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Periods, logarithms and multiple zeta values

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Abstract. This is a survey article discussing classical periods, logarithms and multiple zeta values and some connections between them. We also report the recent progress of the parallel theories and problems in the setting of function fields in positive characteristic.

1. Classical logartihms

1.1. Logarithms of algebraic numbers. Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. Let $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ be the usual exponential function. This is the exponential map of the commutative algebraic group $\mathbb{G}_m$, and one has the short exact sequence

$$0 \to \mathbb{Z} \cdot 2\pi\sqrt{-1} \to \mathbb{C} \twoheadrightarrow \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}) \to 1.$$

Given any nonzero complex number z, Hermite and Lindemann showed in the late 19th century that one at least of z, e^z is transcendental over $\mathbb{Q}$. As a consequence, one derives the transcendence of λ with $0 \neq \lambda \in \mathbb{C}$ for which $e^\lambda \in \overline{\mathbb{Q}}$. Such an λ is called a logarithm of algebraic number.

In the beginning of the twentieth century, Hilbert raised a famous list of 23 problems, part of which was devoted to number theory and Diophantine geometry, and there have been some wonderful developments since then. Hilbert's seventh problem, which is referred to the seventh among the list mentioned above, is about the linear independence of two logarithms of algebraic numbers. More precisely, Hilbert asked about the transcendence of α^β for $\alpha \neq 0, 1$ an algebraic number, and an irrational algebraic number β. It was believed by Hilbert that the proof of this transcendence question would be very difficult and that the solution of this question would lead valuable approaches.

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Surprisingly, it turns out that Hilbert's seventh problem was solved by Gelfond and Schneider independently in 1934. The theorem of Gelfond and Schneider asserts that given two nonzero algebraic numbers α_1, α_2 so that $\log\alpha_1$ and $\log\alpha_2$ are linearly independent over the rational numbers, then we have

$$\beta_1\log\alpha_1+\beta_2\log\alpha_2\neq 0$$

for arbitrary nonzero algebraic numbers β_1, β_2. Baker's celebrated theorem on linear forms in logarithms fully generalizes the work of Gelfond-Schneider: if the given nonzero algebraic numbers $\alpha_1,\ldots,\alpha_n$ satisfy that $\log\alpha_1,\ldots,\log\alpha_n$ are linearly independent over $\mathbb{Q}$, then $1,\log\alpha_1,\ldots,\log\alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$. It is well known that Baker's theorem has many applications in number theory. One typical example is the solution of Gauss' class number one problem using Baker's theorem. Concerning algebraic relations over $\overline{\mathbb{Q}}$ among the logarithms of algebraic numbers, Gelfond's conjecture asserts that $\log\alpha_1,\ldots,\log\alpha_n$ are algebraically independent over $\overline{\mathbb{Q}}$ under the hypothesis that $\log\alpha_1,\ldots,\log\alpha_n$ are linearly independent over $\mathbb{Q}$. However, Gelfond's conjecture is still open in the classical transcendence theory. For more details, we refer the reader to the book [**BW07**] and the survey articles [**W02, Wa08**].

1.2. Periods of elliptic curves and logarithms. Let Λ be a lattice of $\mathbb{C}$ with two generators λ_1, λ_2. We assume that

$$g_2:=g_2(\Lambda):=\frac{1}{60}\sum_{0\neq\lambda\in\Lambda}\frac{1}{\lambda^4},\ g_3:=g_3(\Lambda):=\frac{1}{140}\sum_{0\neq\lambda\in\Lambda}\frac{1}{\lambda^6}$$

are algebraic numbers. We let $\wp_\Lambda$ be the Weierstrass $\wp$-function associated to Λ, and one knows that $\wp_\Lambda(z)$, $\wp'_\Lambda(z)$ parametrize the elliptic curve

$$E_\Lambda: y^2=4x^3-g_2x-g_3$$

defined over $\overline{\mathbb{Q}}$. Elements of Λ are called periods of the elliptic curve E_Λ and they can be expressed as

$$\int_\gamma\frac{dx}{y}$$

for some cycle $\gamma\in H_1(E_\lambda(\mathbb{C}),\mathbb{Z})$. Note that $\frac{dx}{y}$ is a differential form of the first kind, and so elements of Λ are also called periods of the first kind of E_Λ. The transcendence of nonzero periods of the first kind of the elliptic curve E_Λ was first proved by Siegel in the case of complex multiplication, and by Schneider in the general case.

We consider $\frac{xdx}{y}$, which is a differential form of the second kind and the period integral

$$\int_\gamma\frac{xdx}{y}$$

is called period of the second kind or quasi-period of E_Λ. These quasi-periods can be expressed in terms of the Weierstrass ζ-function ζ_Λ of the lattice Λ, which satisfies the differential equation

$$\zeta_\Lambda' = -\wp_\Lambda.$$

The function ζ_Λ is quasi-periodic in the sense that for $\lambda \in \Lambda$, we have

$$\zeta_\Lambda(z+\lambda) = \zeta_\Lambda(z) + \eta(\lambda) \text{ for some } \eta(\lambda) \in \mathbb{C}.$$

We pick cycles $\gamma_1, \gamma_2 \in H_1(E_\Lambda(\mathbb{C}), \mathbb{Z})$ for which

$$\int_{\gamma_i} \frac{dx}{y} = \lambda_i$$

and so we have

$$\int_{\gamma_i} \frac{xdx}{y} = \eta(\lambda_i) \text{ for } i = 1, 2.$$

Note that nonzero quasi-periods of the elliptic curve E_Λ are known to be transcendental numbers by the work of Schneider. The following

$$P_\Lambda := \begin{pmatrix} \lambda_1 & \eta(\lambda_1) \\ \lambda_2 & \eta(\lambda_2) \end{pmatrix}$$

is called the period matrix of the elliptic curve E_Λ. The Legendre relation asserts that $\det P_\Lambda = \pm 2\pi\sqrt{-1}$, which implies the comparison isomorphism between deRham and Betti cohomologies:

$$H^1_{dR}(E_\Lambda) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong H^1_B(E_\Lambda) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The periods conjecture in this case asserts that

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(\lambda_1, \lambda_2, \eta(\lambda_1), \eta(\lambda_2)) = \begin{cases} 2 & \text{if } E_\Lambda \text{ has complex multiplication,} \\ 4 & \text{otherwise.} \end{cases}$$

This conjecture was proved by Chudnovsky in the case when E_Λ has complex multiplication (CM), but it is still open in the non-CM case, in which case Masser showed the $\overline{\mathbb{Q}}$-linear independence of 1, $2\pi\sqrt{-1}$, λ_1, λ_2, $\eta(\lambda_1)$, $\eta(\lambda_2)$.

Along the direction of Baker's theorem on linear independence of logarithms, one can also consider the elliptic logarithms and quasi-logarithms. Let $u_1, \ldots, u_m \in \mathbb{C} \setminus \Lambda$ satisfy $\exp_{E_\Lambda}(u_i) \in E_\Lambda(\overline{\mathbb{Q}})$, and note that each $\zeta_\Lambda(u_i)$ is well-defined as $u_i \notin \Lambda$. Wüstholz showed that if $u_1, \ldots, u_m$ are linearly independent over $\operatorname{End}(E_\Lambda)$, then the following $2+2m$ values

$$1, 2\pi\sqrt{-1}, u_1, \ldots, u_m, \zeta_\Lambda(u_1), \ldots, \zeta_\Lambda(u_m)$$

are linearly independent over $\overline{\mathbb{Q}}$. We mention that the elliptic analogue of Gelfond's conjecture asserts that the $2m$ values

$$u_1, \ldots, u_m, \zeta_\Lambda(u_1), \ldots, \zeta_\Lambda(u_m)$$

are algebraically independent over $\overline{\mathbb{Q}}$ under the hypothesis above.

Wüstholz's result generalizes the previous works on linear independence of elliptic logarithms due to Masser in the CM case, and Bertrand-Masser in the non-CM case. The linear independence result above is a consequence of the powerful machinery when dealing with generalized logarithms at algebraic points, the so-called Wüstholz's analytic subgroup theorem.

Theorem 1.2.1 (Wüstholz [W89]). *Let G be a commutative algebraic group defined over $\overline{\mathbb{Q}}$, and let $\exp_G : \operatorname{Lie} G(\mathbb{C}) \to G(\mathbb{C})$ be the exponential map when regarding $G(\mathbb{C})$ as a Lie group. Let $\mathbf{u} \in \operatorname{Lie} G(\mathbb{C})$ satisfy $\exp_G(\mathbf{u}) \in G(\overline{\mathbb{Q}})$, and let $V_{\mathbf{u}} \subset \operatorname{Lie} G(\mathbb{C})$ be the smallest linear subspace that contains $\mathbf{u}$ and that is defined over $\overline{\mathbb{Q}}$. Then we have*

$$V_{\mathbf{u}} = \operatorname{Lie} H(\mathbb{C})$$

for some algebraic subgroup H of G defined over $\overline{\mathbb{Q}}$.

The spirit of Wüstholz's analytic subgroup theorem is that the $\overline{\mathbb{Q}}$-linear relations among the coordinates of $\mathbf{u}$ are explained by the defining equations of $\operatorname{Lie} H$. To see more applications of Theorem 1.2.1 we refer the reader to the book of Baker-Wüstholz [BW07], where one also sees the historical developments.

2. Multiple zeta values

2.1. Real-valued MZV's. Classical real-valued multiple zeta values (abbreviated as MZV's) are generalizations of the special values of Riemann ζ-function at positive integers at least 2. We refer the reader to the pioneering paper of Zagier [Za94]. An index is an r-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r)$, and we call $\mathfrak{s}$ 'admissible'if $s_1 \geq 2$. The MZV at $\mathfrak{s}$ is defined by the following multiple zeries

$$\zeta(\mathfrak{s}) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}^\times$$

We call $\operatorname{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ and $\operatorname{dep}(\mathfrak{s}) := r$ the weight and depth of the presentation $\zeta(\mathfrak{s})$ respectively. When $r = 1$, these values are special Riemann zeta values. MZV's have many interesting connections. For example, we have the following.

