Frobenius Difference Equations and Difference Galois Groups

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1. Introduction

This is a survey article on recent progress concerning transcendence problems over function fields in positive characteristic. We are interested in some special values that occur in the following two ways. One is the special values of certain special transcendental functions, e.g., Carlitz ζ-values at positive integers, which are specialization of Goss' two-variable ζ-function, arithmetic (resp. geometric) Γ-functions at proper fractions (resp. proper rational functions), which are specialization of Goss' two-variable Γ-function, and Drinfeld logarithms at algebraic points, etc. The other is from algebro-geometric objects that are defined over algebraic function fields. The suitable geometric objects here are Drinfeld modules and the special values are the entries of the period matrix of a Drinfeld module that is related to the comparison between the de Rham and Betti cohomologies of the given Drinfeld module. A natural question concerns the transcendence of these special values.

In the 1980s and 1990s, Yu successfully developed methods of Gelfond-Schneider-Lang type, which can be applied to prove many important results on transcendence of the special values mentioned above. The breakthrough from transcendence of single values to linear independence of several special values is Yu's sub-t-module theorem [38], which is an analogue of the subgroup theorem of Wüstholz [33]. Here t-modules are higher-dimension analogues of Drinfeld modules introduced by Anderson [1] and they play the analogous role of commutative algebraic groups in classical transcendence theory. The key ingredient when applying Yu's sub-t-module theorem is to relate the special values in question to periods of certain t-modules. For more details we refer the readers to [38].

In 2004, Anderson-Brownawell-Papanikolas [3] developed a linear independence criterion over function fields, the so-called ABP criterion. It results from a system of Frobenius difference equations, which can be thought of as analogues of classical first-order linear differential equations. Passing from rigid analytically trivial abelian t-modules to rigid analytically trivial dual t-motives in the terminology of

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[3], we naturally have a system of Frobenius difference equations which parameterize the dual t-motives in question. As remarked in [3, §1.3.4], following this direction the ABP criterion may be regarded as a t-motivic translation of Yu's sub-t-module theorem.

In [22], Papanikolas developed a Tannakian formulation for certain kinds of Frobenius difference modules, which are called rigid analytically trivial pre-t-motives. Note that there is a fully faithful functor $\mathscr{F}$ from the category of rigid analytically trivial dual t-motives up to isogeny to the Tannakian category $\mathscr{R}$ of rigid analytically trivial pre-t-motives (see [22, Thm. 3.4.9]). The category of t-motives in the terminology of [22] is the strictly full Tannakian subcategory of $\mathscr{R}$ generated by the essential image of $\mathscr{F}$.

Given a system of Frobenius difference equations occurring from a rigid analytically trivial pre-t-motive M, Papanikolas further developed a Picard-Vessiot theory for this system and constructed its difference Galois group explicitly. He further proved that the difference Galois group in question is isomorphic to the Galois group of M from Tannakian duality.

Using ABP criterion and Picard-Vessiot theory, Papanikolas achieved an analogue of Grothendieck's periods conjecture for abelian varieties: the dimension of the Galois group of a rigid analytically trivial pre-t-motive that is an image of $\mathscr{F}$ is equal to the transcendence degree of the period matrix of the pre-t-motive. From such an equality, we shall say that this pre-t-motive has the **GP** (Grothendieck periods) property. This property is the central spirit of our t-motivic transcendence program. We will review this t-motivic transcendence theory in §2.

From §3 to §5, we will review the recent algebraic independence results on special ζ-values, Γ-values, and periods and logarithms of Drinfeld modules by using these t-motivic techniques. In §6, we will review a refined version of the ABP criterion investigated by the author of the present article. We will see that not only rigid analytically trivial pre-t-motives that are images of $\mathscr{F}$ have the **GP** property, but that there is a bigger class of pre-t-motives that have the property. We will also review its application to transcendence problems concerning Carlitz ζ-values with varying finite constant fields.

Finally, we mention that in order to avoid some unnecessary confusions with the terminology of t-motives in [1] and [3, 22], in this article we will use the terminology of rigid analytically trivial pre-t-motives that have the **GP** property instead of using the terminology of t-motives in [22] or dual t-motives in [3].

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2. t-motivic transcendence theory

2.1. Notation and Frobenius twisting. Throughout this article, we adopt the following notation.

$\mathbb{F}_q$	= the finite field of q elements with characteristic p.
t, θ	= independent variables.
A	= $\mathbb{F}_q[\theta]$ = the polynomial ring in the variable θ over $\mathbb{F}_q$.
A_+	= the set of all monic polynomials in A.
k	= $\mathbb{F}_q(\theta)$ = the fraction field of A.
k_∞	= $\mathbb{F}_q(\!(1/\theta)\!)$, the completion of k with respect to the place at infinity.
$\overline{k_\infty}$	= a fixed algebraic closure of k_∞.
$\bar{k}$	= the algebraic closure of k in $\overline{k_\infty}$.
$\mathbb{C}_\infty$	= the completion of $\overline{k_\infty}$ with respect to the canonical extension of ∞.
$\lvert\cdot\rvert_\infty$	= a fixed absolute value for the completed field $\mathbb{C}_\infty$ with $\lvert\theta\rvert_\infty = q$.
$\mathbb{T}$	= $\{f \in \mathbb{C}_\infty[\![t]\!] : f$ converges on $\lvert t\rvert_\infty \leq 1\}$ (the Tate algebra of $\mathbb{C}_\infty$).
$\mathbb{L}$	= the fraction field of $\mathbb{T}$.
$\mathbb{G}_a$	= the additive group.
$\mathrm{GL}_{r/F}$	= for a field F, the F-group scheme of invertible $r \times r$ matrices.
$\mathbb{G}_m$	= GL_1 = the multiplicative group.

For $n \in \mathbb{Z}$, given a Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty(\!(t)\!)$ we define the n-fold Frobenius twist of f by the rule $f^{(n)} := \sum_i a_i^{q^n} t^i$. For each n, the Frobenius twisting operation is an automorphism of the Laurent series field $\mathbb{C}_\infty(\!(t)\!)$ stabilizing several subrings, e.g., $\bar{k}[\![t]\!]$, $\bar{k}[t]$ and $\mathbb{T}$. More generally, for any matrix B with entries in $\mathbb{C}_\infty(\!(t)\!)$ we define $B^{(n)}$ by the rule ${B^{(n)}}_{ij} := B_{ij}^{(n)}$.

A power series $f = \sum_{i=0}^\infty a_i t^i \in \mathbb{C}_\infty[\![t]\!]$ that satisfies

$$\lim_{i\to\infty} \sqrt[i]{\lvert a_i\rvert_\infty} = 0 \quad \text{and} \quad [k_\infty(a_0, a_1, a_2, \cdots) : k_\infty] < \infty$$

is called an entire power series. As a function of t, such a power series f converges on whole $\mathbb{C}_\infty$ and, when restricted to $\overline{k_\infty}$, f takes values in $\overline{k_\infty}$. The ring of entire power series is denoted by $\mathbb{E}$.

Let $A_i \in \mathrm{Mat}_{m_i}(\mathbb{L})$ for $i = 1, \ldots, n$, and $m := m_1 + \cdots + m_n$. We define $\oplus_{i=1}^n A_i \in \mathrm{Mat}_m(\mathbb{L})$ to be the block diagonal matrix, i.e., the matrix with $A_1, \ldots, A_n$ down the diagonal and zeros elsewhere.

2.2. Tannakian formulation. In this section we follow [22] for relative background and terminology. Let $\bar{k}(t)[\sigma, \sigma^{-1}]$ be the noncommutative ring of Laurent polynomials in σ with coefficients in $\bar{k}(t)$, subject to the relation

$$\sigma f = f^{(-1)}\sigma, \quad \forall f \in \bar{k}(t).$$

The Laurent series field $\mathbb{C}_\infty(\!(t)\!)$ carries the natural structure of a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module by setting $\sigma(f) = f^{(-1)}$. As such, the subfields $\mathbb{L}$ and $\bar{k}(t)$ are $\bar{k}(t)[\sigma, \sigma^{-1}]$-submodules. For any subfield F of $\mathbb{C}_\infty(\!(t)\!)$ that is invariant under σ, we denote by

F^σ the subfield consisting of all elements in F fixed by σ. Note that we have

$$\mathbb{L}^\sigma = \bar{k}(t)^\sigma = \mathbb{F}_q(t).$$

See [22, Lem. 3.3.2] for more details.

A left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module that is finite dimensional over $\bar{k}(t)$ is called a *pre-t-motive M*. We let $\mathscr{P}$ be the category of pre-t-motives by defining morphisms in $\mathscr{P}$ as left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module homomorphisms of pre-t-motives. Conversely, every matrix in $\mathrm{GL}_r(\bar{k}(t))$ defines a pre-t-motive of dimension r over $\bar{k}(t)$ in the way above.

Given a pre-t-motive M of dimension r over $\bar{k}(t)$, let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ comprise a $\bar{k}(t)$-basis of M. Then multiplication by σ on M is given by

$$\sigma\mathbf{m} = \Phi\mathbf{m}$$

for some matrix $\Phi \in \mathrm{GL}_r(\bar{k}(t))$.

There are several important objects in the category $\mathscr{P}$:

(i) *Tensor products of pre-t-motives.* Given two pre-t-motives M_1 and M_2, we define $M_1 \otimes M_2$ to be the pre-t-motive whose underlying $\bar{k}(t)$-vector space is $M_1 \otimes_{\bar{k}(t)} M_2$, on which σ acts diagonally.

(ii) *The Carlitz motive.* We define the *Carlitz motive* to be the pre-t-motive C whose underlying $\bar{k}(t)$-space is $\bar{k}(t)$ itself, on which σ acts by

$$\sigma f = (t - \theta) f^{(-1)} \text{ for } f \in \bar{k}(t).$$

(iii) *Internal Hom.* Given two pre-t-motives M_1 and M_2, we set

$$\mathrm{Hom}(M_1, M_2) := \mathrm{Hom}_{\bar{k}(t)}(M_1, M_2).$$

Then $\mathrm{Hom}(M_1, M_2)$ is a $\bar{k}(t)$-vector space and we define a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module structure on $\mathrm{Hom}(M_1, M_2)$ by setting

$$\sigma \cdot \rho := \sigma \circ \rho \circ \sigma^{-1} \text{ for } \rho \in \mathrm{Hom}(M_1, M_2).$$

(iv) *Identity object.* We let $\mathbf{1} := \bar{k}(t)$ and give a σ-action on it by

$$\sigma f = f^{(-1)} \text{ for } f \in \bar{k}(t).$$

It has the properties:

- For any $M \in \mathscr{P}$, the natural isomorphisms , $M \otimes_{\bar{k}(t)} \mathbf{1} \cong \mathbf{1} \otimes_{\bar{k}(t)} M \cong M$, are isomorphisms of pre-t-motives ;
- $\mathrm{End}_{\mathscr{P}}(\mathbf{1}) = \mathbb{F}_q(t)$.

(v) *Duals.* Given any $M \in \mathscr{P}$, we define

$$M^\vee := \mathrm{Hom}(M, \mathbf{1}).$$

It has the property that $(M^\vee)^\vee \cong M$.

Finally, we define the notion of rigid analytic trivialization. Let M be a pre-t-motive and let $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ be the matrix representing the multiplication by σ on M with respect to a $\bar{k}(t)$-basis $\mathbf{m}$ of M. We say that M is rigid analytically trivial if there exists $\Psi \in \mathrm{GL}_r(\mathbb{L})$ so that

$$\Psi^{(-1)} = \Phi\Psi.$$

The matrix Ψ is called a rigid analytic trivialization for Φ. It is unique up to right multiplication by a matrix in $\mathrm{GL}_r(\mathbb{F}_q(t))$ (see [22, §4.1.6]).

An example of a rigid analytically trivial pre-t-motive is the Carlitz motive C. Throughout this article we fix a choice of $(q-1)$-th root of $-\theta$ and define

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty}(1 - \frac{t}{\theta^{q^i}}) \in \mathbb{E}. \tag{1}$$

Then we have $\Omega^{(-1)} = (t-\theta)\Omega$ and hence Ω is a rigid analytic trivialization for $(t-\theta)$. Note that the value $\tilde{\pi} := \frac{-1}{\Omega(\theta)}$ is a fundamental period of the Carlitz $\mathbb{F}_q[t]$-module (see [3, Cor. 5.4.1]). Such as the transcendence of $2\pi\sqrt{-1}$, $\tilde{\pi}$ is known to be transcendental over k by the work of Wade [31]. For more details and relative background, see [7].

Given a pre-t-motive $(M, \Phi, \mathbf{m})$ as above, we consider $M^\dagger := \mathbb{L} \otimes_{\bar{k}(t)} M$, where we give $M^\dagger$ a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module structure by letting σ act diagonally:

$$\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m, \quad \forall\, f \in \mathbb{L}, m \in M.$$

Define

$$M^B := (M^\dagger)^\sigma := \{\mu \in M^\dagger : \sigma\mu = \mu\}.$$

Then M^B is a vector space over $\mathbb{F}_q(t)$ since $\mathbb{L}^\sigma = \mathbb{F}_q(t)$. Note that the natural map $\mathbb{L} \otimes_{\mathbb{F}_q(t)} M^B \to M^\dagger$ is an isomorphism if and only if M is rigid analytically trivial (see [22, §3.3]). In this situation, the entries of $\Psi^{-1}\mathbf{m}$ comprise an $\mathbb{F}_q(t)$-basis of M^B, where Ψ is a rigid analytic trivialization for Φ.

