Midterm Exam 2

1 Let A be the following 4 by 4 matrix.

$$A = \begin{bmatrix} 36 & 6 & -12 & 6 \\ 6 & 17 & 2 & -3 \\ -12 & 2 & 30 & 12 \\ 6 & -3 & 12 & 36 \end{bmatrix}$$

i (3 points) Show that A is symmetric positive definite. You must state your reason clearly.
Answer: It is easy to see A is a symmetric matrix with positive diagonal elements. By checking $36 > 6 + 12 + 6$, $17 > 6 + 2 + 3$, $30 > 12 + 2 + 12$, and $36 > 6 + 3 + 12$, we know A is also strictly diagonally dominant. By the theorem in the text book, A is symmetric positive definite.

ii (5 points) Find the Cholesky decomposition of A ($LL^T = A$).
Answer: By solving directly, we have

$$L = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -2 & 1 & 5 & 0 \\ 1 & -1 & 3 & 5 \end{bmatrix}$$

and $A = LL^T$.

iii (2 points) Find the determinant of A.
Answer: $\det(A) = \det(L)\det(L^T) = 600^2 = 360000$

2 Let A be a 3×3 strictly diagonally dominant matrix and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

i (5 points) Show that if we perform Guassian elimination on A with scaled partial pivoting strategy, then a_{11} is the pivot element for the first pass.
Answer: Since A is S.D.D., the scaled vector s is $[|a_{11}], |a_{22}|, |a_{33}|]^T$. Therefore we have

$$\frac{|a_{11}|}{|a_{11}|} = 1, \frac{|a_{21}|}{|a_{22}|} < 1, \frac{|a_{31}|}{|a_{33}|} < 1.$$

This shows a_{11} is the pivot element.

ii (5 points) Suppose after the first pass, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix}.$$

Show that

$$B = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$$

is also a strictly diagonally dominant matrix.

Answer: By some calculation, we have

$$b_{22} = a_{22} - a_{12}\frac{a_{21}}{a_{11}}, \quad b_{23} = a_{23} - a_{13}\frac{a_{21}}{a_{11}},$$
$$b_{32} = a_{32} - a_{12}\frac{a_{31}}{a_{11}}, \quad b_{33} = a_{33} - a_{13}\frac{a_{31}}{a_{11}}$$

Since A is S.D.D., we have $|a_{11}| > |a_{12}| + |a_{13}|$, $|a_{22}| > |a_{21}| + |a_{23}|$, $|a_{33}| > |a_{31}| + |a_{32}|$. Therefore,

$$|b_{22}| \geq |a_{22}| - |a_{21}|\frac{|a_{12}|}{|a_{11}|} > |a_{21}| + |a_{23}| - |a_{21}|\frac{|a_{11}| - |a_{13}|}{|a_{11}|} = |a_{23}| + |a_{21}|\frac{|a_{13}|}{|a_{11}|} \geq |b_{23}|$$
$$|b_{33}| \geq |a_{33}| - |a_{31}|\frac{|a_{13}|}{|a_{11}|} > |a_{31}| + |a_{32}| - |a_{31}|\frac{|a_{11}| - |a_{12}|}{|a_{11}|} = |a_{32}| + |a_{31}|\frac{|a_{12}|}{|a_{11}|} \geq |b_{32}|$$

This shows B is S.D.D.

3 (10 points) Let $P(x) = a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$ be a degree 5 polynomial. Suppose $P(x)$ passes through $(-3, 15), (-2, 10), (-1, 5), (1, -5), (2, 8), (3, 12)$. Find a_0. (Hint: $a_0 = P(0)$).

Answer: By Neville's algorithm, we have

x_i	$f(x_i)$					
-3	15					
-2	10	0				
-1	5	0	0			
1	-5	0	0	0		
2	8	-18	-6	-3	-1.8	
3	12	0	-27	-11.25	-6.3	-4.05

Hence $a_0 = P(0) = -4.05$.

4 Let $x_0 = -2$, $x_1 = -1$, $x_2 = 1$, $x_3 = 2$ and $f(x) = x^4 - x^3 + 1$. Let $P_3(x)$ be the unique interpolating polynomial of degree at most 3 interpolates f at x_0, x_1, x_2, x_3.

i (5 points) Construct the Lagrange form of $P_3(x)$.

Answer: $f(-2) = 25$, $f(-1) = 3$, $f(1) = 1$, $f(2) = 9$. Therefore

$$P_3(x) = 25\frac{(x+1)(x-1)(x-2)}{-12} + 3\frac{(x+2)(x-1)(x-2)}{6} + \frac{(x+2)(x+1)(x-2)}{-6} + 9\frac{(x+2)(x+1)(x-1)}{12}$$

ii (5 points) Construct the Newton form of $P_3(x)$.