- MZV's are periods of mixed Tate motives by Terasoma [Te02] and Goncharov [Gon02].
- Double zeta values (MZV's of depth two) have connections with modular forms by Gangl-Kaneko-Zagier [GKZ06].
- MZV's have connections in arithmetic geometry (see [Br14]).

For relevant references on this topic, we refer the reader to the following books [An04, Zh16, BGF18].

2.2. Regularized double shuffle relations. In the classical theory, there are many $\mathbb{Q}$-linear relations among the same weight MZV's produced by the machinery of regularized double shuffle relations. To describe it, we let $\mathfrak{H} := \mathbb{Q} < x, y >$ be the non-commutative algebra generated by the (non-commutative) variables x, y over $\mathbb{Q}$. We then consider the following two subalgebras

$$\mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y \subset \mathfrak{H}^1 := \mathbb{Q} + \mathfrak{H}y \subset \mathfrak{H}.$$

There are two new products (multiplication laws), called stuffle product (denoted by $\star$) and shuffle product (denoted by Ш) on $\mathfrak{H}^1$ for which $\mathfrak{H}^1$ becomes a commutative $\mathbb{Q}$-algebra under $\bullet = \star, \text{Ш}$ (see [Ho97]). For any admissible index $\mathfrak{s} = (s_1, \dots, s_r)$, we have the corresponding *admissible word* $x^{s_1-1}yx^{s_2-1}y\cdots x^{s_r-1}y$, and one has the following evaluation map: for $\bullet = \star$ or Ш,

$$Z_\bullet := \left(x^{s_1-1}yx^{s_2-1}y\cdots x^{s_r-1}y \mapsto \zeta(s_1,\dots,s_r)\right) : \mathfrak{H}^0 \to \mathbb{R},$$

and defines the map on whole $\mathfrak{H}^0$ by linearity. In fact, the map above is a $\mathbb{Q}$-algebra homomorphism deriving that the $\mathbb{Q}$-vector space spanned by MZV's have two $\mathbb{Q}$-algebra structures. The classical theory of double shuffle relations asserts that for any two admissible words $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{H}^0$, we have

$$Z_\star(\mathbf{w}_1 \star \mathbf{w}_2) = Z_{\text{Ш}}(\mathbf{w}_1 \text{Ш} \mathbf{w}_2).$$

The identity above produces some non-trivial $\mathbb{Q}$-linear relations among MZV's of the same weight. The algebra homomorphism $Z_\bullet$ comes from the series presentation of MZV's for $\bullet = \star$, and from integral presentation for $\bullet = \text{Ш}$.

The regularized double shuffle relations were developed by Racinet [Ra02] and Ihara-Kaneko-Zagier [IKZ06] and we briefly describe the machinery as follows. First, for $\bullet = \star$ or $\bullet = \text{Ш}$ we denote by $\mathfrak{H}^1_\bullet$ the associated $\mathbb{Q}$-algebra under the product $\bullet$ which contains $\mathfrak{H}^0_\bullet$ as subalgebra. By [Ho97, Re93] one knows that $\mathfrak{H}^1_\bullet$ is freely generated by y over $\mathfrak{H}^0_\bullet$ for $\bullet = \star, \text{Ш}$. That is, any element g of $\mathfrak{H}^1_\bullet$ can be uniquely written as

$$g = g_0 + g_1 \bullet y + \cdots + g_r \bullet y^{\bullet r}$$

for some $g_0, \dots, g_r \in \mathfrak{H}^0_\bullet$ with $g_r \neq 0$. Let T be an indeterminate. We then extend the algebra homomorphism $Z_\bullet : \mathfrak{H}^0_\bullet \to \mathbb{R}$ to

$$\hat{Z}_\bullet : \mathfrak{H}^1_\bullet \to \mathbb{R}[T]$$

by putting $\hat{Z}_\bullet(y) := T$ and $\hat{Z}_\bullet|_{\mathfrak{H}^0_\bullet} = Z_\bullet$. Precisely, for the expansion of $g \in \mathfrak{H}^1_\bullet$ above we have

$$\hat{Z}_\bullet(g) = g_0 + Z_\bullet(g_1)T + \cdots + Z_\bullet(g_r)T^r.$$

Define

$$A(u) := \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n)u^n\right)$$

and define the $\mathbb{R}$-linear map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ by

$$\rho(\exp(Tu)) = A(u)(\exp(Tu)).$$

The regularized double shuffle relations for MZV's are the following identity on $\mathfrak{H}^1$:

$$\hat{Z}_{\text{Ш}} = \rho \circ \hat{Z}_\star.$$

We mention that Goncharov [Gon97] predicted that there are no non-trivial $\mathbb{Q}$-linear relations among different weight MZV's.

Conjecture 2.2.1 (Goncharov). *For any integer $n \geq 2$, we let $\mathfrak{Z}_n$ be the $\mathbb{Q}$-vector space spanned by the MZV's of weight n, and let $\mathfrak{Z} := \sum_{n\geq 2} \mathfrak{Z}_n$ be the $\mathbb{Q}$-vector space spanned by all MZV's. Then $\mathfrak{Z}$ forms a graded algebra in the sense that*

$$\mathfrak{Z} = \bigoplus_{n\geq 2} \mathfrak{Z}_n.$$

The regularized double shuffle relations can produce many $\mathbb{Q}$-linear relations among the same weight MZV's, and conjecturally they account for all $\mathbb{Q}$-linear relations.

Conjecture 2.2.2 (Ihara-Kaneko-Zagier [IKZ06]). *The regularized double shuffle relations generate all the $\mathbb{Q}$-linear relations among the same weight MZV's.*

As we have $\mathfrak{Z}_{n_1} \cdot \mathfrak{Z}_{n_2} \subset \mathfrak{Z}_{n_1+n_2}$, according to the Ihara-Kaneko-Zagier conjecture above, understanding the $\mathbb{Q}$-algebraic relations among MZV's is theoretically reduced to understanding the $\mathbb{Q}$-linear relations among the same weight MZV's. This is why the following Zagier's dimension conjecture is the core problem in the theory of MZV's.

Conjecture 2.2.3 (Zagier). *Let $d_0 := 1$, $d_1 := 0$, $d_2 := 1$ and $d_n := d_{n-2} + d_{n-3}$ for integers $n \geq 3$. Then for each integer $n \geq 2$, we have*

$$\dim_{\mathbb{Q}} \mathfrak{Z}_n = d_n.$$

The best known result is that $\dim_{\mathbb{Q}} \mathfrak{Z}_n \leq d_n$ for all integers $n \geq 2$ by the works of Terasoma [Te02] and Goncharov [Gon02].

2.3. p-adic MZV's. In what follows, we fix a prime number p and let $\mathbb{Q}_p$ be the field of p-adic numbers. Let $\mathbb{C}_p$ be the p-adic completion of a fixed algebraic closure of $\mathbb{Q}_p$. Fix an admissible index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and consider the one-variable multiple polylogarithm

$$\mathrm{Li}_{\mathfrak{s}}(z) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{Q}[\![z]\!]$$

and notice that the specialization of $\mathrm{Li}_{\mathfrak{s}}(z)$ at $z = 1$ gives the MZV $\zeta(\mathfrak{s})$. Denote by $\mathrm{Li}_{\mathfrak{s}}(z)_p$ the same series as $\mathrm{Li}_{\mathfrak{s}}(z)$ but we regard it as in $\mathbb{C}_p[\![z]\!]$ and consider its p-adic convergence.

To define p-adic MZV's, Furusho first adapted Coleman's p-adic integration theory [Co82] to show that $\mathrm{Li}_{\mathfrak{s}}(z)_p$ can be analytically continued to $\mathbb{C}_p \setminus \{1\}$, and showed that the limit value $\lim'_{z\to 1} \mathrm{Li}_{\mathfrak{s}}(z)_p$ exists and is independent of the branch choices of the p-adic logarithm when applying Coleman's theory. Here the notation $\lim'_{z\to 1}$ is referred to take any sequence $\{z_1, \ldots, z_n, \ldots\} \subset \overline{\mathbb{Q}_p}$ converging to 1 and satisfying that the ramification indexes $\{e(\mathbb{Q}_p(z_1, \ldots, z_n)/\mathbb{Q}_p)\}_{n=1}^{\infty}$ is bounded. Furusho's p-adic MZV's are defined to be

$$\zeta_p(\mathfrak{s}) := \lim_{z\to 1}{}' \mathrm{Li}_{\mathfrak{s}}(z)_p$$

for admissible indexes $\mathfrak{s}$. The weight and the depth of the presentation $\zeta_p(\mathfrak{s})$ are defined to be $\mathrm{wt}(\mathfrak{s})$ and $\mathrm{dep}(\mathfrak{s})$ respectively.

We mention that in the depth one case, Furusho's p-adic single zeta value $\zeta_p(s)$ and the Kubota-Leopoldt p-adic zeta value at s just differ by a rational multiple. In [**FJ07**], Furusho and Jafari proved that p-adic MZV's satisfy the regularized double shuffle relations. As Conjecture 2.2.2 asserts that the regularized double shuffle relations generate all the $\mathbb{Q}$-linear relations among MZV's, it leads to the following formulation creating a route from the world of real-valued MZV's to the world of p-adic MZV's.

CONJECTURE 2.3.1. *For any integer $n \geq 2$, we let $\mathfrak{Z}_{n,p}$ be the $\mathbb{Q}$-vector space spanned by the p-adic MZV's of weight n. Then the following*

$$\phi_p := (\zeta(\mathfrak{s}) \mapsto \zeta_p(\mathfrak{s})) : \mathfrak{Z}_n \to \mathfrak{Z}_{n,p}$$

is a well-defined $\mathbb{Q}$-linear map.

In the following contexts, we will report the state of the art results in the setting of function fields in positive characteristic.