Theorem 2.1. (Papanikolas, [22, Thm. 3.3.15]) *The category of rigid analytically trivial pre-t-motives $\mathscr{R}$ forms a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor $M \mapsto M^B$.*

Given any $M \in \mathscr{R}$, let $\mathscr{R}_M$ be the strictly full Tannakian subcategory of $\mathscr{R}$ generated by M. That is, $\mathscr{R}_M$ consists of all objects of $\mathscr{R}$ isomorphic to subquotients of finite direct sums of $M^{\otimes u} \otimes (M^\vee)^{\otimes v}$ for various u, v. By Tannakian duality there is an affine algebraic group scheme Γ_M over $\mathbb{F}_q(t)$ so that $\mathscr{R}_M$ is equivalent to the category of finite dimensional representations of Γ_M over $\mathbb{F}_q(t)$. The algebraic group Γ_M is called the (motivic) Galois group of M. In the next section, we will see that the Galois group Γ_M and the faithful representation

$$\Gamma_M \hookrightarrow \mathrm{GL}(M^B)$$

coming from Tannakian duality can be described explicitly.

2.3. Difference Galois groups. From now on, we denote by $(M, \Phi, \Psi, \mathbf{m})$ the object $M \in \mathscr{R}$ endowed with the system of difference equations $\Psi^{(-1)} = \Phi\Psi$ for a given $\bar{k}(t)$-basis $\mathbf{m}$ of M described above. Let r be the dimension of M over $\bar{k}(t)$ and let X be an $r \times r$ matrix with r^2 independent variables X_{ij}. Define the $\bar{k}(t)$-algebra homomorphism

$$\begin{array}{rlcl} \nu_\Psi : & \bar{k}(t)[X, 1/\det(X)] & \to & \mathbb{L} \\ & X_{ij} & \mapsto & \Psi_{ij}. \end{array}$$

Put $Z_\Psi := \text{Spec Im}\nu_\Psi$ and note that Z_Ψ is a closed $\bar{k}(t)$-subscheme of $\mathrm{GL}_{r/\bar{k}(t)}$.

We define two matrices $\Psi_1, \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$ by

$$(\Psi_1)_{ij} := \Psi_{ij} \otimes 1, \ (\Psi_2)_{ij} := 1 \otimes \Psi_{ij}.$$

Put $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$ and define the $\mathbb{F}_q(t)$-algebra homomorphism

$$\begin{array}{rlcl} \mu_\Psi : & \mathbb{F}_q(t)[X, 1/\det(X)] & \to & \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L} \\ & X_{ij} & \mapsto & \widetilde{\Psi}_{ij}. \end{array}$$

Define

$$\Gamma_\Psi := \text{Spec Im}\mu_\Psi. \tag{2}$$

Then Γ_Ψ is a closed subscheme of $\mathrm{GL}_{r/\mathbb{F}_q(t)}$. Finally, we denote by $\bar{k}(t)(\Psi)$ the field generated by all the entries of Ψ over $\bar{k}(t)$.

Theorem 2.2. (Papanikolas, [22]) *Given $(M, \Phi, \Psi, \mathbf{m}) \in \mathscr{R}$, let Z_Ψ, Γ_Ψ be defined as above. Then we have:*

(a) Γ_Ψ *is an affine algebraic group scheme over* $\mathbb{F}_q(t)$.

(b) Γ_Ψ *is smooth over* $\overline{\mathbb{F}_q(t)}$ *and is geometrically connected.*

(c) Z_Ψ *is a torsor for* $\Gamma_\Psi \times_{\mathbb{F}_q(t)} \bar{k}(t)$ *over* $\bar{k}(t)$.

(d) $\dim \Gamma_\Psi = \mathrm{tr.deg}_{\bar{k}(t)}\ \bar{k}(t)(\Psi)$.

(e) Γ_Ψ *is isomorphic to the Galois group* Γ_M *of* M.

Moreover, the faithful representation $\Gamma_M \hookrightarrow \mathrm{GL}(M^B)$ is described as follows: for any $\mathbb{F}_q(t)$-algebra R,

$$\begin{array}{rcl} \Gamma_M(R) & \hookrightarrow & \mathrm{GL}(R \otimes_{\mathbb{F}_q(t)} M^B) \\ \gamma & \mapsto & (1 \otimes \Psi^{-1}\mathbf{m} \mapsto (\gamma^{-1} \otimes 1)(1 \otimes \Psi^{-1}\mathbf{m})). \end{array}$$

2.4. ABP criterion and connection with difference Galois groups. We recall the linear independence criterion developed by Anderson-Brownawell-Papanikolas [3], the so-called ABP criterion.

Theorem 2.3. (Anderson-Brownawell-Papanikolas, [3, Thm. 3.1.1])
Let $\Phi \in \mathrm{Mat}_r(\bar{k}[t])$ be given so that $\det \Phi = c(t-\theta)^s$ for some $c \in \bar{k}^{\times}$. Fix a column vector $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{T})$ satisfying $\psi^{(-1)} = \Phi\psi$. For every $\rho \in \mathrm{Mat}_{1\times r}(\bar{k})$ such that $\rho\psi(\theta) = 0$, there exists a vector $P \in \mathrm{Mat}_{1\times r}(\bar{k}[t])$ so that

$$P\psi = 0 \text{ and } P(\theta) = \rho.$$

In the situation of Theorem 2.3, we first note that by [3, Prop. 3.1.3] the condition of Φ implies $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{E})$. We further mention that the spirit of the ABP criterion is that every $\bar{k}$-linear relation among the entries of $\psi(\theta)$ can be lifted to a $\bar{k}[t]$-linear relation among the entries of ψ. Although the theorem above is a kind of linear independence criterion, by taking tensor products one is able to pass linear independence to algebraic independence. The ideas due to Papanikolas are presented as the following.

Let Φ and ψ be given in Theorem 2.3. For any $n \geq 1$, we consider the Kronecker tensor product $\psi^{\otimes n}$. Then the entries of $\psi^{\otimes n}$ comprise all monomials of total degree n in the entries of ψ. Fix any $d \geq 1$ and take $\widetilde{\psi} \in \mathrm{Mat}_{N\times 1}(\mathbb{E})$ to be the column vector whose entries are the concatenation of 1 and each of the columns of $\psi^{\otimes n}$ for $n \leq d$. (Here $N = (r^{d+1}-1)/(r-1)$). We define $\widetilde{\Phi} \in \mathrm{Mat}_N(\bar{k}[t]) \cap \mathrm{GL}_N(\bar{k}(t))$ to be the block diagonal matrix

$$\widetilde{\Phi} := [1] \oplus \Phi \oplus \Psi^{\otimes 2} \oplus \ldots \oplus \Psi^{\otimes d},$$

then it follows that

$$\widetilde{\psi}^{(-1)} = \widetilde{\Phi}\widetilde{\psi}.$$

Note that $\widetilde{\Phi}$ and $\widetilde{\psi}$ satisfy the conditions of the ABP criterion. Thus, every $\bar{k}$-polynomial relation among the entries of $\psi(\theta)$ can be lifted to a $\bar{k}[t]$-polynomial relation among the entries of ψ. By calculating the Hilbert series in question, Papanikolas showed that

$$\mathrm{tr.deg}_{\bar{k}(t)}\, \bar{k}(t)(\psi) = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\psi(\theta)),$$

where $\bar{k}(\psi(\theta))$ is the field generated by all entries of $\psi(\theta)$ over $\bar{k}$. Combining this identity with Theorem 2.2, one has the following important equality.

Theorem 2.4. (Papanikolas, [22]) *Suppose we are given $(M, \Phi, \Psi, \mathbf{m}) \in \mathscr{R}$ and suppose that $\Phi \in \mathrm{Mat}_r(\bar{k}[t])$, $\det \Phi = c(t-\theta)^s$, $c \in \bar{k}^{\times}$, and that $\Psi \in \mathrm{GL}_r(\mathbb{T})$. Then we have*

$$\dim \Gamma_M = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\Psi(\theta)).$$

For the Ψ in the theorem above, we shall call $\Psi^{-1}(\theta)$ the period matrix of M and note that it is unique up to left multiplication by a matrix in $\mathrm{GL}_r(k)$. In §5.1, we will see an explicit connection between $\Psi^{-1}(\theta)$ and period matrices of Drinfeld modules. Therefore, Theorem 2.4 can be regarded as an analogue of Grothendieck's periods conjecture for abelian varieties. For those $(M, \Phi, \Psi, \mathbf{m}) \in \mathscr{R}$ having the two properties:

- each entry of Ψ is regular at θ,

- $\mathrm{tr.deg}_{\bar{k}(t)}\, \bar{k}(t)(\Psi) = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\Psi(\theta))$,

we say that M has the **GP** property (Grothendieck's periods property), since it follows that

$$\dim \Gamma_M = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\Psi(\theta)). \tag{3}$$

So using t-motivic transcendence theory displays in the following way. Suppose we are given a set S of certain special values in $\mathbb{C}_\infty$. If S has t-motivic interpretation in the sense that there is an object $(M, \Phi, \Psi, \mathbf{m}) \in \mathscr{R}$ which has the **GP** property and $\bar{k}(\Psi(\theta)) \supseteq S$, then we may have hope to figure out all the $\bar{k}$-algebraic relations among the entries of S, since we have the equality (3). However, computing the dimension of Γ_M in terms of the known relations among the special values in question might be difficult in general.

3. Carlitz polylogarithms and special ζ-values

3.1. Carlitz polylogarithms. The first application of Theorem 2.4 is the breakthrough on algebraic independence of Carlitz logarithms due to Papanikolas. (For the background of Carlitz module, we refer the reader to [7]).

Theorem 3.1. (Papanikolas, [11, Thm. 1.2.6]) *Let $\mathscr{C}$ be the Carlitz $\mathbb{F}_q[t]$-module and $\exp_{\mathscr{C}}(z)$ be its exponential function. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy $\exp_{\mathscr{C}}(\lambda_i) \in \bar{k}$ for all $1 \le i \le m$. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over k, then they are algebraically independent over $\bar{k}$.*

The theorem above is an analogue of the classical Gelfond's conjecture on algebraic independence of logarithms of algebraic numbers.

Conjecture 3.2. *Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^\times$ satisfy $e^{\lambda_i} \in \overline{\mathbb{Q}}$ for all $1 \le i \le m$. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{Q}$, then they are algebraically independent over $\overline{\mathbb{Q}}$.*

Under the assumptions in the conjecture above, one only knows the $\overline{\mathbb{Q}}$-linear independence of $1, \lambda_1, \ldots, \lambda_m$ by the celebrated work of Baker in the 1960s. The analogue of Baker's work for Drinfeld modules of arbitrary rank was established by Yu [38]. We will discuss the algebraic independence results for Drinfeld modules of higher rank in §5.

Back to Theorem 3.1, in order to apply Theorem 2.4 suitably we have to give a t-motivic interpretation of the Carlitz logarithms as well as the Carlitz polylogarithms. Classically if one considers the motive which is an extension of the trivial one by the n-th Tate twist for $n \in \mathbb{N}$, then its periods can be given in terms of the classical n-th polylogarithms, which is the classical logarithm when $n = 1$. In the positive characteristic case, we have the same phenomena as the following.

Given a positive integer n, the n-th Carlitz polylogarithm is defined as

$$\log_{\mathscr{C}}^{[n]}(z) := z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q)^n (\theta - \theta^{q^2})^n \cdots (\theta - \theta^{q^i})^n}.$$

Note that $\log_{\mathscr{C}}^{[1]}(z)$ is the Carlitz logarithm (see [4, 7]). Let $\alpha \in \bar{k}^{\times}$ satisfy $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for which the series $\log_{\mathscr{C}}^{[n]}(\alpha)$ converges. Let M be the pre-t-motive which is of dimension 2 over $\bar{k}(t)$, and on which multiplication by σ is represented by

$$\Phi := \begin{pmatrix} (t-\theta)^n & 0 \\ \alpha^{(-1)}(t-\theta)^n & 1 \end{pmatrix}.$$

Then M fits into the short exact sequence of pre-t-motives

$$0 \to C^{\otimes n} \to M \twoheadrightarrow \mathbf{1} \to 0,$$

where $C^{\otimes n}$ is the n-th tensor power of the Carlitz motive C.

To solve the system of difference equations $\Psi^{(-1)} = \Phi\Psi$, we define the following power series

$$L_{\alpha,n}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t-\theta^q)^n(t-\theta^{q^2})^n \cdots (t-\theta^{q^i})^n}. \tag{4}$$

Specializing $L_{\alpha,n}$ at $t=\theta$, one sees that $L_{\alpha,n}(\theta)$ is exactly the n-th Carlitz polylogarithm at α.

Let Ω be given in (1). Then one has the following identity

$$(\Omega^n L_{\alpha,n})^{(-1)} = \alpha^{(-1)}(t-\theta)^n\Omega^n + \Omega^n L_{\alpha,n} \tag{5}$$

Defining

$$\Psi := \begin{pmatrix} \Omega^n & 0 \\ \Omega^n L_{\alpha,n} & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{T}),$$

then it is a solution matrix satisfying the desired system of difference equations $\Psi^{(-1)} = \Phi\Psi$, and so M is rigid analytically trivial and has the **GP** property because of Theorem 2.4.

