Exam 1 Solution

1.(10 points) Let $p_n = \dfrac{3n^2 - 3n + 1}{(n+1)(n-2)}$. Compute the limit $\lim_{n\to\infty} p_n$ and determine the (best) rate of convergence.

Answer:

$$p_n = \frac{3n^2 - 3n + 1}{(n+1)(n-2)} = \frac{3n^2 - 3n + 1}{n^2 - n - 2} = 3 + \frac{7}{n^2 - n - 2}$$

Therefore $\lim_{n\to\infty} p_n = 3$. From the above equation, we have

$$|p_n - 3| = \frac{7}{n^2 - n - 2} \le \frac{7}{n^2/2} = \frac{14}{n^2}$$

for any $n > 4$. Hence the rate of convergence is $O(n^{-2})$.

2. Order of convergence.

i (5 points) Give the definition of order of convergence of a sequence $\{p_n\}$ that converges to p.

ii (10 points) Give an example of a sequence $\{p_n\}$ that converges to 1 of order 4. (Hint: You can consider p_n is of the form $p_n = 1 + q_n$ with an appropriate sequence q_n.)

Answer: (i) If $\lim_{n\to\infty} \dfrac{|p_n - p|}{|p_{n+1} - p|^{\alpha}} = \lambda$ for some $\lambda > 0$, then we say p_n coverges to p of order α with asymptotic error constant λ.

(ii) Let $p_n = 1 + \dfrac{1}{2^{(4^n)}}$

3. Floating point Arithmetic.

i (8 points) Compute $\sqrt{2001} - \sqrt{2000}$ in the floating point number system $\mathbf{F}(10, 5, -10, 10)$ with rounding. Use $\sqrt{2001} = 44.7325384926900\cdots$, and $\sqrt{2000} = 44.7213595499957\cdots$.

ii (2 points) How many significant decimal digits are lost in the performing the subtraction?

iii (5 points) Explain how you would rearrange your computation to obtain a more accurate answer. Just show your method but do not compute the final answer.

Answer: (i) $\mathrm{fl}(\sqrt{2001}) = 0.44733 \times 10^2$, $\mathrm{fl}(\sqrt{2000}) = 0.44721 \times 10^2$. Hence

$$\sqrt{2001} -_{fl} \sqrt{2000} = \mathrm{fl}(0.44733 \times 10^2 - 0.44721 \times 10^2) = \mathrm{fl}(0.00012 \times 10^2) = 0.12000 \times 10^{-1}$$

(ii) 3 digits.

(iii) $\sqrt{2001} - \sqrt{2000} = \dfrac{1}{\sqrt{2001} + \sqrt{2000}} = \dfrac{1}{0.44733 \times 10^2 + 0.44721 \times 10^2}$

4. (10 points) Let p be a fix point of $g(x)$, and a be a number close but not equal to p. Show that if $|g'(x)| \ge k > 1$ for all x in between a and p, then we have

$$|g(a) - p| > |a - p|.$$

Answer: By $g(p) = p$ and mean value theorem, we have

$$|g(a) - p| = |g(a) - g(p)| = |g'(\xi)||a - p|$$

for some ξ in between a and p. Since $|g'(x)| \geq k > 1$, we have

$$|g(a) - p| = |g'(\xi)||a - p| \leq k|a - p| > |a - p|$$

5. Let $g(x) = 2x(1 - x)$.

i (5 points) Find all fixed points of the function $g(x)$.

ii (10 points) Which fixed point we should expect that fixed point iteration, starting with a value close enough to the fixed point, will converge toward it? Which will fail? Why?

Answer: (i) Solve $g(x) = x$ we get $x = 0$ or $2(1 - x) = 1$. Therefore, the fixed points of $g(x)$ are 0 and $\frac{1}{2}$.

(ii) Since $g'(x) = 2 - 2x$ we have $|g'(0)| = 2 > 1$ and $|g'(1/2)| = 0 < 1$. From the book theorem, we know fixed point iteration will converge to $1/2$ with suitable initial condition close to $1/2$. But from 4, we know fixed point iteration will not converge to 0 for most initial conditions.

6. (5 points) What is the theorem we used to show the simple enclosure methods (the bisection method and the method of false position) work? You can either state the name of the theorem or describe the theorem explicitly if you do not know the name.
Answer: Intermediate value theorem.

7. $f(x) = 2x^2 + 3x - 1$ has a simple root in $[0, 1]$. Use the following methods to find the approximations for the root. For each method, compute the approximation until p_2.

i (5 points) Bisection method with $a_1 = 0$ and $b_1 = 1$.

ii (5 points) Newton's method with $p_0 = 1$.

iii (5 points) Secant method with $p_0 = 0$ and $p_1 = 1$.

iv (5 points) Which of above methods is best for finding the root of $f(x)$ and why?

Answer: (i) $f(0) = -1 < 0$, $f(1) = 4 > 0/$ $p_1 = (0 + 1)/2 = \frac{1}{2}$. $f(p_1) = 1 > 0$. Hence $p_2 = (0 + p_1)/2 = 1/4$

(ii) $g(x) = x - f(x)/f'(x) = x - (2x^2 + 3x - 1)/(4x + 3) = (2x^2 + 1)/(4x + 3)$. $p_1 = g(p_0) = 3/7$. $p_2 = g(p_1) = (18/49 + 1)/(12/7 + 3) = 67/231$

(iii) $p_2 = p_1 - f(p_1)(p_1 - p_0)/(f(p_1) - f(p_0)) = 1 - 4 \times 1/5 = 1/5$.

(iv) Newton's method will be the best because the root is simple root and therefore the order of convergence for Newton's method is quadratic which is higher than the order of convergence for other two methods. This means the sequence generated by Newton's method will converges faster.

8. Read the following Matlab code. Skip the code.
Question:

i (5 points) What is the code for? What dose the variable "err" mean in the code?

ii (5 points) What is the stopping condition used in the code?

Answer: (i) The code is to find the root of $f(x) = 0$ by the method of false position with initial interval $[a, b]$. The epsilon is the tolerance. The variable "err" means

$$\left|\frac{\lambda}{1-\lambda}\right| |p_n - p_{n-1}|$$

which is used to estimate the error $|e_n|$.

(ii) The stopping condition is to measure if the absolute error $|e_n|$ smaller than the tolerance epsilon. We check if err $<$ epsilon. If it does, the code stops.