3. Abelian logarithms in positive characteristic

We let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over the finite field $\mathbb{F}_q$ of q elements with characteristic p, and let k be the quotient field of A. Let $|\cdot|_\infty$ be the non-archimedean absolute value corresponding to the infinite place ∞ for which $|\theta|_\infty = q$. We let k_∞ be the completion of k with respect to $|\cdot|_\infty$, and note that $|\cdot|_\infty$ can be uniquely extended to a fixed algebraic closure $\overline{k_\infty}$ of k_∞. For convenience, we still denote by $|\cdot|_\infty$ the extended absolute value on $\overline{k_\infty}$. We let $\mathbb{C}_\infty$ be the completion of $\overline{k_\infty}$ with respect to $|\cdot|_\infty$, and it turns out that $\mathbb{C}_\infty$ is a complete and algebraically closed field. Finally we denote by $\bar{k}$ the elements of $\mathbb{C}_\infty$ that are algebraic over k. Therefore we have the following analogies:

$$\mathbb{Z} \leftrightarrow A,\ \mathbb{Q} \leftrightarrow k,\ \mathbb{R} \leftrightarrow k_\infty,\ \mathbb{C} \leftrightarrow \mathbb{C}_\infty,\ \overline{\mathbb{Q}} \leftrightarrow \bar{k}.$$

3.1. Anderson's t-modules. In the positive characteristic world, the t-modules introduced by Anderson [**A86**] are generalizations of Drinfeld modules [**Dr74**] that play the analogous role of commutative algebraic groups in the characteristic zero world. To distinguish the different roles, we denote by t a new variable independent from the field $\mathbb{C}_\infty$ and will use $\mathbb{F}_q[t]$ for the role of operators on additive groups.

In what follows, we introduce the notion of Anderson's t-modules, but for most interest to us, we work over the field of definition inside $\bar{k}$. For any A-algebra $A \subset R \subset \bar{k}$, we denote by $\mathbb{G}_{a/R}$ the additive group scheme over R. By a d-dimensional t-module defined over R, we mean a pair $E = (\mathbb{G}_{a/R}^d, \rho)$, where ρ is an $\mathbb{F}_q$-linear ring homomorphism

$$\rho : \mathbb{F}_q[t] \to \mathrm{End}_{\mathbb{F}_q}\left(\mathbb{G}_{a/R}^d\right)$$

for which $\partial\rho_t - \theta I_d$ is a nilpotent matrix. Here $\mathrm{End}_{\mathbb{F}_q}\left(\mathbb{G}_{a/R}^d\right)$ is referred to the non-commutative ring of all $\mathbb{F}_q$-linear endomorphisms of the algebraic group scheme $\mathbb{G}_{a/R}^d$, and for $a \in \mathbb{F}_q[t]$, $\partial\rho_a$ is the induced homomorphism of ρ_a on the Lie algebra of $\mathbb{G}_{a/R}^d$. Therefore, for any R-algebra K the K-valued points $E(K)$ has an $\mathbb{F}_q[t]$-module structure via ρ.

Fix such a d-dimensional t-module E over R as above. Anderson [**A86**] showed that there is an *exponential* function of the t-module E. Precisely, it is the unique vector-valued power series in the variable $z_1, \ldots, z_d$ of the form

$$\exp_E(\mathbf{z}) = \sum_{i=0}^{\infty} e_i \mathbf{z}^{(i)},$$

where $e_0 = I_d$, $e_i \in \mathrm{Mat}_d(\bar{k})$ for all i, and $\mathbf{z}^{(i)} := (z_1^{q^i}, \ldots, z_d^{q^i})^{\mathrm{tr}}$, satisfying the identity

$$\exp_E \circ \partial\rho_a = \rho_a \circ \exp_E \text{ for all } a \in \mathbb{F}_q[t].$$

Anderson further showed that $\exp_E : \mathrm{Lie}\, E(\mathbb{C}_\infty) \to E(\mathbb{C}_\infty)$ is an entire $\mathbb{F}_q[t]$-linear map and its kernel $\Lambda_E := \mathrm{Ker} \exp_E$, called the period lattice of the t-module E, is a discrete, finitely generated $\mathbb{F}_q[t]$-submodule of $\mathrm{Lie}\, E(\mathbb{C}_\infty)$. We mention that in [**A86**], Anderson gave an example of a t-module for which its exponential map is not surjective. If $\exp_E$ is surjective, then we call the t-module E *uniformizable* and in this case we have $E(\mathbb{C}_\infty) \cong \mathbb{C}_\infty^d/\Lambda_E$ as $\mathbb{F}_q[t]$-modules.

The *logarithm* of E is defined to be the vector-valued power series $\log_E$ which is the formal inverse of $\exp_E$. That is,

$$\exp_E \circ \log_E = \text{identity} = \log_E \circ \exp_E .$$

Note that $\log_E$ can be expressed as

$$\log_E(\mathbf{z}) = \sum_{i=0}^{\infty} \ell_i \mathbf{z}^{(i)},$$

where $\ell_0 = I_d$ and $\ell_i \in \mathrm{Mat}_d(\bar{k})$ for all i. We further mention that as a function, the convergence domain of $\log_E$ may not be the whole $\mathrm{Lie}\, E(\mathbb{C}_\infty)$ in general.

The t-module E is called *trivial* if $\rho_a = a(\theta) I_d$ for any $a \in \mathbb{F}_q[t]$ in the sense that the $\mathbb{F}_q[t]$-module structure on $E(K)$ just arises from the scalar multiplication of A on K^d when replacing t by θ. Now let us discuss the case of one-dimensional non-trivial t-modules. In fact, these t-modules are called Drinfeld modules nowadays, which were introduced by Drinfeld and he called them *elliptic modules* in his seminal paper [**Dr74**].

We fix a Drinfeld $\mathbb{F}_q[t]$-module $E = (\mathbb{G}_{a/\overline{k}}, \rho)$ defined over $\overline{k}$. Since ρ_t is an $\mathbb{F}_q$-linear endomorphism of the algebraic group $\mathbb{G}_{a/\overline{k}}$, as a polynomial map it can be written as

$$\rho_t(X) = \theta X + a_1 X^q + \cdots + a_r X^{q^r} \tag{3.1.1}$$

for all $a_i \in \overline{k}$ and $a_r \neq 0$ for some positive integer r, since E is a non-trivial $\mathbb{F}_q[t]$-module. In terms of (3.1.1), we say that the Drinfeld $\mathbb{F}_q[t]$-module is of rank r. In this situation, Drinfeld showed that the period lattice Λ_E is a discrete, free A-submodule of rank r inside $\mathbb{C}_\infty$. Note that in this one-dimensional case, for $a \in \mathbb{F}_q[t]$, the operator $\partial\rho_a$ is equal to the scalar multiplication by $a(\theta)$ on $\operatorname{Lie} E(\mathbb{C}_\infty) = \mathbb{C}_\infty$. In this case, the exponential function $\exp_E$ has the infinite product expansion as follows:

$$\exp_E(z) = z \prod_{0\neq\lambda\in\Lambda_E} \left(1 - \frac{z}{\lambda}\right),$$

We mention that in the case of $r = 1$ and $a_1 = 1$, the Drinfeld module above is called the Carlitz module $\mathbf{C}$ introduced by Carlitz [**Ca35**]. Its exponential and logarithm can be written explicitly as:

$$\exp_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}, \text{ and } \log_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{L_i}, \tag{3.1.2}$$

where $D_0 = 1$, $L_0 := 1$, and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j})$, $L_i := \prod_{j=1}^{i}(\theta - \theta^{q^j})$ for $i = 1, 2, \ldots$. As mentioned above, the period lattice of $\mathbf{C}$ is a free A-module of rank one, and its generator, which is denoted by $\tilde{\pi}$ unique to a multiple in $\mathbb{F}_q^\times$, can be expressed as the following infinite product:

$$\tilde{\pi} = (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1} \in \overline{k_\infty}, \tag{3.1.3}$$

where $(-\theta)^{\frac{1}{q-1}}$ is any fixed choice of $(q-1)$th root of $-\theta$ throughout this article. Note that in the function field setting, $\mathbf{C}$ plays the analogue of $\mathbb{G}_m$ and $\tilde{\pi}$ is analogous to $2\pi\sqrt{-1}$. Such as the classical result on the transcendence of $2\pi\sqrt{-1}$, $\tilde{\pi}$ is known to be transcendental over k by Wade [**Wad41**]. For more details, we refer the reader to [**Go96, T04**].

3.2. Drinfeld logarithms. In what follows, we fix a Drinfeld $\mathbb{F}_q[t]$-module $E = (\mathbb{G}_{a/\overline{k}}, \rho)$ defined over $\overline{k}$. In [**Yu86**], Yu developed an analogous theory of Schneider-Lang on what he called E_q functions. As a consequence, he showed that nonzero periods of the Drinfeld module E are transcendental over k. Let $\operatorname{End}(E) := \{\alpha \in \mathbb{C}_\infty | \alpha\Lambda_E \subset \Lambda_E\}$, which can be identified with the ring of endomorphisms of E over $\overline{k}$ and which can be shown to be a free A-module of rank dividing r (see [**Go96, T04**]). In [**Yu97**], Yu established an analogue of Baker's theorem on linear forms in logarithms for Drinfeld modules.

THEOREM 3.2.1 (Yu [**Yu97**]). *Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy that $\exp_E(\lambda_i) \in \overline{k}$ for all i and that $\lambda_1, \ldots, \lambda_m$ are linearly independent over* $\operatorname{End}(E)$*. Then $1, \lambda_1, \ldots, \lambda_m$ are linearly independent over $\overline{k}$.*

The theorem above is a consequence of the following Yu's sub-t-module theorem, which plays the analogue of Wüstholz's analytic subgroup theorem.