More generally, given m nonzero elements $\alpha_1, \cdots, \alpha_m \in \bar{k}^{\times}$ with $|\alpha_i|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$, we let $L_{\alpha_i,n}(t)$ be the series as in (4) for $i = 1, \cdots, m$. We define

$$\Phi_n = \Phi(\alpha_1, \cdots, \alpha_m) := \begin{pmatrix} (t-\theta)^n & 0 & \cdots & 0 \\ \alpha_1^{(-1)}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_m^{(-1)}(t-\theta)^n & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{Mat}_{m+1}(\bar{k}[t]), \tag{6}$$

$$\Psi_n = \Psi(\alpha_1, \cdots, \alpha_m) := \begin{pmatrix} \Omega^n & 0 & \cdots & 0 \\ \Omega^n L_{\alpha_1,n} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^n L_{\alpha_m,n} & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{GL}_{m+1}(\mathbb{T}), \tag{7}$$

then by (5) we have

$$\Psi_n^{(-1)} = \Phi_n \Psi_n. \tag{8}$$

Therefore Φ_n defines a rigid analytically trivial pre-t-motive M_n that has the **GP** property. Note that M_n fits into the short exact sequence of pre-t-motives

$$0 \to C^{\otimes n} \to M_n \twoheadrightarrow \mathbf{1}^{\oplus m} \to 0.$$

From the definition of Γ_{Ψ_n}, which is identified with Γ_{M_n} by Theorem 2.2, we see that

$$\Gamma_{M_n} \subseteq \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{GL}_{m+1/\mathbb{F}_q(t)} \right\}.$$

Note that by Theorem 2.2 the Galois group of $C^{\otimes n}$ is isomorphic to $\mathbb{G}_m$ since Ω^n is transcendental over $\bar{k}(t)$. Since $C^{\otimes n}$ is a sub-pre-t-motive of M_n, we have a surjective map

$$\pi : \Gamma_{M_n} \twoheadrightarrow \Gamma_{C^{\otimes n}} \cong \mathbb{G}_m.$$

More precisely, for any $\mathbb{F}_q(t)$-algebra R the restriction of the action of any element $\gamma \in \Gamma_{M_n}(R)$ to $R \otimes_{\mathbb{F}_q(t)} (C^{\otimes n})^B$ is the same as the action of the upper left corner of γ. That is, π is the projection on the upper left corner of any element of Γ_{M_n}.

As we have that $\bar{k}(\Psi_n(\theta)) = \bar{k}(\tilde{\pi}^n, \log_{\mathscr{C}}^{[n]}(\alpha_1), \ldots, \log_{\mathscr{C}}^{[n]}(\alpha_m))$ and that M_n has the **GP** property, Theorem 3.1 is a consequence of the following (for $n = 1$).

Theorem 3.3. ([22, Thm. 6.3.2], [16, Thm. 3.1]) *Let notation and assumptions be as above. Set* $N_n = k\text{-Span}\left\{\tilde{\pi}^n, \log_{\mathscr{C}}^{[n]}(\alpha_1), \ldots, \log_{\mathscr{C}}^{[n]}(\alpha_m)\right\}$*. Then we have*

$$\dim \Gamma_{\Psi_n} = \operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\tilde{\pi}^n, \log_{\mathscr{C}}^{[n]}(\alpha_1), \ldots, \log_{\mathscr{C}}^{[n]}(\alpha_m)\right) = \dim_k N_n.$$

The first equality of the theorem above is from Theorem 2.4. To prove the second equality, it suffices to consider the case when $\dim \Gamma_{\Psi_n} < m + 1$. We sketch a proof due to Papanikolas as the following steps.

(I) The defining equations of Γ_{Ψ_n} are given by degree one polynomials over $\mathbb{F}_q(t)$.

(II) By (I) and Theorem 2.2, the defining equations of Z_{Ψ_n} (cf. §2.3) are given by degree one polynomials over $\bar{k}(t)$.

(III) From the definition of Z_{Ψ_n}, we have $\bar{k}(t)$-linear relations among the functions $1, \Omega^n, \Omega^n L_{\alpha_1,n}, \ldots, \Omega^n L_{\alpha_m,n}$. By specializing at $t = \theta$ of these functions, we are able to obtain k-linear relations among $\tilde{\pi}^n, \log_{\mathscr{C}}^{[n]}(\alpha_1), \ldots, \log_{\mathscr{C}}^{[n]}(\alpha_m)$.

Finally, using (III) we can show that $\dim_k N_n \leq \dim \Gamma_{\Psi_n}$. As we have

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}(\tilde{\pi}^n, \log_{\mathscr{C}}^{[n]}(\alpha_1), \ldots, \log_{\mathscr{C}}^{[n]}(\alpha_m)) \leq \dim_k N_n,$$

the result of Theorem 3.3 follows.

3.2. Special ζ-values. To motivate the contents of this section, let us consider the classical Riemann ζ-function $\zeta(s)$. The values $\zeta(n)$ for integers $n \geq 2$ are called special ζ-values. A natural question is to ask the nature of such special values. The Euler relations, i.e., $\zeta(2m)/(2\pi\sqrt{-1})^{2m} \in \mathbb{Q}$ for $m \in \mathbb{N}$, imply the transcendence of the ζ-function at even positive integers and hence answer half part of the question above. However, one believes that $\zeta(2m+1)$ for $m \in \mathbb{N}$ should be transcendental numbers, but it is still an open question. Conjecturally one further expects that all the $\overline{\mathbb{Q}}$-algebraic relations among the special ζ-values are generated by Euler relations.

Conjecture 3.4. *Given an integer $s > 2$, we have*

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\left(\zeta(2), \zeta(3), \ldots, \zeta(s)\right) = s - \lfloor s/2 \rfloor.$$

We turn to the positive characteristic world. The primary goal of this section is to explain all the algebraic relations among the following characteristic p ζ-values:

$$\zeta_C(n) := \sum_{a \in A_+} \frac{1}{a^n} \in \mathbb{F}_q((1/\theta)), \qquad n = 1, 2, 3, \cdots. \tag{9}$$

These ζ-values were introduced in 1935 by L. Carlitz [9], where he obtained the Euler-Carlitz relations: if n is divisible by $q-1$, then $\zeta_C(n)/\tilde{\pi}^n \in k$. We call the positive integer n **even** provided it is a multiple of $q-1$, since $q-1$ is the cardinality of the units $A^\times$. Since $\tilde{\pi}$ is transcendental over k, $\zeta_C(n)$ is transcendental over k for n **even**. Therefore, the situation of these positive characteristic ζ-values at **even** positive integers is completely analogous to the situation of the Riemann ζ-function at even positive integers.

In the classical case, the special zeta value $\zeta(n)$ is the specialization of the n-th polylogarithm at 1. However, this simple connection between zeta values and polylogarithms becomes more subtle in the function fields setting. In [4], Anderson and Thakur proved that $\zeta_C(n)$ is a k-linear combination of $\log_{\mathscr{C}}^{[n]}$ at integral points (see Theorem 3.6). They further gave a logarithmic interpretation of the Carlitz ζ-values. More precisely, let $\mathscr{C}^{\otimes n}$ be the n-th tensor power of the Carlitz module. Then they showed that for each positive integer n, $\zeta_C(n)$ occurs (up to a multiple in A) as the last coordinate of the logarithm of $\mathscr{C}^{\otimes n}$ at an explicitly constructed integral point of $\mathscr{C}^{\otimes n}$. In [37], Yu developed his theory of the so-called E_q-functions to show the transcendence of the last coordinate of the logarithm of $\mathscr{C}^{\otimes n}$ at any nonzero algebraic point, whence deriving the transcendence of $\zeta_C(n)$ for all positive integers n, in particular for **odd** n (i.e., n is not divisible by $q-1$).

Later in [38], Yu developed the so-called sub-t-module theorem, which can be applied to determine all $\bar{k}$-linear relations among these Carlitz ζ-values and powers of the fundamental period $\tilde{\pi}$. As expected the Euler-Carlitz relations for n divisible by $q-1$ are the only $\bar{k}$-linear relations among these transcendental values. Since we are in the characteristic p fields, the Frobenius p-th power relations naturally occur: for positive integers m and n,

$$\zeta_C(p^m n) = \zeta_C(n)^{p^m}. \tag{10}$$

Using Papanikolas' theory described in §2, Chang and Yu [16] demonstrated that these, i.e., Euler-Carlitz relations and p-th power relations, account for all the algebraic relations among the ζ-values $\zeta_C(n), n = 1, 2, 3, \cdots$.

Theorem 3.5. (Chang-Yu, [16, Cor. 4.6]) *For any positive integer s, we have*

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}(\tilde{\pi}, \zeta_C(1), \cdots, \zeta_C(s)) = s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/p(q-1) \rfloor + 1.$$

3.2.1. Anderson-Thukar formula. As is mentioned above, the classical special ζ-value at an integer $n \geq 2$ is the n-th polylogarithm at 1, but this is not valid in general in the function field setting. In their seminal work, Anderson and Thakur established a formula that $\zeta_C(n)$ is a k-linear combination of the n-th Carlitz polylogarithms of $1, \theta, \cdots, \theta^{l_n}$ with $l_n < \frac{nq}{q-1}$. We will see later that this formula enables us to give a t-motivic interpretation for $\zeta_C(n)$.

Theorem 3.6. (Anderson-Thakur, [4, §3.9]) *Given any positive integer n, one can find a finite sequence $h_{n,0}, \cdots, h_{n,l_n} \in k$ with $l_n < \frac{nq}{q-1}$, such that the following identity holds*

$$\zeta_C(n) = \sum_{i=0}^{l_n} h_{n,i} L_{\theta^i,n}(\theta). \tag{11}$$

3.2.2. The Galois group of ζ-motive. In this section, we assume $q > 2$ since all positive integers are **even** for $q = 2$, and in this case all Carlitz ζ-values are rational multiples of powers of $\tilde{\pi}$ by the formula of Carlitz. We first briefly sketch the strategy of proving Theorem 3.5 as follows. First, we recall that we have established:

- Theorem 3.3;
- Theorem 3.6.

We then use these two theorems above to list the following major steps to prove Theorem 3.5.

- We make change of basis for N_n in Theorem 3.3 so that $\zeta_C(n)$ occurs as an element of a basis for N_n.
- We take direct sum of the pre-t-motives M_n with varying n and compute its Galois group exactly.

More details in the two steps are given as the following.

Given a positive integer n not divisible by $q-1$ we set

$$N_n := k\text{-span}\{\tilde{\pi}^n, L_{1,n}(\theta), L_{\theta,n}(\theta), \cdots, L_{\theta^{l_n},n}(\theta)\}. \tag{12}$$

Note that according to (11) we have $\zeta_C(n) \in N_n$ and so $m_n + 2 := \dim_k N_n \geq 2$ since $\zeta_C(n)$ and $\tilde{\pi}^n$ are linearly independent over k. (Note that $\tilde{\pi}^n \notin k_\infty$ for

$(q-1) \nmid n)$. It follows that we can pick a k-basis of V_n which involves $\tilde{\pi}^n$ and $\zeta_C(n)$. More precisely, for each **odd** n we fix once for all a finite subset

$$\{\alpha_{n0}, \cdots, \alpha_{nm_n}\} \subseteq \{1, \theta, \cdots, \theta^{l_n}\}$$

such that both

$$\{\tilde{\pi}^n, \mathscr{L}_{n0}(\theta), \cdots, \mathscr{L}_{nm_n}(\theta)\}$$

and

$$\{\tilde{\pi}^n, \zeta_C(n), \mathscr{L}_{n1}(\theta), \cdots, \mathscr{L}_{nm_n}(\theta)\}$$

are bases of N_n over k, where $\mathscr{L}_{nj}(t) := L_{\alpha_{nj},n}(t)$ for $j = 0, \cdots, m_n$.

To each such **odd** integer n, we define M_n to be the pre-t-motive which is of dimension $m_n + 2$ over $\bar{k}(t)$, and on which multiplication by σ is represented by the matrix

$$\Phi_n = \Phi(\alpha_{n0}, \cdots, \alpha_{nm_n}) \text{ as in (6).}$$

Theorem 3.3 implies that the Galois group of M_n has dimension $m_n + 2$. Since M_n has the **GP** property, $m_n + 2$ also equals the transcendence degree over $\bar{k}$ of

$$\bar{k}(\tilde{\pi}^n, \mathscr{L}_{n0}(\theta), \cdots, \mathscr{L}_{nm_n}(\theta)) = \bar{k}(\tilde{\pi}^n, \zeta_C(n), \mathscr{L}_{n1}(\theta), \cdots, \mathscr{L}_{nm_n}(\theta)).$$

In particular, the elements

$$\tilde{\pi}^n, \zeta_C(n), \mathscr{L}_{n1}(\theta), \cdots, \mathscr{L}_{nm_n}(\theta)$$

are algebraically independent over $\bar{k}$.

Given any positive integer s, we define $U(s) := \{1 \leq n \leq s \mid p \nmid n, q-1 \nmid n\}$. We further define the block diagonal matrices

$$\Phi_{(s)} := \oplus_{n \in U(s)} \Phi_n,$$
$$\Psi_{(s)} := \oplus_{n \in U(s)} \Psi_n.$$

The matrix $\Phi_{(s)}$ defines a pre-t-motive $M_{(s)} := M_{\Phi_{(s)}}$, which is the direct sum of the pre-t-motives M_n with $n \in U(s)$. Note that $\Psi_{(s)}$ is a rigid analytic trivialization for $\Phi_{(s)}$ and $M_{(s)}$ has the **GP** property by Theorem 2.4. We shall call $M_{(s)}$ a ζ-motive as we have seen that $\cup_{n=1}^{s} \{\zeta_C(n)\} \subseteq \bar{k}\left(\Psi_{(s)}(\theta)\right)$. In [16], the authors computed the Galois group $\Gamma_{(s)}$ of this ζ-motive explicitly.