Answer: First we compute the divided difference,

x_i	$f(x_i)$			
-2	25			
-1	3	-22		
1	1	-1	7	
2	9	8	3	-1

Therefore

$$P_3(x) = 25 - 22(x+2) + 7(x+2)(x+1) - (x+2)(x+1)(x-1)$$

iii (5 points) Show that

$$\max_{x\in[-2,2]} |f(x) - P_3(x)| \le 4$$

Answer: By the error analysis in the textbook, we have

$$r(x) = f(x) - P(x) = \frac{f^4(\xi)}{24}(x+2)(x+1)(x-1)(x-2) = (x^2-4)(x^2-1) = x^4 - 5x^2 + 4$$

The critical points are $x = 0, \pm\sqrt{5/2}$. Since $r(2) = r(-2) = 0$, $r(0) = 4$ and $r(\pm\sqrt{5/2}) = -9/4$, we have $-\frac{9}{4} \le r(x) \le 4$. Therefore

$$\max_{x\in[-2,2]} |f(x) - P_3(x)| = \max_{x\in[-2,2]} |r(x)| \le 4$$

iv (5 points) Find another set of interpolation points $\tilde{x}_0$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ so that the interpolating polynomial $\tilde{P}_3(x)$ has smaller interpolation error. That is

$$\max_{x\in[-2,2]} |f(x) - \tilde{P}_3(x)| \le 2$$

You only need to list the set of interpolation points.

Answer: The best choices of the interpolating points for maximum norm are the roots of Chebysheve polynomial under suitable linear transformation. Therefore we should choose

$$\tilde{x}_0 = 2\cos(\frac{\pi}{8}),\ \tilde{x}_1 = 2\cos(\frac{3\pi}{8}),\ \tilde{x}_2 = 2\cos(\frac{5\pi}{8}),\ \tilde{x}_3 = 2\cos(\frac{7\pi}{8}).$$

5 Let $s(x)$ be a cubic spline interpolant of f relative to the partition

$$a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b.$$

Suppose on each interval $[x_j, x_{j+1}]$, s is written as

$$s(x) = s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

and $s(x_j) = a_j = f(x_j)$. Let $h_j = x_{j+1} - x_j$.

i (5 points) What equations can you get from continuity of $s(x)$, $s'(x)$ and $s''(x)$?

Answer: Continuity of s gives

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3, \quad j = 0, 1, 2, 3$$

Continuity of s' gives

$$b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2, \quad j = 0, 1, 2, 3$$

Continuity of s' gives

$$2c_{j+1} = 2c_j + 6d_j h_j \quad (\text{or } c_{j+1} = c_j + 3d_j h_j), \quad j = 0, 1, 2, 3$$

ii (5 points) Derive the equation that involves c_{j-1}, c_j, c_{j+1}, a_{j-1}, a_j, a_{j+1} only by using equations obtained in i.

Answer: See the textbook for detail. For $j = 1, 2, 3, 4$, we have

$$h_{j-1}c_{j-1} + 2(h_j - 1 + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$

iii (5 points) Suppose s satisfies the nature boundary condition $s''(a) = 0$ and $s''(b) = 0$. What equation does the boundary condition translate to in terms of the coefficients? Write down the complete system for determining the c_j.

Answer: $c_0 = 0$ and $c_5 = 0$. Hence the linear system is

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 & 0 \\ h_1 & 2(h_1+h_2) & h_2 & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 \\ 0 & 0 & h_3 & 2(h_3+h_4) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 3 \begin{bmatrix} \frac{a_2-a_1}{h_1} - \frac{a_1-a_0}{h_0} \\ \frac{a_3-a_2}{h_2} - \frac{a_2-a_1}{h_1} \\ \frac{a_4-a_3}{h_3} - \frac{a_3-a_2}{h_2} \\ \frac{a_5-a_4}{h_4} - \frac{a_4-a_3}{h_3} \end{bmatrix}$$

iv (5 points) Show that the nature cubic spline s defined in iii satisfies the minimum curvature property: Let g be any C^2-function on $[a,b]$ which interpolates f over the partition

$$a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b.$$

Then

$$\int_a^b [s''(x)]^2\, dx \le \int_a^b [g''(x)]^2\, dx.$$

Moreover, the equality holds only if $g(x) = s(x)$. You must provide the detail of your proof.

Answer: Let g br a C^2-function and $r(x) = g(x) - s(x)$. Obviously $r(x)$ is a C^2-function.

$$\begin{aligned} \int_a^b s''(x)r''(x)\, dx &= \sum_{i=0}^{4} \int_{x_i}^{x_{i+1}} s''(x)r''(x)\, dx = \sum_{i=0}^{4} \left(s''(x)r'(x)\Big|_{x_i}^{x_i+1} - \int_{x_i}^{x_{i+1}} s_i'''(x)r'(x)\, dx \right) \\ &= s''(b)r'(b) - s''(a)r'(a) + \sum_{i=0}^{4} \left(s_i'''(x)r(x)\Big|_{x_i}^{x_i+1} - \int_{x_i}^{x_{i+1}} s_i^{(4)}(x)r(x)\, dx \right) \\ &= 0 \end{aligned}$$