Theorem 3.2.2 ([**Yu97**, Thm. 0.1]). *Let (G, ϕ) be a regular t-module defined over $\overline{k}$. Let Z be a vector in* $\operatorname{Lie} G(\mathbb{C}_\infty)$ *such that* $\exp_G(Z) \in G(\overline{k})$. *Then the smallest linear subspace in* $\operatorname{Lie} G(\mathbb{C}_\infty)$ *defined over $\overline{k}$, which is invariant under $\partial\phi_t$ and contains Z, is the tangent space at the origin of a sub-t-module of G over $\overline{k}$.*

Here the notion of 'regular'for G means that there exists a positive integer n so that the a-torsion submodule of $G(\overline{k})$ is free of rank n over $\mathbb{F}_q[t]/(a)$ for any nonzero $a \in \mathbb{F}_q[t]$. By a sub-t-module of G over $\overline{k}$, we mean a connected algebraic subgroup of G defined over $\overline{k}$ which is invariant under the action of ϕ_a for all $a \in \mathbb{F}_q[t]$.

The breakthrough from linear independence to algebraic independence for Drinfeld logarithms was first achieved by Papanikolas in the rank one case.

Theorem 3.2.3 (Papanikolas [**P08**]). *Let $\mathbf{C}$ be the Carlitz $\mathbb{F}_q[t]$-module and let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy that* $\exp_{\mathbf{C}}(\lambda_i) \in \overline{k}$ *for all i. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over k, then $\lambda_1, \ldots, \lambda_m$ are algebraically independent over $\overline{k}$.*

Along this direction for Drinfeld modules of higher rank, it was first generalized by Papanikolas and the present author in [**CP11**], where they showed the algebraic independence result for logarithms at algebraic points in the case of rank two Drinfeld $\mathbb{F}_q[t]$-modules without complex multiplication under the assumption that the characteristic of k is odd. Later on they successfully employed the theory of Galois representations on t-adic Tate module of the Drinfeld module in question, and connected its image to the t-motivic Galois group in question. Using Pink's result [**Pin97**] on the size of Galois image and Papanikolas' theory [**P08**] they established the analogue of Gelfond's conjecture for Drinfeld modules of arbitrary rank.

Theorem 3.2.4 (Chang-Papanikolas [**CP12**]). *Let E be a Drinfeld $\mathbb{F}_q[t]$-module of rank r defined over $\overline{k}$. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy that* $\exp_E(\lambda_i) \in \overline{k}$ *for all i. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over* $\operatorname{End}(E)$, *then they are algebraically independent over $\overline{k}$.*

3.3. Quasi-logarithms. For any A-algebra R, we denote by $\tau := (x \mapsto x^q) : R \to R$ the Frobenius qth power operator. We let $R[\tau]$ be the non-commutative algebra generated by τ over R subject to the relation:

$$\tau\alpha = \alpha^q \tau \text{ for } \alpha \in R.$$

Fix a Drinfeld $\mathbb{F}_q[t]$-module $E = (\mathbb{G}_{a/\overline{k}}, \rho)$ of rank $r \geq 2$ that is defined over $\overline{k}$. In what follows, we describe the theory of de Rham group for E developed by Anderson, Deligne, Gekeler and Yu in the 1980's. We consider $\mathbb{C}_\infty[\tau]$ as $\mathbb{F}_q[t]$-bimodule with left action as scalar multiplication by $a(\theta)$, and right action given by ρ_a for $a \in \mathbb{F}_q[t]$. A biderivation is an $\mathbb{F}_q$-linear map

$$\delta : \mathbb{F}_q[t] \to \mathbb{C}_\infty[\tau]\tau$$

satisfying the identity:

$$\delta_{ab} = a(\theta)\delta_b + \delta_a \rho_b \text{ for all } a, b \in \mathbb{F}_q[t].$$

We observe that by the definition each biderivation δ is uniquely determined by δ_t. The set of all biderivations is denoted by $D(\rho)$, and we say that $\delta \in D(\rho)$ is defined over $\overline{k}$ if the image of δ lies in $\overline{k}[\tau]\tau$. In analogy with the Weierstrass ζ-function ζ_Λ satisfying the algebraic differential equation $\zeta'_\Lambda = -\wp_\Lambda$, for a biderivation δ we consider the following algebraic difference equations

- $F(\theta z) - \theta F(z) = \delta_t(\exp_E(z))$,
- $F(z) \equiv 0$ (mod deg q),

and one can solve these equations to obtain a unique function $F_\delta(z)$ satisfying above. One then easily shows that $F_\delta(z)$ satisfies the follwoing: for all $a \in \mathbb{F}_q[t]$,

- $F(a(\theta)z) - a(\theta)F(z) = \delta_a(\exp_E(z))$.
- $F(z) \equiv 0$ (mod deg q).

Furthermore, $F_\delta(z)$ can be shown to be an entire function on $\mathbb{C}_\infty$. We call F_δ the *quasi-periodic function* of E associated to δ as when we restrict to Λ_E, $F_\delta|_{\Lambda_E} : \Lambda_E \to \mathbb{C}_\infty$ is A-linear. We call the values $F_\delta(\lambda)$, for $\lambda \in \Lambda_E$, *quasi-periods* of E, and following Anderson we use the integration notation

$$\int_\lambda \delta := F_\delta(\lambda).$$

A biderivation δ is said to be inner if there exists $m \in \mathbb{C}_\infty[\tau]$ so that $\delta_a = a(\theta)m - m\rho_a$ for all $a \in \mathbb{F}_q[t]$, in which case we denote this biderivation by $\delta^{(m)}$. As in [**Ge89**, **Yu90**, **BP02**], we put

$$\begin{aligned} D_{\mathrm{si}}(\rho) &= \left\{\delta^{(m)} | m \in \mathbb{C}_\infty[\tau]\tau\right\} \text{ (strictly inner)}, \\ H_{dR}(\rho) &= D(\rho)/D_{\mathrm{si}}(\rho) \text{ (de Rham)}, \end{aligned}$$

where $H_{dR}(\rho)$ is called the de Rham group of E. We observe that $\delta^{(1)} : a \mapsto a(\theta) - \rho_a$ is a biderivation and its associated quasi-periodic function is given by

$$F_{\delta^{(1)}}(z) = z - \exp_E(z),$$

and hence

$$\int_\lambda \delta^{(1)} = \lambda \text{ for } \lambda \in \Lambda_E.$$

So we shall view $\delta^{(1)}$ as the differential form of the first kind for E. We further mention that for $\delta \in D_{\mathrm{si}}(\rho)$, one can show that $\int_\lambda \delta = F_\delta(\lambda) = 0$, whence having the well-defined pairing

$$\begin{array}{ccc} H_{dR}(\rho) \times \Lambda_E & \to & \mathbb{C}_\infty \\ (\delta, \lambda) & \mapsto & \int_\lambda \delta. \end{array}$$

The deRham isomorphism established by Anderson and Gekeler [**Ge89**] is that the above is a perfect paring. It follows that as a $\mathbb{C}_\infty$-vector space,

$\dim_{\mathbb{C}_\infty} H_{dR}(\rho) = r$. Note that we have a standard basis for $H_{dR}(\rho)$ presented by the classes of biderivations $\{\delta_1, \ldots, \delta_r\}$, where we put $\delta_1 := \delta^{(1)}$ and $\delta_i : t \mapsto \tau^{i-1}$ for $i = 2, \ldots, r$. We think of that $\delta_2, \ldots, \delta_r$ are differential forms of the second kind for E. The fundamental theorem of Yu [**Yu86, Yu90**] asserts that nonzero periods and quasi-periods of E are transcendental over k. Yu's works are precisely the function field analogue of Siegel-Schneider's theorem for elliptic curves.

Fix an A-basis $\{\lambda_1, \ldots, \lambda_r\}$ of the period lattice Λ_E. The period matrix of E is referred to the $r \times r$ matrix

$$P_E := \left(\int_{\lambda_i} \delta_j \right)_{1 \leq i,j \leq r}, \tag{3.3.1}$$

and by Anderson one has the analogue of Lengdre's relations for elliptic curves:

$$\det P_E = c_E \tilde{\pi}$$

for some constant $c_E \in \overline{k}^\times$ (see [**Pe07**]). Let s be the rank of $\mathrm{End}(E)$ over A, and note that s divides r. Using Yu's sub-t-module theorem Brownawell [**B01**] showed that all the $\overline{k}$-linear relations among the entries of the period matrix P_E are those induced from the endomorphisms of E; in particular, the dimension of the $\overline{k}$-vector space spanned by the entries of P_E is equal to r^2/s. Brownawell-Yu's periods conjecture for Drinfeld modules asserts that all the $\overline{k}$-algebraic relations among the periods and quasi-periods of E arise in the way mentioned above, and it is indeed the case from the following theorem.

Theorem 3.3.2 (Chang-Papanikolas [**CP12**]). *Let E be a Drinfeld $\mathbb{F}_q[t]$-module of rank $r \geq 2$ defined over $\overline{k}$, and let P_E be the period matrix defined in (3.3.1). Let s be the rank of $\mathrm{End}(E)$ over A, and $\overline{k}(P_E)$ be the field generated by the entries of P_E over $\overline{k}$. Then we have*

$$tr.deg_{\overline{k}}\overline{k}\,(P_E) = \frac{r^2}{s}.$$

In fact, one can take quasi-logarithms into account. That is, we consider the quasi-periodic functions evaluated at the Drinfeld logarithms of algebraic points. With the help of Conrad on removing 'separability'hypothesis for the multiplication field of the Drinfeld module in question, the authors of [**CP12**] showed the algebraic independence result when ruling out the nature linear relations arising from the endomorphisms of the given Drinfeld module.

Theorem 3.3.3 (Chang-Papanikolas-Conrad [**CP12**]). *Let E be a Drinfeld $\mathbb{F}_q[t]$-module of rank r defined over $\overline{k}$. Let the classes of $\delta_1, \ldots, \delta_r$ represent a basis of $H_{dR}(\rho)$ defined over $\overline{k}$. Let $u_1, \ldots, u_n \in \mathbb{C}_\infty$ satisfy $\exp_E(u_i) \in \overline{k}$ for $i = 1, \ldots, n$. If $u_1, \ldots, u_n$ are linearly independent over $\mathrm{End}(E)$, then the rn quasi-logarithms*

$$\cup_{i=1}^{n} \cup_{j=1}^{r} \left\{ F_{\delta_j}(u_i) \right\}$$

are algebraically independent over $\overline{k}$.