Theorem 3.7. (Chang-Yu, [16, Thm. 4.5])

Fix any $s \in \mathbb{N}$. Then we have an exact sequence of algebraic groups over $\mathbb{F}_q(t)$:

$$1 \to V_{(s)} \to \Gamma_{(s)} \twoheadrightarrow \mathbb{G}_m \to 1,$$

where $V_{(s)}$ is isomorphic to the vector group $\prod_{n \in U(s)} \mathbb{G}_a^{m_n+1}$. In particular, we have

$$\dim \Gamma_{(s)} = 1 + \sum_{n \in U(s)} (m_n + 1).$$

By the theorem above we find that $1+\sum_{n\in U(s)}(m_n+1)$ is exactly the transcendence degree over $\bar{k}$ of the following field:

$$\begin{aligned}&\bar{k}(\tilde{\pi},\textstyle\bigcup_{n\in U(s)}\{\mathscr{L}_{n0}(\theta),\cdots,\mathscr{L}_{nm_n}(\theta)\})\\ =\,&\bar{k}(\tilde{\pi},\textstyle\bigcup_{n\in U(s)}\{\zeta_C(n),\mathscr{L}_{n1}(\theta),\cdots,\mathscr{L}_{nm_n}(\theta)\}).\end{aligned}$$

It follows that the set

$$\{\tilde{\pi}\}\bigcup_{n\in U(s)}\{\zeta_C(n),\mathscr{L}_{n1}(\theta),\cdots,\mathscr{L}_{nm_n}(\theta)\}$$

is algebraically independent over $\bar{k}$, hence also $\{\zeta_C(n)\mid n\in U(s)\}$ is algebraically independent over $\bar{k}$. Counting the cardinality of $U(s)$ shows Theorem 3.5.

4. Special values of geometric and arithmetic Γ-functions

4.1. Geometric Γ-function. We first mention the classical case as our motivation for this section. We consider the classical Euler Γ-function at proper fractions (note that Γ has poles at non-positive integers and gives rational values at positive integers) and call them the special Γ-values. The celebrated Chowla-Selberg formula expresses nonzero periods of CM elliptic curves defined over $\overline{\mathbb{Q}}$ as products of special Γ-values. For a CM elliptic curve over $\overline{\mathbb{Q}}$, Chudnovsky showed the algebraic independence of a nonzero period and a nonzero quasi-period of such a curve, and hence deriving the transcendence of special Γ-values at those proper fractions having denominators $2,4,6$. However, every special Γ-value is expected to be a transcendental number and this problem is still wild open. For more details, see [32].

We further discuss the question on the $\overline{\mathbb{Q}}$-algebraic relations among these special Γ-values. As the Γ-function satisfies the translation formula $\Gamma(z+1)=z\Gamma(z)$, reflection formula

$$\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z),$$

and Gauss multiplication identities

$$\Gamma(z)\Gamma(z+\frac{1}{n})\cdots\Gamma(z+\frac{n-1}{n})=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz)\text{ for an integer }n\geq 2.$$

Specializations at proper fractions of the identities above give rise to natural families of algebraic relations among these special Γ-values. The Rohrlich-Lang conjecture asserts that all $\overline{\mathbb{Q}}$-algebraic relations among the special Γ-values and $2\pi\sqrt{-1}$ are explained by the identities satisfied by the Γ-function (see [32]). However, Rorhlich-Lang conjecture can be also formulated as the assertion that all $\overline{\mathbb{Q}}$-linear relations among the monomials of special Γ-values and $2\pi\sqrt{-1}$ follow linearly from the two-term relations provided by the Deligne-Koblitz-Ogus criterion. Its transcendence degree formulation is conjectured as the following.

Conjecture 4.1. *For any integer $n > 2$ the transcendence degree of the field generated by the set*

$$\{2\pi\sqrt{-1}\} \cup \left\{\Gamma(r); r \in \frac{1}{n}\mathbb{Z} \setminus \mathbb{Z}_{\leq 0}\right\}$$

over $\overline{\mathbb{Q}}$ is $1 + \frac{1}{2} \cdot \#(\mathbb{Z}/n)^{\times}$.

Now we turn to the function field setting. In his Harvard PhD thesis, Thakur studied the geometric Γ-function over A (see [27]), which is a specialization of the two-variable Γ-function of Goss [20],

$$\Gamma(z) := \frac{1}{z} \prod_{n \in A_+} \left(1 + \frac{z}{n}\right)^{-1}, \quad z \in \mathbb{C}_\infty.$$

Note that we still use the notation Γ for the Thakur Γ-function in characteristic p setting when there are no confusions. The function is a function field analogue of the classical Euler Γ-function. It is meromorphic on $\mathbb{C}_\infty$ with poles at zero and $-n \in -A_+$ and satisfies several functional equations, which are analogous to the translation, reflection, and Gauss multiplication identities satisfied by the classical Γ-function.

In analogy with classical special Γ-values, which are values of the Euler Γ-function at proper fractions, we consider the *special geometric* Γ*-values* $\Gamma(r)$ for $r \in k \setminus A$. For x, $y \in \mathbb{C}_\infty^\times$ we denote by $x \sim y$ when $x/y \in \bar{k}^\times$. Then we have the following three types of $\bar{k}$-algebraic relations among the special geometric Γ-values obtained from specializations at the identities satisfied by the Γ-function: for all $r \in k \setminus A$, $a \in A$, $g \in A_+$ with $\deg g = d$, we have:

- $\Gamma(r + a) \sim \Gamma(r)$;
- $\prod_{\xi \in \mathbb{F}_q^\times} \Gamma(\xi r) \sim \tilde{\pi}$;
- $\prod_{a \in A/(g)} \Gamma(\frac{r+a}{g}) \sim \tilde{\pi}^{\frac{q^d-1}{q-1}} \Gamma(r)$.

In the characteristic p word, the transcendence of special geometric Γ-values was first observed by Thakur [27] in the case $q = 2$. Thakur showed that when $q = 2$, all values of $\Gamma(r)$, $r \in k \setminus A$, are $\bar{k}$-multiples of $\tilde{\pi}$ and hence are transcendental over k. Thakur also related some special geometric Γ-values to periods of Drinfeld modules, and hence deduced their transcendence using the work of Yu [34].

Compared with the classical transcendence question on special Γ-values, Sinha [26] gave an answer for a large class of special geometric Γ-values. Precisely, he proved the transcendence of $\Gamma(\frac{a}{f}+b)$ whenever a, $f \in A_+$, $\deg a < \deg f$, and $b \in A$. The basic strategy of Sinha is to relate the special Γ-values in question to periods of certain t-modules with complex multiplication by Carlitz cyclotomic fields (such t-modules are typical examples of the Hilbert-Blumental-Drinfeld modules). The construction of Sinha's t-modules is to use Anderson's soliton functions [2], which play the analogous roles of Coleman's functions for Fermat curves. Having such

t-mdoules at hand, Sinha applied Yu's theorem of Schneider-Lang type [35] to deduce the transcendence of the special Γ-values mentioned above.

Concerning the algebraic relations among special geometric Γ-values, a natural question is to determine which algebraic relations arise from the functional equations of the Γ-function. To answer this question, Thakur [27] adapted the approach of Deligne-Koblitz-Ogus to devise a diamond bracket criterion. Such criterion can determine whether a given geometric Γ-monomial is in $\bar{k}$. Here a geometric Γ-monomial is a monomial in $\tilde{\pi}$ and special geometric Γ-values with positive or negative exponents.

In [6], Brownawell and Papanikolas not only proved the transcendence of any special geometric Γ-value, but also proved the linear independence in the sense that all the $\bar{k}$-linear relations among $1, \tilde{\pi}$ and special geometric Γ-values are those generated by diamond-bracket relations. The main ingredient is to extend Sinha's approach to relate the Γ-values in question to the coordinates of periods and quasi-periods of certain t-modules with complex multiplication. Then the next step is to analyze the structure of the t-module and appeal to Yu's sub-t-module theorem [38] to show the desired result.

Later on in [3], Anderson, Brownawell and Papanikolas extended the approaches in [26, 6] to create rigid analytically trivial pre-t-motives whose period matrices contain the geometric Γ-monomials in question. They further applied the ABP-criterion to analyze the interplay between the relations among the geometric Γ-monomials and the isogeny relations among the simple quotients of the pre-t-motives in question. The detailed study allows them to show that all algebraic relations over $\bar{k}$ among special geometric Γ-values arise from diamond bracket relations among geometric Γ-monomials, and thus showed that all algebraic relations among special geometric Γ-values can be explained by the standard functional equations. As a consequence, the transcendence degree of the field generated by special geometric Γ-values in question can be obtained explicitly and this is the precise function field analogue of the Rohrlich-Lang conjecture.

Theorem 4.2. (Anderson-Brownawell-Papanikolas, [3, Cor. 1.2.2]) *For any $f \in A_+$ of positive degree, the transcendence degree of the field*

$$\bar{k}\left(\{\tilde{\pi}\} \cup \left\{\Gamma(r); r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\right\}\right)$$

over $\bar{k}$ is $1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times$.

4.1.1. *t*-motivic interpretation of special geometric Γ-values. Fix an $f \in A_+$ with positive degree. Let $\mathscr{A}_f$ be the free abelian group on symbols of the form $[x]$, where $x \in \frac{1}{f}A/A$. Every $\mathbf{a} \in \mathscr{A}_f$ is expressed as the form

$$\mathbf{a} = \sum_{a \in A, \deg a < \deg f} m_a[a/f], \quad m_a \in \mathbb{Z},$$

and $\mathbf{a}$ is called effective if all coefficients m_a are non-negative.

Note that the group $(A/f)^\times$ has a natural action on the symbols from $\frac{1}{f}A/A$ via

$$a \star [x] = [ax] \text{ for } \quad a \in (A/f)^\times, x \in \frac{1}{f}A/A.$$

This gives rise to the unique automorphism $(\mathbf{a} \mapsto a \star \mathbf{a}) : \mathscr{A}_f \to \mathscr{A}_f$ of abelian groups extending the action of $(A/f)^\times$ on those symbols $[x]$. Finally, for each $\mathbf{a} = \sum_{a \in A, \deg a < \deg f} m_a [a/f] \in \mathscr{A}_f$ we define the geometric Γ-monomial

$$\Gamma(\mathbf{a}) = \prod_{a \in A, \deg a < \deg f} \Gamma(\frac{a}{f})^{m_a}.$$

Let r be the cardinality of $(A/f)^\times$. A crucial procedure in [3] is to give an explicit construction of a rigid analytically trivial pre-t-motive that has the **GP** property and whose period matrix is given by the geometric Γ-monomials in question. We briefly sketch the construction as follows.

We first define a function $\mathrm{wt} : \mathscr{A}_f \to \mathbb{Z}[1/(q-1)]$, which is the unique group homomorphism such that for $x \in \frac{1}{f}A/A$,

$$\mathrm{wt}[x] = \begin{cases} 0 & \text{if } x \in A, \\ \frac{1}{q-1} & \text{if } x \notin A, \end{cases}$$

and then fix an effective $\mathbf{a} \in \mathscr{A}_f$ with positive weight, i.e., $\mathrm{wt}\,\mathbf{a} > 0$. We consider the f-th Carlitz cyclotomic extension of $\mathbb{F}_q(t)$ and let X_f be its corresponding smooth projective curve over $\mathbb{F}_q$. Denote by $\overline{X}_f$ the base change of X to $\bar{k}$. For an integer n, we define the n-fold twisting operation on the function field of $\overline{X}_f$ to be the unique automorphism that is the q^n-th power automorphism when restricted to $\bar{k}$, and that fixes every element of the function field of X_f. Then one explicitly defines the generalized Coleman function $g_{\mathbf{a}}$ associated to $\mathbf{a}$, and note that its divisors are given explicitly by

$$(g_{\mathbf{a}}) = -(\mathrm{wt}\mathbf{a}) \cdot \infty_{X_f} + \xi_{\mathbf{a}} + W_{\mathbf{a}}^{(1)} - W_{\mathbf{a}}.$$

For the definitions of the notation above, see [3, §6.3].

Let $\overline{\mathbf{L}}$ be the integral closure of $\bar{k}[t]$ in $\bar{k}(\overline{X}_f)$. Put $\overline{U} := \mathrm{Spec}\ \overline{\mathbf{L}}$ and define $H(\mathbf{a}) := H^0(\overline{U}, \mathscr{O}_{\overline{X}_f}(-W_{\mathbf{a}}^{(1)}))$, which is an $\overline{\mathbf{L}}$-submodule of the space $\overline{\mathbf{L}}$ of global sections of the structure sheaf on $\overline{U}$. Then we equip $H(\mathbf{a})$ with a σ-action given by

$$\sigma h := g_{\mathbf{a}} h^{(-1)} \text{ for } h \in H(\mathbf{a}).$$

Anderson-Brownawell-Papanikolas showed that $M_{\mathbf{a}} := \bar{k}(t) \otimes_{\bar{k}[t]} H(\mathbf{a})$ is a rigid analytic trivial pre-t-motive that has the **GP** property and whose period matrix contains the desired geometric Γ-monomials. We state the result as follows.

Theorem 4.3 (Anderson-Brownawell-Papanikolas [3, Prop. 6.4.4]). *Let* $\mathbf{a} \in \mathscr{A}_f$ *be effective with* $\mathrm{wt}\,\mathbf{a} > 0$. *Then there exists* (*via explicit construction*) *an object*

$(M_{\mathbf{a}}, \Phi_{\mathbf{a}}, \Psi_{\mathbf{a}}, \mathbf{m}) \in \mathscr{R}$ *with* $\Phi_{\mathbf{a}} \in \mathrm{Mat}_r(\bar{k}[t])$, $\det \Phi_{\mathbf{a}} = c(t-\theta)^s$, $c \in \bar{k}^{\times}$ *and* $\Psi_{\mathbf{a}} \in \mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_r(\mathbb{E})$, *satisfying the property that the sets*

$$\{\Psi_{\mathbf{a}}(\theta)_{ij} \mid i,j = 1,\ldots,r\}, \quad \{\Gamma(a \star \mathbf{a})^{-1} \mid a \in A,\ (a,f) = 1\},$$

span the same $\bar{k}$*-subspace in* $\overline{k_\infty}$.

