end{pmatrix}.$$

Therefore, by [24, § 4.1.6] we have that

$$\gamma\Psi_{\mathbf{s}} = \begin{pmatrix} \Psi_{\mathbf{s}'} & \\ & 1 \end{pmatrix} \begin{pmatrix} I_r & \\ f_1, \ldots, f_r & 1 \end{pmatrix}$$

for some $f_1,\dots,f_r\in\mathbb{F}_q(t)$. Specializing the equation above at $t=\theta^{q^N}$ will give rise to

$$\zeta(s_1,\dots,s_r)/\tilde{\pi}^{s_1+\cdots+s_r}\in k.$$

The discussion above provides the idea of proving the direction ($\Leftarrow$) of Theorem 3.5, where the overall proof is just a slight generalization.

The direction ($\Rightarrow$) of the proof of Theorem 3.5 will use the ABP-criterion to create a matrix $\gamma\in GL_{r+1}(\bar{k}[t])$ and an $a\in\mathbb{F}_q[t]$ satisfying the following equality

$$\gamma^{(-1)}\begin{pmatrix} \Phi_{\mathbf{s}'} & \\ 0,\dots,aH_{s_r-1}^{(-1)}(t-\theta)^{s_r} & 1\end{pmatrix}=\begin{pmatrix}\Phi_{\mathbf{s}'} & \\ & 1\end{pmatrix}\gamma.$$

Since the object $a*M_{\mathbf{s}}$ is defined by the matrix

$$\begin{pmatrix} \Phi_{\mathbf{s}'} & \\ 0,\dots,aH_{s_r-1}^{(-1)}(t-\theta)^{s_r} & 1\end{pmatrix},$$

it follows that $M_{\mathbf{s}}$ is an $a$-torsion element in $\mathrm{Ext}^1_{\mathfrak{F}}(\mathbf{1},M_{\mathbf{s}'})$. The detailed construction of $\gamma$ and $a$ above is referred to [19].

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