4. Positive characteristic MZV's

4.1. Single zeta values. Let A_+ be the set of all monic polynomials in A. In analogy with the special values of Riemann ζ-function at positive integers, Carlitz [**Ca35**] initiated the following positive characteristic zeta values: for $s \in \mathbb{N}$,

$$\zeta_A(s) := \sum_{a \in A_+} \frac{1}{a^s} \in k_\infty^\times.$$

Note that since the absolute value $|\cdot|_\infty$ is non-archimedean, $\zeta_A(s)$ converges for any positive integer s. One interesting theory of Carlitz is to derive an analogue of Euler's formula: for any positive integer s divisible by $q-1$, we have

$$\zeta_A(s) = \frac{BC_s}{\Gamma_{s+1}} \tilde{\pi}^s \tag{4.1.1}$$

where $\Gamma_{s+1} \in A$ is the Carlitz factorial defined by $\Gamma_{s+1} := \prod_{i=0}^{\infty} D_i^{s_i}$ for writing the q-adic expansion $s = \sum_{i=0}^{\infty} s_i q^i$, and $BC_s \in k$ is the sth Bernoulli-Carlitz number defined from the generating function:

$$\frac{z}{\exp_{\mathbf{C}}(z)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Gamma_{n+1}} z^n.$$

For an integer $i \geq 0$, we denote by $A_{i,+}$ the set of monic polynomials of degree i in A. Note that for any positive integer s, the power sum $\sum_{a \in A_{i,+}} a^s$ vanishes for $i \gg 0$ by the work of Goss [**Go79**]. Therefore, the analogue of Riemann zeta function at non-positive integers is defined as follows: for integer $s \geq 0$,

$$\zeta_A(-s) := \sum_{i=0}^{\infty} \left(\sum_{a \in A_{i,+}} a^s \right) \in k.$$

Moreover, Goss showed a precise analogue of the classical situation for the vanishing of Riemann zeta function at even negative integers: for any positive integer s divisible by $q-1$, one has $\zeta_A(-s) = 0$.

Let v be a monic irreducible polynomial of A. Goss [**Go79**] further carried out that for non-negative integer s,

$$\zeta_A(-s)_v := (1 - v^s) \zeta_A(-s)$$

interpolates to a continuous A_v-valued function on $\mathbb{S}_v := \mathbb{Z}/(q^{\deg v} - 1) \times \mathbb{Z}_p$. As a consequence, an equivalent definition of Goss' v-adic zeta values is given as follows: for any integer s,

$$\zeta_A(s)_v := \sum_{i=0}^{\infty} \left(\sum_{a \in A_{i,+}, v \nmid a} \frac{1}{a^s} \right) \in A_v.$$

Given a positive integer s, Goss showed that $\zeta_A(s)_v = 0$ if and only if s is divisible by $q-1$.

In the seminar paper [**AT90**], Anderson and Thakur gave logarithmic interpretations both for ∞-adic and v-adic single zeta values. To state their result, we introduce the notion of tensor powers of Carlitz module in [**AT90**]. Fix a positive integer s. The sth tensor power of the Carlitz module denoted by $\mathbf{C}^{\otimes s} = (\mathbb{G}_{a/A}^{s}, [\cdot]_s)$ is the s-dimensional t-module defined over A given by the $\mathbb{F}_q$-linear ring homomorphism

$$[\cdot]_s : \mathbb{F}_q[t] \to \mathrm{Mat}_s(A[\tau])$$

with

$$[t]_s = \theta I_s + N_s + E_s\tau,$$

where

$$N_s := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad E_s := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 0 \end{pmatrix}.$$

We denote by $\exp_{\mathbf{C}^{\otimes s}}$ and $\log_{\mathbf{C}^{\otimes s}}$ the exponential and the logarithm of $\mathbf{C}^{\otimes s}$ respectively. Note that since $\log_{\mathbf{C}^{\otimes s}}$ is a vector-valued power series with coefficients in k, it makes sense to consider its v-adic convergence and we denote by $\log_{\mathbf{C}^{\otimes s}}(\mathbf{x})_v$ whenever $\log_{\mathbf{C}^{\otimes s}}$ converges v-adically at the point $\mathbf{x}$.

Theorem 4.1.2 (Anderson-Thakur [**AT90**, Theorem 3.8.3]). *Let s be a positive integer, and v be a monic irreducible polynomial of A. Then there exists a special point $\mathbf{v}_s \in \mathbf{C}^{\otimes s}(A)$ satisfying the following.*

(1) *There exists a vector $Z_s \in \mathrm{Lie}\,\mathbf{C}^{\otimes s}(\mathbb{C}_\infty)$ with last coordinate given by $\Gamma_s\zeta_A(s)$ for which $\exp_{\mathbf{C}^{\otimes s}}(Z_s) = \mathbf{v}_s$.*
(2) *The logarithm of $\mathbf{C}^{\otimes s}$ converges v-adically at $[v^s - 1]_s\mathbf{v}_s$ and the last coordinate of $\log_{\mathbf{C}^{\otimes s}}([v^s-1]_s\mathbf{v}_s)_v$ gives $v^s\Gamma_s\zeta_v(s)$.*

In [**Yu91**], Yu developed a transcendence theory for the last coordinate logarithms of $\mathbf{C}^{\otimes s}$ at algebraic points. Together with Theorem 4.1.2, one has the following transcendence results which have surpassed the classical status.

Theorem 4.1.3 (Yu [**Yu91**]). *Let s be a positive integer, and v be a finite place of k. Then the following hold.*

(1) *$\zeta_A(s)$ is transcendental over k.*
(2) *$\zeta_v(s)$ is transcendental over k if and only if s is not divisible by $q-1$.*

Using the t-motivic transcendence theory developed by Anderson, Brownawell and Papanikolas [**ABP04**] and Papanikolas [**P08**], Chang and Yu [**CY07**] demonstrated that all the $\bar{k}$-algebraic relations among the Carlitz zeta values are those arising from the Euler-Carlitz relations (4.1.1) and the Frobenius pth power relations: for $s, n \in \mathbb{N}$,

$$\zeta_A(sp^n) = \zeta_A(s)^{p^n}.$$

An equivalent formulation is the following identity in terms of transcendence degree.

THEOREM 4.1.4 (Chang-Yu [**CY07**]). *Let s be a positive integer. Then we have the following:*

$$\operatorname{tr.deg}_{\overline{k}} \overline{k}\left(\tilde{\pi}, \zeta_A(1), \ldots, \zeta_A(s)\right) = s + 1 - \left\lfloor \frac{s}{p} \right\rfloor - \left\lfloor \frac{s}{q-1} \right\rfloor + \left\lfloor \frac{s}{p(q-1)} \right\rfloor.$$

In the following contexts, we will report the results of [**CM17**] generalizing Theorem 4.1.2 and their applications.

4.2. ∞-adic MZV's. In [**T04**], Thakur defined the ∞-adic MZV's that are generalizations of Carlitz zeta values: for any index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$,

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty,$$

where $a_1, \ldots, a_r$ run over all monic polynomials in A for which $|a_1|_\infty > \cdots > |a_r|_\infty$. As same as the terminology in the classical theory, the weight and the depth of the presentation $\zeta_A(\mathfrak{s})$ are defined as $\operatorname{wt}(\mathfrak{s})$ and $\operatorname{dep}(\mathfrak{s})$ respectively. We shall mention that for higher depth MZV's, it is not straightforward to see whether they are non-vanishing or not, but Thakur [**T09**] showed that every ∞-adic MZV is non-vanishing.

For these ∞-adic MZV's, we have the following properties that are parallel to some of classical real-valued MZV's:

(1) *Periods interpretation.* In [**AT09**], Anderson and Thakur showed that for every ∞-adic MZV $\zeta_A(\mathfrak{s})$, one is able to explicitly construct a rigid analytically trivial dual t-motive in the sense of [**ABP04**] so that up an explicit multiple in A, $\zeta_A(\mathfrak{s})$ occurs as certain entry of the period matrix of the t-motive in question. This result can be viewed as a function field analogue of the work of Terasoma [**Te02**] and Goncharov [**Gon02**].
(2) *Sum shuffle relations.* In [**T10**], Thakur showed that the product of two ∞-adic MZV's can be expressed as an $\mathbb{F}_p$-linear combination of certain MZV's with the same weight, and Thakur called these *sum shuffle* relations. In the special case for the product of two Carlitz zeta values, the coefficients of the relations mentioned above are explicitly worked out by Chen [**Ch15**].
(3) *Double Eisenstein series.* In [**Ch17**], for any MZV of depth two, called double zeta value, Chen explicitly constructed the so-called double Eisenstein series for which the double zeta value in question occurs as the constant term of the double Eisenstein series under the 't-expansion'. Furthermore, Chen established sum shuffle relations for the product of two single Eisenstein series in terms of linear

combinations of some double Eisenstein series of the same weight. Chen's lifting result can be regarded as a partial analogue of the work of Gangl-Kaneko-Zagier [**GKZ06**].

(4) *Dimension conjecture.* In [**To18**], Todd created some linear relations among the same weight ∞-adic MZV's and predicted that these account for all the k-linear relations.

Conjecture 4.2.1 (Todd's dimension conjecture). *Let n be a positive integer, and let $\mathcal{Z}_n$ be the k-vector space spanned by all ∞-adic MZV's of weight n. Then one has*

$$\dim_k \mathcal{Z}_n = \begin{cases} 2^{n-1} & \text{if } 1 \leq n < q, \\ 2^{n-1} - 1 & \text{if } n = q \\ \sum_{i=1}^{q} \dim_k \mathcal{Z}_{n-i} & \text{if } n > q. \end{cases}$$

We let $\mathcal{Z} := \sum_{n=1}^{\infty} \mathcal{Z}_n$ be the vector space spanned by all ∞-adic MZV's over k. According to Thakur's sum shuffle relations [**T10**] mentioned above, we have that $\mathcal{Z}_{n_1} \cdot \mathcal{Z}_{n_2} \subset \mathcal{Z}_{n_1+n_2}$ and so $\mathcal{Z}$ has a k-algebra structure. Using the work of Anderson-Thakur on periods interpretation for ∞-adic MZV's and the linear independence criterion developed by Anderson-Brownawell-Papanikolas [**ABP04**], the so-called ABP criterion, we have the following result as a stronger analogue of Goncharov's conjecture including the Baker-Wüstholz-Yu philosophy.