4.1.2. The Galois group of geometric Γ-motive. Let $\mathbf{a} \in \mathscr{A}_f$ and $M_{\mathbf{a}}$ be given in the theorem above, we shall call $M_{\mathbf{a}}$ a geometric Γ-motive associated to the geometric Γ-monomial $\Gamma(\mathbf{a})$. From the construction of $M_{\mathbf{a}}$ above, we see that the endomorphism ring $\mathrm{End}_{\mathscr{R}}(M_{\mathbf{a}})$ contains the f-th Carlitz cyclotomic extension of $\mathbb{F}_q(t)$, denoted by R_f, which is a Galois extension of $\mathbb{F}_q(t)$ with Galois group isomorphic to $(A/f)^{\times}$. In other words, the geometric Γ-motive $M_{\mathbf{a}}$ has *geometric complex multiplication* by R_f. Using this property and the fact that the faithful representation $\Gamma_{M_{\mathbf{a}}} \hookrightarrow \mathrm{GL}(M_{\mathbf{a}}^B)$ is functorial in $M_{\mathbf{a}}$, the authors of [14] showed that the Galois group $\Gamma_{M_{\mathbf{a}}}$ is contained inside the Weil restriction of scalars $\mathrm{Res}_{R_f/\mathbb{F}_q(t)}(\mathbb{G}_{m/R_f})$, whence a torus. For more details, see [14, Prop. 3.3.1].

Let $E_f = \bar{k}\left(\{\tilde{\pi}\} \cup \left\{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\right\}\right)$. Since we have known the transcendence degree over $\bar{k}$ of E_f by the work of Anderson-Brownawell-Papanikolas, we would know the dimension of the t-motivic Galois group of a pre-t-motive M_f which has the **GP** property and whose period matrix generates the same field as E_f over $\bar{k}$. The idea of constructing such a M_f is to take direct sum of geometric Γ-motives, and the precise construction is given as follows.

We first pick a finite subset B_f of $\mathscr{A}_f$ whose elements are effective and of positive weight so that

$$E_f = \bar{k}\left(\cup_{\mathbf{a} \in B_f} \{\Gamma(a \star \mathbf{a})^{-1} \mid a \in A,\ (a,f) = 1\}\right). \tag{13}$$

For each $\mathbf{a} \in B_f$, let $\Phi_{\mathbf{a}}$ and $\Psi_{\mathbf{a}}$ be given as in Theorem 4.3 and then define

$$M_f := \oplus_{\mathbf{a} \in B_f} M_{\mathbf{a}}.$$

So the multiplication by σ on M_f is represented by the block diagonal matrix $\Phi_f := \oplus_{\mathbf{a} \in B_f} \Phi_{\mathbf{a}}$ and M_f has rigid analytic trivialization $\Psi_f := \oplus_{\mathbf{a} \in B_f} \Psi_{\mathbf{a}}$. Moreover, M_f has the **GP** property because of Theorems 4.3 and 2.4. Since each Galois group $\Gamma_{M_{\mathbf{a}}}$ is a torus, by the definition of Γ_{Ψ_f} we see that $\Gamma_{M_f} \subseteq \prod_{\mathbf{a} \in B_f} \Gamma_{M_{\mathbf{a}}}$, whence a tours over $\mathbb{F}_q(t)$. On the other hand, Theorem 4.3 and (13) imply that $\bar{k}(\Psi_f(\theta)) = E_f$. Thus we have the following summary by Theorem 4.2 and Theorem 2.4.

Lemma 4.4. *The Galois group* Γ_{M_f} *is a torus over* $\mathbb{F}_q(t)$*, which is split over* R_f*. Moreover, its dimension is equal to*

$$1 + \frac{q-2}{q-1} \cdot \#(A/f)^{\times}.$$

4.2. Arithmetic Γ-function. Let

$$D_n := \prod_{i=0}^{n-1} (\theta^{q^n} - \theta^{q^i}), \quad \overline{D_n} := D_n/(\theta^{\deg D_n}).$$

The Carlitz factorial of n is defined to be $\prod D_i^{n_i} \in \mathbb{F}_q[\theta]$ for $n = \sum n_i q^i \in \mathbb{N}$, $0 \le n_i < q$. The arithmetic Γ-function is defined by

$$\Gamma_a := \left(n \mapsto \prod \overline{D_i}^{n_i}\right) : \mathbb{Z}_p \to k_\infty \text{ for } n = \sum n_i q^i,\ 0 \le n_i < q,$$

that is the interpolation of its unit part for $n \in \mathbb{Z}_p$ (due to Goss [19]). The values $\Gamma_a(r)$ for $r \in \mathbb{Q} \cap (\mathbb{Z}_p \backslash \mathbb{Z})$, are called *special arithmetic* Γ*-values.*

When r is a non-negative integer, from the definition we see that $\Gamma_a(r) \in k$. For r a negative integer, $\Gamma_a(r)$ is a $k^\times$-multiple of $\tilde{\pi}$ (see [27, p. 34]), and it is thus transcendental over k. Moreover, for $r \in \mathbb{Q} \cap (\mathbb{Z}_p \setminus \mathbb{Z})$, $\Gamma_a(r)$ depends up to multiplication by a factor in $\bar{k}$ only on r modulo $\mathbb{Z}$ (see [27]). Thus, to study the nature of special arithmetic Γ-values, we may without loss of generality focus on those $\Gamma_a(r)$ with $-1 < r < 0$.

Given such an r, we write $r = \frac{a}{b}$, where a and b are integers and b is not divisible by p. Fermat's little theorem implies that r can be written in the form

$$r = \frac{c}{1 - q^\ell} \text{ for some } 0 < c < q^\ell - 1.$$

Write $c = \sum_{i=0}^{\ell-1} c_i q^i$, with $0 \le c_i < q$. It follows from the definition of $\Gamma_a(r)$ that

$$\Gamma_a(r) = \prod_{i=0}^{\ell-1} \Gamma_a\left(\frac{q^i}{1-q^\ell}\right)^{c_i}. \tag{14}$$

Each $\Gamma_a(\frac{q^i}{1-q^\ell})$ is known to be transcendental over k by the work of Thakur [28].

Note that according to (14), to determine all the algebraic relations among

$$\left\{\Gamma_a\left(\frac{1}{1-q^\ell}\right), \Gamma_a\left(\frac{2}{1-q^\ell}\right), \ldots, \Gamma_a\left(\frac{q^\ell - 2}{1-q^\ell}\right)\right\},$$

it suffices to determine those relations among the ℓ values

$$\left\{\Gamma_a\left(\frac{1}{1-q^\ell}\right), \Gamma_a\left(\frac{q}{1-q^\ell}\right), \ldots, \Gamma_a\left(\frac{q^{\ell-1}}{1-q^\ell}\right)\right\}.$$

The first main result in [13] is to show the algebraic independence of the above set.

Theorem 4.5. (Chang-Papanikolas-Thakur-Yu, [13, Cor. 3.3.3]) *Fix a positive integer ℓ. Then the ℓ values*

$$\Gamma_a\left(\frac{1}{1-q^\ell}\right), \Gamma_a\left(\frac{q}{1-q^\ell}\right), \ldots, \Gamma_a\left(\frac{q^{\ell-1}}{1-q^\ell}\right)$$

are algebraically independent over $\bar{k}$. Particularly, we have

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\Gamma_a\left(\frac{1}{1-q^\ell}\right), \Gamma_a\left(\frac{2}{1-q^\ell}\right), \ldots, \Gamma_a\left(\frac{q^\ell-2}{1-q^\ell}\right)\right) = \ell.$$

Remark 4.6. We shall mention that in [27, §7] and [29, §4.12], Thakur gave a general framework for arithmetic and geometric Γ-functions. He also gave a "bracket criterion"for the transcendence of monomials of special arithmetic Γ-values. It is discussed in [13] that Theorem 4.5 implies that all the algebraic relations among the special arithmetic Γ-values are generated by the bracket relations.

4.2.1. Drinfeld modules of Carlitz type. For a fixed positive integer ℓ, we recall the Carlitz $\mathbb{F}_{q^\ell}[t]$-module, denoted by $\mathscr{C}_\ell$, which is given by the $\mathbb{F}_{q^\ell}$-linear ring homomorphism

$$\begin{array}{rccl} \mathscr{C}_\ell : & \mathbb{F}_{q^\ell}[t] & \rightarrow & \operatorname{End}_{\mathbb{F}_{q^\ell}}(\mathbb{G}_a) \\ & t & \mapsto & (x \mapsto \theta x + x^{q^\ell}). \end{array}$$

Its exponential function is given by

$$\exp_{\mathscr{C}_\ell}(z) = z \prod_{0 \neq a \in \mathbb{F}_{q^\ell}[\theta]} (1 - \frac{z}{a\tilde{\pi}_\ell}),$$

where

$$\tilde{\pi}_\ell := \theta(-\theta)^{\frac{1}{q^\ell-1}} \prod_{i=1}^{\infty} (1 - \frac{\theta}{\theta^{q^{\ell i}}})^{-1} \tag{15}$$

is a fundamental period of $\mathscr{C}_\ell$ over $\mathbb{F}_{q^\ell}[t]$ in the sense that

$$\operatorname{Ker} \exp_{\mathscr{C}_\ell} = \mathbb{F}_{q^\ell}[\theta] \cdot \tilde{\pi}_\ell.$$

Throughout this article we fix a choice of $\overline{(-\theta)^{\frac{1}{q^\ell-1}}}$ so that $\tilde{\pi}_\ell$ is a well-defined element in $\overline{k_\infty}$. We also choose these roots in a compatible way so that $\tilde{\pi}_1 = \tilde{\pi}$ fixed in §2 and when $\ell | \ell'$ the number $\overline{(-\theta)^{\frac{1}{q^\ell-1}}}$ is a power of $\overline{(-\theta)^{\frac{1}{q^{\ell'}-1}}}$. So we note that $\mathscr{C}_1 = \mathscr{C}$.

We can regard $\mathscr{C}_\ell$ also as a Drinfeld $\mathbb{F}_q[t]$-module, and then it is of rank ℓ with complex multiplication by $\mathbb{F}_{q^\ell}[t]$ for $\ell \geq 2$ (see [21], and [29]). Note that there is a fully faithful functor from the category of Drinfeld $\mathbb{F}_q[t]$-modules defined over $\bar{k}$ up to isogeny to the category $\mathscr{R}$ (see [7, §4.4-§4.5]). The image of $\mathscr{C}_\ell$ under this functor, denoted by M_ℓ, is given as follows.

We consider the free left $\bar{k}[\sigma]$-module of rank one $\mathscr{M}_\ell := \bar{k}[\sigma]$, and make it a left $\bar{k}[t]$-module structure by defining

$$t \cdot 1 = \theta + \sigma^\ell.$$

Then $\mathscr{M}_\ell$ is a free $\bar{k}[t]$-module of rank ℓ with the natural basis $\{1, \sigma, \ldots, \sigma^{\ell-1}\}$. Put $M_\ell := \bar{k}(t) \otimes_{\bar{k}[t]} \mathscr{M}_\ell$, then multiplication by σ on the basis above is represented by

$$\Phi_\ell := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ (t-\theta) & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_\ell(\bar{k}(t)) \cap \mathrm{Mat}_\ell(\bar{k}[t]).$$

Note that $\Phi_\ell := (t-\theta)$ if $\ell = 1$. Now, we are going to give a rigid analytic trivialization for M_ℓ.

Let ξ_ℓ be a primitive element of $\mathbb{F}_{q^\ell}$ and define $\Psi_\ell := \Omega_\ell$ if $\ell = 1$, and otherwise let

$$\Psi_\ell := \begin{pmatrix} \Omega_\ell & \xi_\ell\Omega_\ell & \cdots & \xi_\ell^{\ell-1}\Omega_\ell \\ \Omega_\ell^{(-1)} & (\xi_\ell\Omega_\ell)^{(-1)} & \cdots & (\xi_\ell^{\ell-1}\Omega_\ell)^{(-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_\ell^{(-(\ell-1))} & (\xi_\ell\Omega_\ell)^{(-(\ell-1))} & \cdots & (\xi_\ell^{\ell-1}\Omega_\ell)^{(-(\ell-1))} \end{pmatrix} \in \mathrm{Mat}_\ell(\mathbb{T}),$$

where

$$\Omega_\ell(t) := (-\theta)^{\frac{-q^\ell}{q^\ell-1}} \prod_{i=1}^{\infty}\left(1 - \frac{t}{\theta^{q^{\ell i}}}\right) \in \overline{k_\infty}[[t]] \subseteq \mathbb{C}_\infty((t)). \tag{16}$$

Observe that Ω_ℓ is an entire function and that $\Omega_\ell(\theta) = \frac{-1}{\tilde{\pi}_\ell}$. Moreover, one has the following functional equation,

$$\Omega_\ell^{(-\ell)} = (t-\theta)\Omega_\ell, \tag{17}$$

which implies $\Psi_\ell^{(-1)} = \Phi_\ell\Psi_\ell$. Since $\{1, \xi_\ell, \ldots, \xi_\ell^{\ell-1}\}$ is a basis of $\mathbb{F}_{q^\ell}$ over $\mathbb{F}_q$, we have that $\Psi_\ell \in \mathrm{GL}_\ell(\mathbb{L})$. Therefore M_ℓ is a rigid analytically trivial pre-t-motive that has the **GP** property. Theorem 4.5 is a consequence of the following Lemma combined with Theorems 4.9 and 4.10.