The first two terms are zero since $s''(b) = s''(a) = 0$. We also use the fact

$$r(x_i) = g(x_i) - s(x_i) = f(x_i) - f(x_i) = 0$$

and $s_i^{(4)} = 0$. Therefore we have

$$\begin{aligned} \int_a^b [g''(x)]^2\, dx &= \int_a^b [s''(x) + r''(x)]^2\, dx = \int_a^b [s''(x)]^2\, dx + 2\int_a^b s''(x)r''(x)\, dx + \int_a^b [r''(x)]^2\, dx \\ &= \int_a^b [s''(x)]^2\, dx + \int_a^b [r''(x)]^2\, dx \end{aligned}$$

Hence

$$\int_a^b [s''(x)]^2\, dx \le \int_a^b [g''(x)]^2\, dx.$$

and when the equality holds, we must have $r''(x) = 0$. This shows $r(x)$ is a linear function. But $r(x_i) = 0$ for $i = 0, 1, 2, 3, 4$. It implies $r(x) = 0$. It follows the equality holds only if $g(x) = s(x)$.

6 Let L be an $n \times n$ lower triangular matrix with nonzero diagonal elements and b be an $n \times 1$ vector.

i (5 points) Write a matlab code to solve the linear system $Lx = b$ by forward substitution. The code should be a function M-file. It talks L, b as inputs and returns x.

Answer:

```
function x = forward_sub(L,b)
[n,m] = size(b);
x = zeros(n,m);
x(1) = b(1)/L(1,1);
  for i = 2:n
    sum = b(i);
    for k = 1:i-1
      sum = sum - L(i,k)*x(k);
    end
  x(i) = sum/L(i,i);
end
```

ii (5 points) What is the operation counts of forward substitution? Your answer should be expressed as a function of n.

Answer: The operation counts are

$$1 + \sum_{i=2}^{n}\left(\sum_{k=1}^{i-1} 2 + 1\right) = 1 + \sum_{i=2}^{n}(2(i-1)+1) = 1 + (n-1)(2n+2)/2 = n^2$$

7 Let $f(x)$ be a C^4-function on $[a, b]$ and $x_0 = a$, $x_1 = b$.

i (5 points) Show that there exists a unique polynomial $P(x)$ of degree at most 3 that satisfies $P(x_i) = f(x_i)$ and $P'(x_i) = f'(x_i)$ for $i = 0, 1$.

Answer: Check Theorem on page 406 of the textbook.

ii (5 points) Let P be the unique polynomial defined in i and $x \in (a, b)$. Define

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t-a)^2(t-b)^2}{(x-a)^2(x-b)^2}$$

Show that $g(a) = g(b) = g(x) = 0$ and $g'(a) = g'(b) = 0$

Answer: Since $P(a) = f(a), P(b) = f(b), P'(a) = f'(a), P'(b) = f'(b)$, we have

$$g(a) = f(a) - P(a) = 0, \quad g(b) = f(b) - P(b) = 0, \quad g(x) = f(x) - P(x) - [f(x) - P(x)] \times 1 = 0$$

$$g'(a) = f'(a) - P'(a) = 0, \quad g'(b) = f'(b) - P'(b) = 0$$

iii (5 points) Use ii and Rolle's theorem to show that there exists a point $\xi \in (a, b)$ such that $g^{(4)}(\xi) = 0$.

Answer: Since $g(a) = g(x) = g(b) = 0$, by Rolle's theorem there exists $\xi_1 \in (a, x)$ and $\xi_2 \in (x, b)$ such that $g'(\xi_1) = g'(\xi_2) = 0$. Since $a < \xi_1 < \xi_2 < b$ and $g'(a) = g'(\xi_1) = g'(\xi_2) = g'(b) = 0$, by Rolle's theorem there exits $\eta_1 \in (a, \xi_1)$, $\eta_2 \in (\xi_1, \xi_2)$, $\eta_3 \in (\xi_2, b)$ such that $g''(\eta_1) = g''(\eta_2) = g''(\eta_3) = 0$. This implies there exits $a < \theta_1 < \theta_2 < b$ such that $g'''(\theta_1) = g'''(\theta_2) = 0$. By Rolle's theorem, it follows there exits $\xi \in (a, b)$ such that $g^{(4)}(\xi) = 0$.

iv (5 points) Use $g^{(4)}(\xi) = 0$ to show that

$$f(x) - P(x) = \frac{f^{(4)}(\xi)}{24}(x-a)^2(x-b)^2$$

Answer: Since $P(t)$ is a cubic polynomial, $P^{(4)}(t) = 0$. Therefore,

$$0 = g^{(4)}(\xi) = f^{(4)}(\xi) - [f(x) - P(x)]\frac{24}{(x-a)^2(x-b)^2}$$

It follows

$$f(x) - P(x) = \frac{f^{(4)}(\xi)}{24}(x-a)^2(x-b)^2$$