Theorem 4.2.2 (Chang [**C14**]). *The following results hold.*

(1) *We have $\mathcal{Z} = \bigoplus_{n=1}^{\infty} \mathcal{Z}_n$. That is, $\mathcal{Z}$ is a graded k-algebra.*
(2) *If the given ∞-adic MZV's $u_1, \ldots, u_m$ are linearly independent over k, then*

$$1, u_1, \ldots, u_m$$

are linearly independent over $\overline{k}$.

As a consequence, we have that every ∞-adic MZV is transcendental over k. One core problem for ∞-adic MZV's is to understand all the $\overline{k}$-algebraic relations among them. As we have $\mathcal{Z}_{n_1} \cdot \mathcal{Z}_{n_2} \subset \mathcal{Z}_{n_1+n_2}$, the structure theorem above reduces the study of $\overline{k}$-algebraic relations among ∞-adic MZV's to the understanding of the k-linear relations among the same weight MZV's. In other words, theoretically one is able to understand all the $\overline{k}$-algebraic relations among ∞-adic MZV's once Conjecture 4.2.1 is true, but it is still far from Todd's conjecture at present. However, once we fix weight and restrict to double zeta values, then there is an effective criterion enabling us to compute the dimensions.

Theorem 4.2.3 (Chang [**C16**]). *Let $n \geq 2$ be a positive integer. Put*

$$\mathcal{V} := \left\{ (s_1, s_2) \in \mathbb{N}^2; s_1 + s_2 = n \text{ and } (q-1) | s_2 \right\}.$$

(1) *For each $\mathfrak{s} \in \mathcal{V}$, we explicitly construct a special point $\Xi_{\mathfrak{s}} \in \mathbf{C}^{\otimes n}(A)$ so that*

$$\begin{aligned}&\dim_k \mathit{Span}_k \{\tilde{\pi}^n, \zeta_A(1, n-1), \zeta_A(2, n-2), \cdots, \zeta_A(n-1, 1)\}\\ &= n - \left\lfloor \frac{n-1}{q-1} \right\rfloor + \operatorname{rank}_{\mathbb{F}_q[t]} \mathit{Span}_{\mathbb{F}_q[t]} \{\Xi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathcal{V}}.\end{aligned}$$

(2) *We establish an effective algorithm for computing the rank*

$$\operatorname{rank}_{\mathbb{F}_q[t]} \mathit{Span}_{\mathbb{F}_q[t]} \{\Xi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathcal{V}}.$$

We refer the reader to some computational data in [**C16**, Sec. 6.3]. By generalizing some methods in [**P08, CY07**], some algebraic independence results for ∞-adic MZV's under certain conditions are obtained by Mishiba in [**M17**].

4.3. v-adic MZV's and state of the art. Let s be a positive integer. We recall the Carlitz logarithm given in (3.1.2) playing the analogue of the classical logarithm. In analogy with the classical polylogarithms, Anderson and Thakur defined the sth Carlitz polylogarithm in [**AT09**]:

$$\mathrm{Li}_s(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{L_i^s}.$$

For convenience, we still use the same symbol Li without confusions with the classical ones, which will not be used below. Another important interpretation for $\zeta_A(s)$ in Theorem 4.1.2 (1) is that $\zeta_A(s)$ can be expressed as an explicit k-linear combination of Li_s at some integral points in A. This important formula is the key step for the authors of [**CY07**] when properly quoting the theory of Papanikolas [**P08**].

To generalize Li_s to the multi-variable version, the author of the present article defined the Carlitz multiple polylogarithms (abbreviated as CMPL's) in [**C14**], and one naturally defines their star version in [**CM17**].

DEFINITION 4.3.1. *Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$. The $\mathfrak{s}$th Carlitz multiple polylogarithm is defined by*

$$\mathrm{Li}_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[\![z_1, \ldots, z_r]\!]$$

and its star version by

$$\mathrm{Li}^{\star}_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[\![z_1, \ldots, z_r]\!].$$

In [**C14**], the author used the interpolation formula of Anderson-Thakur polynomials to generalize Anderson-Thakur's formula on Carlitz zeta values to higher depth MZV's. For any index $\mathfrak{s} \in \mathbb{N}^r$, the MZV $\zeta_A(\mathfrak{s})$ can be expressed as an explicit k-linear combination of $\mathrm{Li}_{\mathfrak{s}}$ at some integral points. As power series, $\mathrm{Li}_{\mathfrak{s}}$ can be written as a linear combination of some $\mathrm{Li}^{\star}_{\mathfrak{s}_\ell}$ (see [**CM17**]). Therefore we can derive the following explicit formula ([**CM17**, Thm. 5.2.5]): for any index $\mathfrak{s} \in \mathbb{N}^r$, there exist some explicit indexes $\mathfrak{s}_\ell$ with $\mathrm{wt}(\mathfrak{s}_\ell) = \mathrm{wt}(\mathfrak{s})$ and $\mathrm{dep}(\mathfrak{s}_\ell) \leq \mathrm{dep}(\mathfrak{s})$, coefficients $\alpha_\ell \in k^\times$, and integral points $\mathbf{u}_\ell \in A^{\mathrm{dep}(\mathfrak{s}_\ell)}$ so that

$$\zeta_A(\mathfrak{s}) = \sum_\ell \alpha_\ell \, \mathrm{Li}^{\star}_{\mathfrak{s}_\ell}(\mathbf{u}_\ell). \tag{4.3.2}$$

In what follows, we fix a monic irreducible polynomial v in A, and let k_v be the completion of k at v. Let $\mathbb{C}_v$ be the v-adic completion of a fixed algebraic closure of k_v. We then fix an embedding of $\overline{k}$ into $\mathbb{C}_v$. For any vector $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}_v^r$, we define

$$\|\mathbf{z}\|_v := \max_i \{|z_i|_v\}.$$

Inspired by the definitions of Furusho's p-adic MZV's, the authors of [**CM17**] prove first about the v-adic analytic continuation of $\mathrm{Li}^{\star}_{\mathfrak{s}}$ and then define the v-adic MZV's.

Lemma 4.3.3. *For any index $\mathfrak{s} \in \mathbb{N}^r$, we define $\mathrm{Li}^{\star}_{\mathfrak{s}}(z_1, \ldots, z_r)_v$ to be the same power series as $\mathrm{Li}^{\star}_{\mathfrak{s}}(z_1, \ldots, z_r)$ but we regard it as in $\mathbb{C}_v[\![z_1, \ldots, z_r]\!]$. Then the series $\mathrm{Li}^{\star}_{\mathfrak{s}}(z_1, \ldots, z_r)_v$ can be analytically continued to the closed unit ball $\left\{\mathbf{z} \in \mathbb{C}_v^r \middle| \|\mathbf{z}\|_v \leq 1\right\}$, and we still denote by $\mathrm{Li}^{\star}_{\mathfrak{s}}(z_1, \ldots, z_r)_v$ the analytically continued function.*

Based on the Lemma above, we have that $\mathrm{Li}^{\star}_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)_v$ is well-defined for those $\mathfrak{s}_\ell$ and $\mathbf{u}_\ell$ occurring in (4.3.2) as $\|\mathbf{u}_\ell\|_v \leq 1$. Furusho's definition of p-adic MZV's inspires us to define the following v-adic MZV's.

Definition 4.3.4. *Fix any index $\mathfrak{s} \in \mathbb{N}^r$. We let α_ℓ, $\mathfrak{s}_\ell$, $\mathbf{u}_\ell$ be those given in (4.3.2). We define the v-adic multiple zeta value $\zeta_A(\mathfrak{s})_v$ by the following formula:*

$$\zeta_A(\mathfrak{s})_v = \sum_\ell \alpha_\ell \, \mathrm{Li}^{\star}_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)_v.$$

The weight and the depth of the presentation $\zeta_A(\mathfrak{s})_v$ are defined to be $\mathrm{wt}(\mathfrak{s})$ and $\mathrm{dep}(\mathfrak{s})$ respectively.

We mention that such as the classical p-adic case, in the depth one case our v-adic single zeta value $\zeta_A(s)_v$ and Goss' v-adic zeta value $\zeta_v(s)$ (for $s \in \mathbb{N}$) just differ by a rational multiple $(1 - v^{-s}) \in k$. See [**AT90**, Thm. 3.8.3 (II)]. The primary result of [**CM17**] is to verify a stronger version of the analogue of Conjecture 2.3.1 in the function fields setting.

Theorem 4.3.5 (Chang-Mishiba [**CM17**]). *Let v be a finite place of k and fix an embedding $\overline{k} \hookrightarrow \mathbb{C}_v$. For any positive integer n, we let $\overline{\mathcal{Z}}_n$ (resp. $\overline{\mathcal{Z}}_{n,v}$) be the $\overline{k}$-vector space spanned by ∞-adic MZV's (resp. v-adic MZV's) of weight n. Then the following*

$$\phi_v := (\zeta_A(\mathfrak{s}) \mapsto \zeta_A(\mathfrak{s})_v) : \overline{\mathcal{Z}}_n \to \overline{\mathcal{Z}}_{n,v}$$

is a well-defined $\overline{k}$-linear map. Moreover, if n is divisible by $q-1$, then the kernel of ϕ_v contains the one-dimensional $\overline{k}$-vector subspace $\overline{k}\cdot\zeta_A(n)$.

Based on the result above, the following question is natural but important.

QUESTION 4.3.6. *Is the kernel of ϕ_v independent of v?*

4.4. Key ideas of the proof of Theorem 4.3.5. In this section, we sketch how we prove Theorem 4.3.5. There are two key ingredients in the proof.