Lemma 4.7. ([13, Lem. 3.2.1]) *For any positive integer ℓ, we let $M_\ell \in \mathscr{R}$ be defined as above. Then its Galois group $\Gamma_{M_\ell} \subseteq \mathrm{GL}_{\ell/\mathbb{F}_q(t)}$ is an ℓ-dimensional torus over $\mathbb{F}_q(t)$, which is spilt over $\mathbb{F}_{q^\ell}(t)$.*

Proof. We briefly sketch the ideas of the proof as the following. Suppose on the contrary that the $\dim \Gamma_{M_\ell} < \ell$. According to Theorem 2.2, it follows that the functions

$$\Omega_\ell, \Omega_\ell^{(-1)}, \ldots, \Omega_\ell^{(-\ell+1)}$$

are algebraically dependent over $\bar{k}(t)$.

To deduce a contradiction, we consider the system of difference equations

$$\sigma^\ell\left(\oplus_{i=1}^{\ell}\Omega_\ell^{(-i+1)}\right) = \left(\oplus_{i=1}^{\ell}(t-\theta)^{(-i+1)}\right)\left(\oplus_{i=1}^{\ell}\Omega_\ell^{(-i+1)}\right).$$

We then let N_ℓ be the pre-t-motive that is the left $\bar{k}(t)[\sigma^\ell, \sigma^{-\ell}]$-module of dimension ℓ over $\bar{k}(t)$, and on which (with respect to a fixed $\bar{k}(t)$-basis) the action of σ^ℓ is presented by the block diagonal matrix

$$\oplus_{i=1}^{\ell}(t-\theta)^{(-i+1)} \in \mathrm{Mat}_\ell(\bar{k}(t)).$$

(Such N_ℓ will be called a pre-t-motive of level ℓ in § 6) We denote by T_ℓ the Galois group of N_ℓ over $\mathbb{F}_{q^\ell}(t)$, and note that it is a split torus over $\mathbb{F}_{q^\ell}(t)$ by (2). One then shows that T_ℓ is isomorphic to Γ_{M_ℓ} over $\mathbb{F}_{q^\ell}(t)$, whence Γ_{M_ℓ} is a torus split over $\mathbb{F}_{q^\ell}(t)$. As the split tours T_ℓ are cut from the kernel of some characters, by the assumption and (2) we see that the functions $\Omega_\ell, \Omega_\ell^{(-1)}, \ldots, \Omega_\ell^{(-\ell+1)}$ satisfy some nontrivial monomial relations, on which arguing the zeros or poles shows the existence of such monomial relations impossible. □

Remark 4.8. More precisely, the Galois group Γ_{M_ℓ} is in fact isomorphic to the restriction of scalars $\mathrm{Res}_{\mathbb{F}_{q^\ell}(t)/\mathbb{F}_q(t)}\left(\mathbb{G}_m/\mathbb{F}_{q^\ell}(t)\right)$ (see [12, Thm 3.5.4]).

4.2.2. *t*-motivic interpretation of special arithmetic Γ-values. The following first theorem is an analogue (see [29, §4.12]) of the Chowla-Selberg formula and the following second theorem is its quasi-periods counterpart

Theorem 4.9. (Thakur, [27]) *For each positive integer ℓ, we have*

$$\frac{\Gamma_a\left(\frac{1}{1-q^\ell}\right)}{\Gamma_a\left(\frac{q^{\ell-1}}{1-q^\ell}\right)^q} \sim \Omega_\ell(\theta). \tag{18}$$

Theorem 4.10. ([13, Thm. 3.3.2]) *Fix an integer $\ell \geq 2$. For each j, $1 \leq j \leq \ell-1$, we have*

$$\frac{\Gamma_a\left(\frac{q^j}{1-q^\ell}\right)}{\Gamma_a\left(\frac{q^{j-1}}{1-q^\ell}\right)^q} \sim \Omega_\ell^{(-(\ell-j))}(\theta). \tag{19}$$

Proofs for both follow in exactly the same fashion by straightforward manipulation. One uses $D_i/D_{i-1}^q = (\theta^{q^i} - \theta)$ and takes unit parts to verify that the left side in each formula is the one-unit part of the corresponding right side.

Remark 4.11. As we have seen that the period matrix of M_ℓ is given in terms of special arithmetic Γ-values, we shall call M_ℓ an arithmetic Γ-motive. By [11, §2.4] the endomorphism ring $\mathrm{End}_{\mathscr{R}}(M_\ell)$ is isomorphic to $\mathbb{F}_{q^\ell}(t)$, that is, M_ℓ has *arithmetic complex multiplication* by $\mathbb{F}_{q^\ell}(t)$.

4.3. Γ-values and ζ-values. Concerning the special values of the Euler Γ-function at proper fractions and the special values of the Riemann ζ-function at positive integers bigger than 1, it is obvious that $2\pi\sqrt{-1}$ is closely related to them. In a conference at Hanoi (2006), Jing Yu provided an open question concerning the algebraic relations among the special Γ-values and the special ζ-values put together. Naturally this question raises the analogous question in the function field setting. In Lemmas 4.4, 4.7 and Theorem 3.7 we have seen that the Galois groups associated to special Γ-values are tori, and the Galois groups associated to ζ-values are extensions of $\mathbb{G}_m$ by vector groups. (Here $\mathbb{G}_m$ is coming from the Carlitz motive, whence related to $\tilde{\pi}$). Since there are no relations between tori and vector groups, there are expectedly no nontrivial algebraic relations among the

Γ-values and ζ-values. (Note that the trivial relations are those coming from the connection with $\tilde{\pi}$). This is indeed the case when taking Γ-motives and ζ-motives at the same stage.

Theorem 4.12. (Chang-Papanikolas-Yu, [14, Thm. 5.12]) *Given $f \in A_+$ with positive degree and s a positive integer, the transcendence degree of the field*

$$\bar{k}\left(\{\tilde{\pi}\} \cup \left\{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\right\} \cup \{\zeta_C(1), \dots, \zeta_C(s)\}\right)$$

over $\bar{k}$ is

$$1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times + s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/(p(q-1)) \rfloor.$$

Theorem 4.13. (Chang-Papanikolas-Thakur-Yu, [13, Thm. 4.2.2])

Given any two positive integers s and ℓ, let E be the field over $\bar{k}$ generated by the set

$$\{\tilde{\pi}, \zeta_C(1), \dots, \zeta_C(s)\} \cup \left\{\Gamma_a\left(\frac{c}{1-q^\ell}\right);\ 1 \le c \le q^\ell - 2\right\}.$$

Then the transcendence degree of E over $\bar{k}$ is

$$s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/p(q-1) \rfloor + \ell.$$

5. Periods and logarithms of Drinfeld modules

One of the main themes in the transcendence theory for Drinfeld modules is the study of the periods, quasi-periods and logarithms of algebraic points, which is inspired from the transcendence theory of elliptic curves. We first mention the classical case as the motivation of the contexts in this section.

Let E be an elliptic curve defined over a number field given by the Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$. We pick two generators $\{\gamma_1, \gamma_2\}$ of $H_1(E(\mathbb{C}), \mathbb{Z})$ and put $\omega_i := \int_{\gamma_i} \frac{dx}{y}$ for $i = 1, 2$. We note that dx/y is a differential form of the first kind, which is a holomorphic differential one form on E. We let Λ be the $\mathbb{Z}$-submodule of $\mathbb{C}$ spanned by ω_1 and ω_2, and this is the so-called period lattice of E. We let $\wp(z)$ be the Weierstrass $\wp$-function associated to the lattice Λ. Then the map $\exp_E : z \mapsto (\wp(z), \wp'(z), 1)$ is the exponential map for E, and one has the short exact sequence of abelian groups

$$0 \to \Lambda \to \mathbb{C} \twoheadrightarrow E(\mathbb{C}) \to 0.$$

Elements λ in $\mathbb{C}$ for which $\exp_E(\lambda) \in E(\overline{\mathbb{Q}})$ are called elliptic logarithms of algebraic points, and it is known by Siegel-Schneider in the 1930s that nonzero elliptic logarithms of algebraic points are transcendental numbers (see [8, 32]).

Now we consider xdx/y, which is a differential form of the second kind for E. Then we have the quasi-period $\eta_i := \int_{\gamma_i} xdx/y$ for $i = 1, 2$, and the matrix

$$P_E := \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$$

is called the period matrix of E. It is well-known that we have the Legendre relation which asserts that $\det P_E = \pm 2\pi\sqrt{-1}$ (note that the Legendre relation implies the de Rham isomorphism for E). The transcendence of nonzero periods and quasi-periods are also known by the work of Siegel and Schneider. Concerning the algebraic relations among the entries of P_E, conjecturally one expects that all the algebraic relations are generated by those linear relations over $\mathrm{End}(E)$. That is, conjecturally one expects that the transcendence degree of $\overline{\mathbb{Q}}(P_E)$ is either 2 if E has complex multiplication or 4 if E has no complex multiplication. Chudnovsky proved this conjecture in the CM case. In the non-CM case, the conjecture is still open (see [32]).

Compared with Conjecture 3.2, one can formulate a conjecture for elliptic logarithms of algebraic points. Let $\zeta(z)$ be the Weierstrass ζ-function associated to the lattice Λ. Suppose that $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ satisfy $\exp_E(\lambda_i) \in E(\overline{\mathbb{Q}})$ for $i = 1, \ldots, m$. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathrm{End}(E)$, then conjecturally

$$\lambda_1, \ldots, \lambda_m, \zeta(\lambda_1), \ldots, \zeta(\lambda_m)$$

are algebraically independent over $\overline{\mathbb{Q}}$. However, the best known results involve only linear independence over $\overline{\mathbb{Q}}$, due to Masser (elliptic logarithms in the CM case), Bertrand-Masser (elliptic logarithms in the non-CM case), and Wüstholz (elliptic integrals of both the first and second kind). We mention that the linear independence results above can be deduced from the subgroup theorem of Wüstholz. See [8] for more details.

Now we turn to the function field setting. It many literatures, Drinfeld modules are regarded as analogues of elliptic curves, and in this section our algebro-geometric objects are Drinfeld $\mathbb{F}_q[t]$-modules ρ defined over $\bar{k}$. Following the point of view of Grothendieck, we consider the period matrix P_ρ of ρ, which is related to the isomorphism between the de Rham and Betti cohomologies of ρ. The theory of de Rham cohomology for Drinfeld modules was well-developed by Anderson, Deligne, Gekeler and Yu in the late 1980s, and the de Rham isomorphism was established by Anderson and Gekeler independently (see [18]).

Such as the classical case, nonzero periods and quasi-periods occur as entries of P_ρ, and they are shown to be transcendental over k by the work of Yu [34, 36]. Determination of the algebraic relations among the entries of P_ρ is the central problem in this direction. In the following contexts, we will review the recent results on this problem as well as the determination of the algebraic relations among the Drinfeld logarithms at algebraic points.

5.1. Drinfeld modules of rank 2. In this section, we will focus on rank 2 Drinfeld $\mathbb{F}_q[t]$-modules ρ defined over $\bar{k}$ and see its analogy with classical transcendence theory for elliptic curves. Let τ be the Frobenius qth power operator on $\mathbb{C}_\infty$. We assume that $\rho : \mathbb{F}_q[t] \to \bar{k}[\tau]$ satisfies $\rho_t = \theta + \kappa\tau + \tau^2, \quad \kappa \in \bar{k}$, since every rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined over $\bar{k}$ is isomorphic to ρ. In what follows, we will explicitly construct the rigid analytically trivial pre-t-motive M_ρ associated to ρ, and see the connection between the rigid analytic trivialization of M_ρ and the period matrix of ρ.

Let $\exp_\rho(z) := z+\sum_{i=1}^{\infty} \alpha_i z^{q^i}$ be the exponential function of ρ, and we fix an A-basis $\{\omega_1, \omega_2\}$ of the period lattice $\Lambda_\rho := \operatorname{Ker} \exp_\rho$. We consider the two t-division sequences $\left\{\exp_\rho(\frac{\omega_j}{\theta^{i+1}}\right\}_{i=0}^{\infty}$ (for $j = 1, 2$), which give rise the natural $\mathbb{F}_q[\![t]\!]$-basis of the t-adic Tate module of ρ:

$$\varprojlim \rho[t^n] \cong \mathbb{F}_q[\![t]\!]^{\oplus 2}.$$

Then we define the Anderson generating functions:

$$f_j(t) := \sum_{i=0}^{\infty} \exp_\rho\left(\frac{\omega_j}{\theta^{i+1}}\right) t^i = \sum_{i=0}^{\infty} \frac{\alpha_i \omega_j^{q^i}}{\theta^{q^i} - t} \in \mathbb{T}. \tag{20}$$

One sees that $f_j(t)$ is a meromorphic function on $\mathbb{C}_\infty$ and has simple poles at θ, $\theta^q, \ldots$ with residues $-\omega_j$, $-\alpha_1\omega_j^q, \ldots$ respectively. Using the functional equation satisfied by the Drinfeld exponential function $\exp_\rho(\theta z) = \rho_t(\exp_\rho(z))$, we have

$$\kappa f_j^{(1)}(t) + f_j^{(2)}(t) = (t - \theta) f_j(t).$$

Such difference equations produce the functional equations:

$$\begin{pmatrix} f_1 & f_1^{(1)} \\ f_2 & f_2^{(1)} \end{pmatrix}^{(1)} = \begin{pmatrix} f_1 & f_1^{(1)} \\ f_2 & f_2^{(1)} \end{pmatrix} \begin{pmatrix} 0 & (t-\theta) \\ 1 & -\kappa \end{pmatrix}. \tag{21}$$

Let F_τ be the quasi-periodic function of ρ associated to the biderivation given by $t \mapsto \tau$ (see [7, 18, 36]). Recall the analogue of the Legendre relation proved by Anderson,

$$\omega_1 F_\tau(\omega_2) - \omega_2 F_\tau(\omega_1) = \tilde{\pi}/\xi \text{ (see [29, Thm. 6.4.6])}, \tag{22}$$

where $\xi \in \overline{\mathbb{F}_q}^{\times}$ satisfies $\xi^{(-1)} = -\xi$. We fix such a ξ throughout this section. Put

$$\Psi_\rho := \xi\Omega \begin{pmatrix} -f_2^{(1)} & f_1^{(1)} \\ \kappa f_2^{(1)} + f_2^{(2)} & -\kappa f_1^{(1)} - f_1^{(2)} \end{pmatrix}.$$

By (21) we have $\Psi_\rho^{(-1)} = \Phi_\rho \Psi_\rho$ and one checks that Ψ_ρ is an invertible matrix. As $\Omega(t)$ has simple zeros at $t = \theta^q, \theta^{q^2}, \ldots$, one sees that the entries of Ψ_ρ are regular on $\mathbb{C}_\infty$. We define M_ρ to be the pre-t-motive which is of dimension 2 over $\bar{k}(t)$, and on which multiplication by σ is represented by Φ_ρ. Then it follows that M_ρ is rigid analytically trivial and has the **GP** property by Theorem 2.4.