(1) Give logarithmic interpretations both for ∞-adic and v-adic MZV's.
(2) Apply Theorem 3.2.2.

The step (1) above is to generalize Theorem 4.1.2 to higher depth MZV's. The precise statement is as follows.

THEOREM 4.4.1. *Fix an index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$ with $\mathrm{wt}(\mathfrak{s})=n$ and v a finite place of k. Then one explicitly constructs a uniformizable, regular t-module $G_{\mathfrak{s}}$ defined over A and a special point $\Xi_{\mathfrak{s}}\in G_{\mathfrak{s}}(A)$ so that the following hold.*

(1) *There exists a vector $Z_{\mathfrak{s}}\in\operatorname{Lie}G_{\mathfrak{s}}(\mathbb{C}_\infty)$ for which $\exp_{G_{\mathfrak{s}}}(Z_{\mathfrak{s}})=\Xi_{\mathfrak{s}}$ and the nth coordinate of $Z_{\mathfrak{s}}$ gives $c_{\mathfrak{s}}\cdot\zeta_A(\mathfrak{s})$ for some explicit nonzero constant $c_{\mathfrak{s}}$ in A.*
(2) *The logarithm of $G_{\mathfrak{s}}$ converges v-adically at $\Xi_{\mathfrak{s}}$ and its nth coordinate gives $c_{\mathfrak{s}}\cdot\zeta_A(\mathfrak{s})_v$.*

The results above are given in [**CM17**, Thm. 1.2.2, Thm. 6.2.4] with the special point $\Xi_{\mathfrak{s}}$ taken to be $[a(t)]\mathbf{v}_{\mathfrak{s}}$ in [**CM17**, Thm. 6.2.4] and the constant $c_{\mathfrak{s}}$ given by $a(\theta)\Gamma_{s_1}\cdots\Gamma_{s_r}$. To achieve the desired results above, one goes back to the formula (4.3.2). There are two parts in the procedure. The first is that for each pair $(\mathfrak{s}_\ell,\mathbf{u}_\ell)$, one explicitly constructs a uniformizable, regular t-module G_ℓ defined over A and a special point $\mathbf{v}_\ell\in G_\ell(A)$ so that $\log_{G_\ell}(\mathbf{v}_\ell)$ converges and its nth coordinate is $\mathrm{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)$. This was carried out in [**CM19**] and the v-adic case can be done in the same fashion of computations on logarithms. The ideas of constructing G_ℓ and $\mathbf{u}_\ell$ go back to [**CPY19**] when considering G_ℓ, $\mathbf{u}_\ell$ as transforming certain Frobenius modules $\mathcal{M}'_\ell$ and $\mathcal{M}_\ell$. Terminology of Frobenius modules are referred to [**CPY19**] and the detailed transformation and correspondence are referred to [**CM17**, Sec. 4].

To describe the second part, we first mention that the nth tensor power of the Carlitz t-motive denoted by $C^{\otimes n}$ is a Frobenius submodule of $\mathcal{M}'_\ell$ for each ℓ. Note that in this setting $\mathcal{M}'_\ell$ and $C^{\otimes n}$ are dual t-motives in the terminology of [**ABP04**] (dual notion of the t-motives in [**A86**]), and $C^{\otimes n}$ is the dual t-motive associated to $\mathbf{C}^{\otimes n}$ (see [**BP20**]). Then we take $\mathcal{M}$ to be the fiber coproduct of $\{\mathcal{M}'_\ell\}_\ell$ over $C^{\otimes n}$ and note that $\mathcal{M}$ is a dual t-motive. Then $G_{\mathfrak{s}}$ is the corresponding t-module of $\mathcal{M}$, and we note that there is a natural morphism

$$\pi:\bigoplus_\ell G_\ell\to G_{\mathfrak{s}}.$$

To simplify the arguments for catching the crucial ideas quickly, we simply *assume* that all the coefficients α_ℓ are 1 (although it is not valid in general). The special point $\mathbf{v}_\mathfrak{s} \in G_\mathfrak{s}(A)$ is image of the point $(\mathbf{v}_\ell)_\ell$ under the morphism π. The special point $\Xi_\mathfrak{s}$ is then taken under suitable $\mathbb{F}_q[t]$-action on the point $\mathbf{v}_\mathfrak{s}$ in the t-module $G_\mathfrak{s}$ in order to ensure the v-adic convergence in question. The desired formulas are then derived from the key Lemma 3.3.2 of [**CM17**], which basically says that the nth coordinate of $\log_{G_\mathfrak{s}}$ is the sum of the nth coordinate of $\log_{G_\ell}$.

Now we briefly describe how to prove Theorem 4.3.5 using Theorem 4.4.1 and Theorem 3.2.2. Suppose that we are given a non-trivial $\overline{k}$-linear relations $\sum_{i=1}^{m} a_i\zeta_A(\mathfrak{s}_i) = 0$ with $\mathrm{wt}(\mathfrak{s}_i) = n$ for all i. Our goal is to show that

$$\sum_{i=1}^{m} a_i\zeta_A(\mathfrak{s}_i)_v = 0.$$

Put $b_i := a_i/c_{\mathfrak{s}_i}$ with $c_{\mathfrak{s}_i}$ given in Theorem 4.4.1. So the original identity is rewritten as

$$\sum_{i=1}^{m} b_i c_{\mathfrak{s}_i}\zeta_A(\mathfrak{s}_i) = 0. \tag{4.4.2}$$

Put $E := \bigoplus_{i=1}^{m} G_{\mathfrak{s}_i}$, $\Xi := (\Xi_{\mathfrak{s}_1}^{\mathrm{tr}}, \ldots, \Xi_{\mathfrak{s}_m}^{\mathrm{tr}})^{\mathrm{tr}} \in E(A)$ and $Z := (Z_{\mathfrak{s}_1}^{\mathrm{tr}}, \ldots, Z_{\mathfrak{s}_m}^{\mathrm{tr}})^{\mathrm{tr}} \in \mathrm{Lie}\, E(\mathbb{C}_\infty)$. By Theorem 4.4.1 we know the following.

- We have $\exp_E(Z) = \Xi$;
- The nth coordinate of each block $Z_{\mathfrak{s}_1}, \ldots, Z_{\mathfrak{s}_m}$ in the vector Z gives $c_{\mathfrak{s}_1} \cdot \zeta_A(\mathfrak{s}_1), \ldots, c_{\mathfrak{s}_m} \cdot \zeta_A(\mathfrak{s}_m)$ respectively.
- The logarithm of E converges v-adic at the special point Ξ, and the nth coordinate of each block $\log_{G_{\mathfrak{s}_1}}(\Xi_{\mathfrak{s}_1})_v, \ldots, \log_{G_{\mathfrak{s}_m}}(\Xi_{\mathfrak{s}_m})_v$ in the vector $\log_E(\Xi)_v$ gives $c_{\mathfrak{s}_1} \cdot \zeta_A(\mathfrak{s}_1)_v, \ldots, c_{\mathfrak{s}_m} \cdot \zeta_A(\mathfrak{s}_m)_v$ respectively.

We fix a natural identification $\mathrm{Lie}\, E = \mathrm{Lie}\, G_{\mathfrak{s}_1} \oplus \cdots \mathrm{Lie}\, G_{\mathfrak{s}_m}$. To simplify the notation, we first fix a coordinate system for each $\mathrm{Lie}\, G_{\mathfrak{s}_i}$ for $i = 1, \ldots, m$ and then let X_i be the nth coordinate of the fixed coordinate system of $\mathrm{Lie}\, G_{\mathfrak{s}_i}$ for each i. It follows that Z satisfies the equation $\sum_{i=1}^{m} b_i X_i = 0$ because of (4.4.2). We denote by ρ the map defining the $\mathbb{F}_q[t]$-module structure on E. We then consider the smallest linear subspace in $\mathrm{Lie}\, E(\mathbb{C}_\infty)$ so that it contains Z, and it is invariant under $\partial\rho_t$-action, and it is defined over $\overline{k}$. By Theorem 3.2.2 this space is exactly equal to $\mathrm{Lie}\, H(\mathbb{C}_\infty)$ for some sub-t-module H of E defined over $\overline{k}$. One shows that the hyperplane defined by $\sum_{i=1}^{m} b_i X_i = 0$ is invariant under $\partial\rho_t$, and hence

$$\sum_{i=1}^{m} b_i X_i \in (\text{Defining equations of } \mathrm{Lie}\, H)\,. \tag{4.4.3}$$

Note that since $Z \in \mathrm{Lie}\, H(\mathbb{C}_\infty)$, $\Xi = \exp_E(Z) \in H(\mathbb{C}_\infty) \cap E(\overline{k})$. One then shows that $\log_E(\Xi)_v$ is equal to $\log_H(\Xi)_v \in \mathrm{Lie}\, H(\mathbb{C}_v)$ and therefore

$\log_E(\Xi)_v$ satisfies the defining equations of $\operatorname{Lie} H$. By (4.4.3) one has the identity $\sum_{i=1}^m b_i c_{\mathfrak{s}_i} \zeta_A(\mathfrak{s}_i)_v = 0$, whence deriving

$$\sum_{i=1}^{m} a_i \zeta_A(\mathfrak{s}_i)_v = 0$$

as claimed.

REMARK 4.4.4. At the end, we point out that the transcendence tool of proving Theorem 4.3.5 is Yu's sub-t-module theorem that is parallel to Wüstholz's analytic subgroup theorem. As realized from our arguments above, logarithmic interpretation of MZV's will provide one possible approach towards Conjecture 2.3.1. In the classical theory one can ask naively whether any real-valued MZV (up to a rational multiple) can be related to certain coordinate of the logarithm of certain commutative algebraic group defined over $\overline{\mathbb{Q}}$ fitting into Wüstholz's theory. However, this question is still unknown.

References

[A86] G. W. Anderson, *t-motives*, Duke Math. J. **53** (1986), no. 2, 457–502. MR 0850546

[ABP04] G. W. Anderson, W. D. Brownawell and M. A. Papanikolas, *Determination of the algebraic relations among special Γ-values in positive characteristic*, Ann. of Math. (2) **160** (2004), no. 1, 237–313. MR 2119721

[AT90] G. W. Anderson and D. S. Thakur, *Tensor powers of the Carlitz module and zeta values*, Ann. of Math. (2) **132** (1990), no. 1, 159–191. MR 1059938

[AT09] G. W. Anderson and D. S. Thakur, *Multizeta values for $\mathbb{F}_q[t]$, their period interpretation, and relations between them*, Int. Math. Res. Not. IMRN (2009), no. 11, 2038–2055. MR 2507109

[An04] Y. André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthéses, **17**. Société Mathématique de France, Paris, 2004. MR 2115000

[BW07] A. Baker and G. Wüstholz, *Logarithmic forms and Diophantine geometry*, New Mathematical Monographs, 9. Cambridge University Press, Cambridge, 2007. MR 2382891

[Br14] F. Brown, *Motivic periods and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$*, Proceedings of the International Congress of Mathematicians-Seoul 2014. Vol. II, 295–318, Kyung Moon Sa, Seoul, 2014. MR 3728616

[B01] W. D. Brownawell, *Minimal extensions of algebraic groups and linear independence*, J. Number Theory **90** (2001), 239–254. MR 1858075

[BP02] W. D. Brownawell and M. A. Papanikolas, *Linear independence of gamma values in positive characteristic*, J. Reine Angew. Math. 549 (2002), 91–148. MR 1916653

[BP20] W. D. Brownawell and M. A. Papanikolas, *A rapid introduction to Drinfeld modules, t-modules, and t-motives*, to appear in: t-Motives: Hodge Structures, Transcendence, and Other Motivic Aspects.

[BGF18] J. I. Burgos Gil and J. Fresán, *Multiple zeta values: from numbers to motives*, to appear in Clay Mathematics Proceedings.

[Ca35] L. Carlitz, *On certain functions connected with polynomials in a Galois field*, Duke Math. J. **1** (1935), no. 2, 137–168. MR 1545872

[C14] C.-Y. Chang, *Linear independence of monomials of multizeta values in positive characteristic*, Compositio Math. **150** (2014), 1789–1808. MR 3279257

[C16] C.-Y. Chang, *Linear relations among double zeta values in positive characteristic*, Camb. J. Math. **4** (2016), no. 3, 289–331. MR 3550284

[CM17] C.-Y. Chang and Y. Mishiba, *Logarithmic interpretation of multiple zeta values in positive characteristic*, arXiv:1710.10849.

[CM19] C.-Y. Chang and Y. Mishiba, *On multiple polylogarithms in characteristic p: v-adic vanishing versus ∞-adic Eulerianness*, Int. Math. Res. Not. IMRN 2019, no. 3, 923–947. https://doi.org/10.1093/imrn/rnx151. MR 3910478

[CP11] C.-Y. Chang and M. A. Papanikolas, *Algebraic relations among periods and logarithms of rank 2 Drinfeld modules*, Amer. J. Math. **133** (2011), 359–391. MR 2797350

[CP12] C.-Y. Chang and M. A. Papanikolas, *Algebraic independence of periods and logarithms of Drinfeld modules. With an appendix by B. Conrad*, J. Amer. Math. Soc. **25** (2012), 123–150. MR 2833480

[CPY19] C.-Y. Chang, M. A. Papanikolas and J. Yu, *An effective criterion for Eulerian multizeta values in positive characteristic*, J. Eur. Math. Soc. JEMS **21** (2019), no. 2, 405–440. arXiv:1411.0124. MR 3896206

[CY07] C.-Y. Chang and J. Yu, *Determination of algebraic relations among special zeta values in positive characteristic*, Adv. Math. **216** (2007), no. 1, 321–345. MR 2353259

[Ch15] H.-J. Chen, *On shuffle of double zeta values for $\mathbb{F}_q[t]$*, J. Number Theory **148** (2015), 153–163. MR 3283173

[Ch17] H.-J. Chen, *On shuffle of double Eisenstein series in positive characteristic*, J. Théor. Nombres Bordeaux **29** (2017), no. 3, 815–825. MR 3745250

[Co82] R. Coleman, *Dilogarithms, regulators and p-adic L-functions*, Invent. Math. **69** (1982), no. 2, 171–208. MR 0674400

[Dr74] V. G. Drinfeld, *Elliptic modules*, Math. USSR-Sb. **23** (1974), 561–592. MR 0384707

[F04] H. Furusho, *p-adic multiple zeta values. I. p-adic multiple polylogarithms and the p-adic KZ equation*, Invent. Math. **155** (2004), no. 2, 253–286. MR 2031428

[FJ07] H. Furusho and A. Jafari, *Regularization and generalized double shuffle relations for p-adic multiple zeta values*, Compositio Math. **143** (2007), 1089–1107. MR 2360311

[GKZ06] H. Gangl, M. Kazeko and D. Zagier, *Double zeta values and modular forms*, Automorphic forms and zeta functions, 71–106, World Sci. Publ., Hackensack, NJ, 2006. MR 2208210

[Ge89] E.-U. Gekeler, *On the de Rham isomorphism for Drinfeld modules*, J. Reine Angew. Math. **401** (1989), 188–208. MR 1018059

[G19] N. Green, *Special zeta values using tensor powers of Drinfeld modules*, Math. Res. Lett. **26** (2019), no. 6, 1629–1676.

[Go79] D. Goss, *v-adic zeta functions, L-series, and measures for function fields. With an addendum*, Invent. Math. **55** (1979), no. 2, 107–119. MR 0553704

[Go96] D. Goss, *Basic structures of function field arithmetic*, Springer-Verlag, Berlin, 1996. MR 1423131

[Gon97] A. B. Goncharov, *The double logarithm and Manin's complex for modular curves*, Math. Res. Lett., **4** (1997), 617–636. MR 1484694

[Gon02] A. B. Goncharov, *Periods and mixed Tate motives*, arXiv:math/0202154. MR 2115000

[Ho97] M. Hoffman, *The algebra of multiple harmonic series*, J. Algebra **194** (1997), 477–495. MR 1467164

[IKZ06] K. Ihara, M. Kaneko and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compositio Math. **142** (2006), 307–338. MR 2218898

[M17] Y. Mishiba, *On algebraic independence of certain multizeta values in characteristic p*, J. Number Theory **173** (2017), 512–528. MR 3581930

[P08] M. A. Papanikolas, Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms, Invent. Math. **171** (2008), no. 1, 123–174. MR 2358057

[Pe07] F. Pellarin, Aspects de l'indépendance algébrique en caractéristique non nulle, Séminaire Bour-baki, Vol. 2006/2007, Astérisque No. 317 (2008), Exp. no. 973, viii, 205–242. MR 2487735

[Pin97] R. Pink, *The Mumford-Tate conjecture for Drinfeld modules*, Publ. Res. Inst. Math. Sci. **33** (1997), 393–425. MR 1474696

[Ra02] G. Racinet, Doubles mélanges des polylogarithmes multiples aux racines de l'unité, Publ. Math. Inst. Hautes Études Sci. **95** (2002), 185–231. MR 1953193

[Re93] C. Reutenauer, *free Lie Algebras*, Oxford Science Publications (1993). MR 1231799

[T92] D. S. Thakur, *Drinfeld modules and arithmetic in the function fields*, Internat. Math. Res. Notices 1992, no. 9, 185–197. MR 1185833

[Te02] T. Terasoma, *Mixed Tate motives and multiple zeta values*, Invent. Math. **149** (2002), 339–369. MR 1918675

[T04] D. S. Thakur, *Function field arithmetic*, World Scientific Publishing, River Edge NJ, 2004. MR 2091265

[T09] D. S. Thakur, *Power sums with applications to multizeta and zeta zero distribution for $\mathbb{F}_q[t]$*, Finite Fields Appl. **15** (2009), no. 4, 534–552. MR 2535594

[T10] D. S. Thakur, *Shuffle relations for function field multizeta values*, Int. Math. Res. Not. IMRN (2010), no. 11, 1973–1980. MR 2646351

[To18] G. Todd, *A Conjectural Characterization for $\mathbb{F}_q(t)$-Linear Relations between Multizeta Values*, J. Number Theory **187**, 264–287 (2018). MR 3766911

[Wad41] L. I. Wade, *Certain quantities transcendental over $GF(p^n, x)$*, Duke Math. J. **8**, 701–729 (1941). MR 0006157

[Wa08] M. Waldschmidt, *Elliptic functions and transcendence*, Surveys in Number Theory, Dev. Math. **17** (2008), 143–188. MR 2462949

[W89] G. Wüstholz, *Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen* (German) [Algebraic points on analytic subgroups of algebraic groups], Ann. of Math. (2) **129** (1989), no. 3, 501–517. MR 0997311

[W02] G. Wüstholz, *One century of logarithmic forms*, A panorama of number theory or the view from Baker's garden (Zürich, 1999), 1–10, Cambridge Univ. Press, Cambridge, 2002. MR 1975441

[Yu86] J. Yu, *Transcendence and Drinfeld modules*, Invent. Math. **83** (1986), 507–517. MR 0827364

[Yu90] J. Yu, *On periods and quasi-periods of Drinfeld modules*, Compositio Math. **74** (1990), 235–245. MR 1055694

[Yu91] J. Yu, *Transcendence and special zeta values in characteristic p*, Ann. of Math. (2) **134** (1991), no. 1, 1–23. MR 1114606

[Yu97] J. Yu, *Analytic homomorphisms into Drinfeld modules*, Ann. of Math. (2) **145** (1997), no. 2, 215–233. MR 1441876

[Za94] D. Zagier, *Values of zeta functions and their applications*, in ECM volume, Progress in Math., **120** (1994), 497–512. MR 1341859

[Zh16] J. Zhao, *Multiple zeta functions, multiple polylogarithms and their special values*, Series on Number Theory and its Applications, 12. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016. MR 3469645

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