Note that the following equation

$$F_\tau(\omega_i) = \sum_{j=0}^{\infty} \exp_\rho\left(\frac{\omega_i}{\theta^{j+1}}\right)^q \theta^j, \quad i = 1, 2, \text{ (see [29, §6.4])}$$

enables us to obtain

$$\Psi_\rho^{-1}(\theta) = \begin{pmatrix} \omega_1 & -F_\tau(\omega_1) \\ \omega_2 & -F_\tau(\omega_2) \end{pmatrix},$$

whence $\bar{k}(\Psi_\rho(\theta)) = \bar{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2))$. When ρ has complex multiplication, Thiery [30] proved that $\operatorname{tr.deg}_{\bar{k}} \bar{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)) = 2$. In [11], the authors dealt with the case without complex multiplication.

Theorem 5.1. (Chang-Papanikolas, [11, Thm. 3.4.1]) *Let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined by $\rho_t = \theta + \kappa\tau + \tau^2$ with $\kappa \in \bar{k}$. Let ω_1, ω_2 generate the period lattice of ρ. If $p \neq 2$ and ρ is without complex multiplication, then we have $\Gamma_{M_\rho} = \mathrm{GL}_2$ and hence*

$$\mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)) = 4.$$

Let notation and assumptions be as in Theorem 5.1, we list the ingredients of the proof as follows.

(I) Establish an analogue of t-motivic version of Tate's conjecture in this setting:

$$\mathrm{End}(M_\rho) \cong \mathrm{Cent}_{\mathrm{Mat}_2(\mathbb{F}_q(t))}(\Gamma_{M_\rho}(\overline{\mathbb{F}_q(t)}))$$

(note that it also holds for arbitrary object in $\mathscr{R}$).

(II) Show that the motivic Galois representation $\Gamma_{M_\rho} \hookrightarrow \mathrm{GL}(M_\rho^B)$ is absolutely irreducible.

(III) Show that the determinant map $\det : \Gamma_M \to \mathbb{G}_m$ is surjective.

Note that the property (II) relies on (I). Having these properties at hand, we are able to prove Theorem 5.1.

Suppose on the contrary that $\Gamma_{M_\rho} \subsetneq \mathrm{GL}_2$. Then we have $\dim \Gamma_{M_\rho} \leq 3$, since Γ_{M_ρ} is connected. We claim that Γ_{M_ρ} is solvable, which contradicts (II) and hence $\Gamma_{M_\rho} = \mathrm{GL}_2$.

To prove the claim above, we let G be the kernel of the determinant map $\det : \Gamma_{M_\rho} \twoheadrightarrow \mathbb{G}_m$ from (III). Let G^0 be the identity component of G. Since G is normal in Γ_{M_ρ}, for any $\gamma \in \Gamma_{M_\rho}$ we have $\gamma^{-1}G\gamma = G$ and hence $\gamma^{-1}G^0\gamma = G^0$. We note that G^0 is solvable since $\dim G^0 \leq 2$, and Γ_{M_ρ}/G^0 is abelian since it is a one-dimensional connected algebraic group. It follows that Γ_{M_ρ} is solvable.

Concerning the ρ-logarithms at algebraic points, in §3.1 we have seen that the t-motivic interpretation of Carlitz logarithms is related to the extensions of direct sums of the identity object **1** by the Carlitz motive. Here the t-motivic interpretation of ρ-logarithms is related the extensions of direct sums of **1** by M_ρ. Following this direction, Chang and Papanikolas proved the following result.

Theorem 5.2. (Chang-Papanikolas [11, Thm. 1.2.4]) *Let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined over $\bar{k}$ without complex multiplication. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy $\exp_\rho(\lambda_i) \in \bar{k}$ for each $1 \leq i \leq m$. Suppose that p is odd. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over k, then the $2m$ quantities*

$$\lambda_1, \ldots, \lambda_m, F_\tau(\lambda_1), \ldots, F_\tau(\lambda_m)$$

are algebraically independent over $\bar{k}$.

5.2. Algebraic independence for Drinfeld modules of arbitrary rank. In §5.1, we have sketched an approach to calculate the Galois group of the pre-t-motive associated to a given Drinfeld module of rank 2 over $\bar{k}$. For Drinfeld modules of arbitrary rank r defined over $\bar{k}$, the authors of [12] proved the Brownawell-Yu conjecture, the so-called periods conjecture for Drinfeld modules. Their method is to make an explicit connection with the t-adic Galois representations attached to Drinfeld modules, which is different from the approach above. Now we state the result and sketch its proof.

Theorem 5.3. (Chang-Papanikolas, [12]) *Let ρ be a Drinfeld $\mathbb{F}_q[t]$-module of rank r defined over $\bar{k}$ and let s be the rank of* $\mathrm{End}(\rho)$ *over A. Let $\omega_1, \ldots, \omega_r$ be an A-basis of the period lattice $\Lambda_\rho := \mathrm{Ker}\exp_\rho$, and F_{τ^j} be the quasi-periodic function of ρ associated to the biderivation given by $t \mapsto \tau^j$, $1 \le j \le r-1$. Then we have*

$$\mathrm{tr.deg}_{\bar{k}}\, \bar{k}\left(\omega_i, F_{\tau^j}(\omega_i); 1 \le i \le r, 1 \le j \le r-1\right) = r^2/s.$$

Proof. Again, without loss of generality we may assume that the leading coefficient of ρ_t in τ is 1. We recall that there is a fully faithful functor H from the category of Drinfeld modules over $\bar{k}$ up to isogeny to $\mathscr{R}$ (see [12]). Let M_ρ be the image of ρ under H with rigid analytic trivialization $\Psi_\rho \in \mathrm{GL}_r(\mathbb{T})$. The first step of proof is to show:

(I) M_ρ has the **GP** property and the field generated over $\bar{k}$ by the entries of the period matrix of M_ρ is equal to $\bar{k}\left(\omega_i, F_{\tau^j}(\omega_i); 1 \le i \le r, 1 \le j \le r-1\right)$.

(Note that we have seen such constructions of M_ρ in the case of rank 2).

It follows that by Theorem 2.4 it suffices to show that $\dim \Gamma_{M_\rho} = r^2/s$. Now we consider the t-adic Tate module of ρ over $\mathbb{F}_q(\!(t)\!)$:

$$\varprojlim \rho[t^n] \otimes_{\mathbb{F}_q[\![t]\!]} \mathbb{F}_q(\!(t)\!) \cong \mathbb{F}_q(\!(t)\!)^{\oplus r}.$$

Then we pick a finite extension K/k so that

- the Drinfeld module ρ is defined over K;
- each endomorphism of ρ is defined over K;
- the variety Z_{Ψ_ρ} is defined over $K(t)$ (see §2.3).

Since the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on the t^n-torsion points of ρ for each n, it naturally gives rise to the t-adic Galois representation

$$\phi_t : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_r(\mathbb{F}_q(\!(t)\!)).$$

Let K_ρ be the fraction field of $\mathrm{End}(\rho)$, which is a finite extension of degree s over k. By the full faithfulness of H we see that K_ρ is isomorphic to $\mathscr{K}_\rho := \mathrm{End}_{\mathscr{R}}(M_\rho)$, which is naturally embedded into $\mathrm{Mat}_r(\mathbb{F}_q(t))$. By the choice of K, we see that in fact the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ under ϕ_t is contained inside $\mathrm{Cent}_{\mathrm{GL}_r(\mathbb{F}_q(\!(t)\!))}(\mathscr{K}_\rho)$.

The second key step is to show the inequalities:

(II) $\phi_t(\mathrm{Gal}(K^{\mathrm{sep}}/K)) \subseteq \Gamma_{\Psi_\rho}(\mathbb{F}_q((t))) \subseteq \mathrm{Cent}_{\mathrm{GL}_r(\mathbb{F}_q((t)))}(\mathscr{K}_\rho)$.

Then we apply the fundamental theorem of Pink [25] that $\phi_t(\mathrm{Gal}(K^{\mathrm{sep}}/K))$ is open inside $\mathrm{Cent}_{\mathrm{GL}_r(\mathbb{F}_q((t)))}(\mathscr{K}_\rho)$. It follows from the dimension argument that $\Gamma_{\Psi_\rho} \cong \mathrm{Cent}_{\mathrm{GL}_r/\mathbb{F}_q(t)}(\mathscr{K}_\rho)$. Note that the latter algebraic group is isomorphic to the restriction of scalars $\mathrm{Res}_{\mathscr{K}_\rho/\mathbb{F}_q(t)}(\mathrm{GL}_{\frac{r}{s}}/\mathscr{K}_\rho)$, and its dimension is the desired quantity we want. □

In other words, the theorem above asserts that all the $\bar{k}$-algebraic relations among the entries of the period matrix of ρ are those linear relations induced by the endomorphisms of ρ. Concerning the algebraic relations among the Drinfeld logarithms at algebraic points, we construct a suitable pre-t-motive which is an extension of direct sums of $\mathbf{1}$ by M_ρ, and which has the **GP** property, and whose period matrix contains the logarithms at algebraic points in question. Based on Theorem 5.3 and inspired from the methods of Hardouin on computation in Ext^1-modules, the authors of [12] further proved the following algebraic independence result concerning the Drinfeld logarithms.

Theorem 5.4. (Chang-Papanikolas, [12]) *Let ρ be a Drinfeld $\mathbb{F}_q[t]$-module of rank r defined over $\bar{k}$. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy $\exp_\rho(\lambda_i) \in \bar{k}$ for all $1 \leq i \leq m$. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over* $\mathrm{End}(\rho)$, *then they are algebraically independent over $\bar{k}$.*

6. Transcendence problems with varying constant fields

In this section, we will investigate the transcendence problem concerning Carlitz ζ-values with varying finite constant fields.

6.1. A refined version of the ABP criterion. Given a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module M that is finite dimensional over $\bar{k}(t)$, we observe that the iteration of the σ-action on M makes M be a left $\bar{k}(t)[\sigma^r, \sigma^{-r}]$-module for any $r \in \mathbb{N}$. This motivates the following definition, which specifies the corresponding operators as powers of σ.

Definition 6.1. Let r and s be positive integers.

(I) A pre-t-motive of level r is a left $\bar{k}(t)[\sigma^r, \sigma^{-r}]$-module M that is finite dimensional over $\bar{k}(t)$.

(II) For a pre-t-motive M of level r, we define its s-th derived pre-t-motive $M^{(s)}$ that is a pre-t-motive of level rs: the underlying space is the same as M, but it is regarded now as a left $\bar{k}(t)[\sigma^{rs}, \sigma^{-rs}]$-module.

Note that pre-t-motives of level 1 here are the pre-t-motives we used in the previous sections. We shall give more precise description of the M in part (II) above. Let $\mathbf{m} \in \mathrm{Mat}_{n\times 1}(M)$ comprise a $\bar{k}(t)$-basis of M and suppose that the matrix representing multiplication by σ^r on M is given by $\sigma^r\mathbf{m} = \Phi\mathbf{m}$ for some

$\Phi \in \mathrm{GL}_n(\bar{k}(t))$. Then the matrix representing multiplication by σ^{rs} on M is given by

$$\sigma^{rs}\mathbf{m} = \Phi^{(-r(s-1))} \dots \Phi^{(-r)}\Phi\mathbf{m}.$$

To be compatible with the definition of rigid analytic triviality of pre-t-motives before, here we say that the given pre-t-motive M of level r is rigid analytically trivial (with respect to the operator σ^r) if the there exists $\Psi \in \mathrm{GL}_n(\mathbb{L})$ so that $\Psi^{(-r)} = \Phi\Psi$. Notice that the category of rigid analytically trivial pre-t-motives of level r is a neutral Tannakian category over $\mathbb{F}_{q^r}(t)$ and notice that the Galois group Γ_M of M is defined over $\mathbb{F}_{q^r}(t)$. Again, we say that M has the **GP** property if there exists a $\bar{k}(t)$-basis $\mathbf{m} \in \mathrm{Mat}_{n\times 1}(M)$ of M so that there exists a rigid analytic trivialization $\Psi \in \mathrm{GL}_n(\mathbb{L})$ of M with respect to $\mathbf{m}$ for which

- all the entries of Ψ are regular at $t = \theta$;
- $\mathrm{tr.deg}_{\bar{k}(t)}\, \bar{k}(t)(\Psi) = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\Psi(\theta))$.

Note that the second property above implies $\dim \Gamma_M = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\Psi(\theta))$. Moreover, the **GP** property is independent of the choices of Ψ for a fixed $\mathbf{m}$.

Following the definition of **GP** property, one has the following property.

Proposition 6.2. *Let M be a rigid analytically trivial pre-t-motive of level r which has the* **GP** *property. For any positive integer s, the s-th derived pre-t-motive $M^{(s)}$ of M is also rigid analytically trivial and has the* **GP** *property.*

Following the proof of the ABP criterion in [3], the author of the present article obtained a refined version of the ABP criterion.

Theorem 6.3. (Chang, [10, Thm. 1.2]) *Fix a positive integer r. Fix a matrix $\Phi = \Phi(t) \in \mathrm{Mat}_\ell(\bar{k}[t])$ such that $\det \Phi$ is a polynomial in t satisfying $\det \Phi(0) \neq 0$. Fix a vector $\psi = [\psi_1(t), \cdots, \psi_\ell(t)]^{\mathrm{tr}} \in \mathrm{Mat}_{\ell\times 1}(\mathbb{T})$ satisfying the functional equation $\psi^{(-r)} = \Phi\psi$. Let $\xi \in \bar{k}^\times$ satisfy $\xi \notin \overline{\mathbb{F}_q}$ and*

$$\det \Phi(\xi^{\frac{1}{q^{ri}}}) \neq 0 \text{ for all } i = 1, 2, 3, \cdots.$$

Then we have:

(1) *For every vector $\rho \in \mathrm{Mat}_{1\times\ell}(\bar{k})$ such that $\rho\psi(\xi) = 0$ there exists a vector $P = P(t) \in \mathrm{Mat}_{1\times\ell}(\bar{k}[t])$ such that $P(\xi) = \rho$, and $P\psi = 0$.*

(2) $\mathrm{tr.deg}_{\bar{k}(t)}\, \bar{k}(t)(\psi_1(t), \cdots, \psi_\ell(t)) = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\psi_1(\xi), \cdots, \psi_\ell(\xi))$.

Note that by Proposition 3.1.3 of [3] the condition $\det\Phi(0) \neq 0$ of the theorem above implies $\psi \in \mathrm{Mat}_{\ell\times 1}(\mathbb{E})$. Theorem 6.3 can be also thought of as a function field analogue of the Siegel-Shidlovskii theorem concerning the E-functions satisfying the linear differential equations (cf. [5]).

A direct consequence of Theorem 6.3 is an extension of Theorem 2.4.

Corollary 6.4. *Suppose that $\Phi \in \mathrm{Mat}_n(\bar{k}[t])$ defines a rigid analytically trivial pre-t-motive M of level r with a rigid analytic trivialization $\Psi \in \mathrm{Mat}_n(\mathbb{T}) \cap \mathrm{GL}_n(\mathbb{L})$. If $\det \Phi(0) \neq 0$ and $\det \Phi(\theta^{\frac{1}{q^{ri}}}) \neq 0$ for all $i = 1, 2, 3, \dots$, then M has the* **GP** *property.*

Moreover, we have the following result which takes the pre-t-motives having the **GP** property with respect to different constant fields at the same stage in a larger constant field.

Corollary 6.5. ([15, Cor. 2.2.4]) *Given an integer $d \geq 2$, we let $\ell := \mathrm{lcm}(1, \dots, d)$. For each $1 \leq r \leq d$, let $\ell_r := \frac{\ell}{r}$ and let $\Phi_r \in \mathrm{Mat}_{n_r}(\bar{k}[t]) \cap \mathrm{GL}_{n_r}(\bar{k}(t))$ define a pre-t-motive M_r of level r with a rigid analytic trivialization $\Psi_r \in \mathrm{Mat}_{n_r}(\mathbb{T}) \cap \mathrm{GL}_{n_r}(\mathbb{L})$. Suppose that each Φ_r satisfies the hypotheses of Theorem 6.3 for $r = 1, \dots, d$. Then the direct sum*

$$M := \oplus_{r=1}^{d} {M_r}^{(\ell_r)}$$

is a rigid analytically trivial pre-t-motive of level ℓ that has the **GP** *property.*

6.2. Application to ζ-values with varying constant fields. In Theorem 3.5, we have determined the algebraic relations among the Carlitz ζ-values (at positive integers) associated to a polynomial ring over a finite field. The aim in this section is to determine the algebraic relations among the Carlitz ζ-values with varying different constant fields:

$$\zeta_r(n) := \sum_{\substack{a \in \mathbb{F}_{q^r}[\theta] \\ a\,:\,\text{monic}}} \frac{1}{a^n} \in \mathbb{F}_{q^r}(\!(1/\theta)\!) \subseteq \overline{\mathbb{F}_q}(\!(1/\theta)\!),$$

where r and n vary over all positive integers.

In 1998, Denis [17] proved the algebraic independence of all fundamental periods $\{\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \dots\}$ as the constant field varies. Therefore one expects that for the bigger set of zeta values,

$$\cup_{r=1}^{\infty} \{\zeta_r(1), \zeta_r(2), \zeta_r(3), \dots\},$$

there shall be no nontrivial algebraic relations among them, and it is indeed the case shown in the following theorem. That is, in view of Theorem 3.5, the Euler-Carlitz relations and the Frobenius p-th power relations in individual family $\{\zeta_r(1), \zeta_r(2), \zeta_r(3), \dots\}$ generate all the algebraic relations.

Theorem 6.6. (Chang-Papanikolas-Yu, [15]) *Given any positive integers s and d, the transcendence degree of the field*

$$\overline{\mathbb{F}_p(\theta)}(\cup_{r=1}^{d} \{\tilde{\pi}_r, \zeta_r(1), \dots, \zeta_r(s)\})$$

over $\overline{\mathbb{F}_p(\theta)}$ is

$$\sum_{r=1}^{d} \left(s - \left\lfloor \frac{s}{p} \right\rfloor - \left\lfloor \frac{s}{q^r - 1} \right\rfloor + \left\lfloor \frac{s}{p(q^r - 1)} \right\rfloor + 1 \right).$$

Now, we sketch the ideas of the proof as the following.

In §3.2 we have already constructed suitable pre-t-motives that fit into our consideration of ζ-values for each fixed polynomial ring over a finite field. As we fix a positive integer r, and consider the finite field $\mathbb{F}_{q^r}$ and its corresponding operator σ^r, it follows that the results and constructions in §3.2 are still valid by replacing $\mathbb{F}_q$ by $\mathbb{F}_{q^r}$, and replacing σ by σ^r. So the strategy of proving Theorem 6.6 is to put such pre-t-motives together when r varies, and then appeal to Corollary 6.5. Since we work with varying r, we shall add one more index of the notation used in §3.2 to indicate that we are in the situation with respect to $\mathbb{F}_{q^r}$ and σ^r. For example, given $n \in \mathbb{N}$ and $\alpha \in \bar{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{\frac{nq^r}{q^r-1}}$, we substitute $L_{\alpha,rn}(t)$ for $L_{\alpha,n}(t)$ with respect to the finite field $\mathbb{F}_{q^r}$ and σ^r; that is,

$$L_{\alpha,rn}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^{ri}}}{(t-\theta^{q^r})^n \cdots (t-\theta^{q^{ri}})^n},$$

which as a function on $\mathbb{C}_\infty$ converges on $|t|_\infty < |\theta|_\infty^{q^r}$. Note that $L_{\alpha,rn}(\theta)$ is the n-th polylogarithm of α associated to the Carlitz $\mathbb{F}_{q^r}[t]$-module $\mathscr{C}_r$, i.e., $L_{\alpha,rn}(\theta) = \log_{\mathscr{C}_r}^{[n]}(\alpha)$. More generally, given a collection of such numbers α, say $\alpha_1, \ldots, \alpha_m$, we have defined the following matrices and objects constructed before, but we add one more index r to specify the operator σ^r:

$$\begin{array}{rcl} \Phi_{rn} := \Phi_{rn}(\alpha_1, \ldots, \alpha_m) & \to & \Phi_n := \Phi_n(\alpha_1, \ldots, \alpha_m) \text{ (cf. (6))} \\ \Psi_{rn} := \Psi_n(\alpha_1, \ldots, \alpha_m) & \to & \Psi_n := \Psi_n(\alpha_1, \ldots, \alpha_m) \text{ (cf. (7))} \\ M_{rn} & \to & M_n \text{ (cf. §3.1).} \end{array}$$

Note that the object M_{rn} defined by Φ_{rn} is a rigid analytically trivial pre-t-motive of level r that has the **GP** property, because of $\Psi_{rn}^{(-r)} = \Phi_{rn}\Psi_{rn}$.

Working with the polynomial ring $\mathbb{F}_{q^r}[\theta]$, we further make the following substitutes that fit in §3.2.1 – §3.2.2:

$$\begin{array}{ccc} \ell_{rn} & \to & \ell_n \\ h_{rn,i} & \to & h_{n,i} \\ N_{rn} & \to & N_n \\ m_{rn} & \to & m_n \\ \alpha_{i,rn} & \to & \alpha_{i,n} \\ \mathscr{L}_{j,rn} & \to & \mathscr{L}_{j,n}. \end{array}$$

Note that $m_{rn} + 2$ is the dimension of N_{rn} over $\mathbb{F}_{q^r}(\theta)$. In the case of $q = 2$ and $r = 1$, the $\mathbb{F}_q(\theta)$-dimension of N_{11} is 1 and we set $m_{11} := -1$.

Definition 6.7. Given any positive integers s and d with $d \geq 2$. For each $1 \leq r \leq d$ we define

$$\begin{array}{ll} U_r(s) = \{1\}, & \text{if } q = 2 \text{ and } r = 1; \\ U_r(s) = \{1 \leq n \leq s; p \nmid n,\ (q^r - 1) \nmid n\}, & \text{otherwise.} \end{array}$$

For each $n \in U_r(s)$, we define that if $q = 2$ and $r = 1$,

$$\begin{array}{l} \Phi_{rn} := (t - \theta) \in \mathrm{GL}_1(\bar{k}(t)), \\ \Psi_{rn} := \Omega_1 \in \mathrm{GL}_1(\mathbb{L}), \end{array}$$

otherwise

$$\Phi_{rn} := \Phi_{rn}(\alpha_{0,rn}, \dots, \alpha_{m_{rn},rn}) \in \mathrm{GL}_{(m_{rn}+2)}(\bar{k}(t)),$$
$$\Psi_{rn} := \Psi_{rn}(\alpha_{0,rn}, \dots, \alpha_{m_{rn},rn}) \in \mathrm{GL}_{(m_{rn}+2)}(\mathbb{L}).$$

Note that we have

$$\dim \Gamma_{\Psi_{rn}} = \mathrm{tr.deg}_{\bar{k}(t)}\, \bar{k}(t)(\Psi_{rn}) = m_{rn} + 2.$$

Put $\Phi_{(rs)} := \oplus_{n \in U_r(s)} \Phi_{rn}$, then $\Phi_{(rs)}$ defines a rigid analytically trivial pre-t-motive $M_{(rs)}$ of level r with rigid analytical trivialization $\Psi_{(rs)} := \oplus_{n \in U_r(s)} \Psi_{rn}$. Moreover, $M_{(rs)}$ has the **GP** property. For clarity, we summarize the notations that fit in §3.2.2:

$$\begin{array}{rcl} \Phi_{rn} & \to & \Phi_n \\ \Psi_{rn} & \to & \Psi_n \\ U_r(s) & \to & U(s) \\ \Phi_{(rs)} & \to & \Phi_{(s)} \\ \Psi_{(rs)} & \to & \Psi_{(s)} \\ M_{(rs)} & \to & M_{(s)}. \end{array}$$

Now, we put $\ell := \mathrm{lcm}(1, \dots, d)$ and $\ell_r := \frac{\ell}{r}$ for $r = 1, \dots, d$. For each $1 \le r \le d$ let $\mathbf{M}_r := M_{(rs)}^{(\ell_r)}$ be the ℓ_r-th derived pre-t-motive of $M_{(rs)}$ defined as above. Note that $\mathbf{M}_r$ is a rigid analytically trivial pre-t-motive of level ℓ. By Proposition 6.2 each $\mathbf{M}_r$ has the **GP** property and by Theorem 3.7 its Galois group $\Gamma_{\mathbf{M}_r}$ has dimension $1 + \sum_{n \in U_r(s)} (1 + m_{rn})$.

Now, put $\mathbf{M} := \oplus_{r=1}^d \mathbf{M}_r$, and the natural rigid analytic triviality of $\mathbf{M}$ is given by $\mathbf{\Psi} := \oplus_{r=1}^d \Psi_{(rs)}$. By Corollary 6.5 we see that $\mathbf{M}$ has the **GP** property. Notice that

$$\bar{k}(\mathbf{\Psi}(\theta)) = \bar{k}\left(\cup_{r=1}^d \cup_{n \in U_r(s)} \{\tilde{\pi}_r^n, \zeta_r(n), \mathscr{L}_{1,rn}(\theta), \dots, \mathscr{L}_{m_{rn},rn}(\theta)\} \right).$$

Theorem 6.6 follows from the following explicit description of $\Gamma_{\mathbf{M}}$.

Theorem 6.8. (Chang-Papanikolas-Yu, [15, Theorem 4.5.1]) *Given any positive integers s and d with $d \ge 2$, let $\mathbf{M}$ be defined as above. Then the Galois group $\Gamma_{\mathbf{M}}$ is an extension of a d-dimensional spit torus by an vector group and its dimension is given by*

$$\dim \Gamma_{\mathbf{M}} = d + \sum_{r=1}^{d} (m_{rn} + 1).$$

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