Midterm Exam 2 Solution

1 (15 points) True or False. T for true and F for false.

i (F) If $a > 0$ and $x > y > 0$, then $\log_a x > \log_a y$.

ii (T) For any real number $|x| \leq 1$, we have $\cos(\arccos x) = x$.

iii (F) If f and g are continuos on $[a, b]$ and $f(x) \geq g(x)$ on $[a, b]$, then $\int_b^a |f(x)|dx \geq \int_b^a |g(x)|dx$.

iv (F) A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$. The *distance* traveled during the time period $1 \leq t \leq 4$ is $-\frac{9}{2}$.

v (T) Assume that f and g are continuous, that $a < b$, and that $\int_a^b f(x)dx > \int_a^b g(x)dx$, then for any partition P of $[a, b]$

$$U_f(P) > \int_a^b g(x)\, dx$$

2 (5 points) Find the upper Riemann sum $U_f(P)$ and the lower Riemann sum $L_f(P)$ of the function $f(x) = 2x^3 - 3x^2 + 1$ on $[-2, 4]$ with $P = \{-2, -1, 2, 4\}$.

Answer: $f'(x) = 6x^2 - 6x = 6x(x - 1)$. Hence $f(x)$ is increasing on $[-2, 0]$ and $[1, 4]$ and decreasing on $[0, 1]$. Therefore

$$U_f(P) = f(-1) \times 1 + f(2) \times 3 + f(4) \times 2 = -4 + 15 + 162 = 173$$

$$L_f(P) = f(-2) \times 1 + f(-1) \times 3 + f(2) \times 2 = -27 - 12 + 10 = -29$$

3 (25 points.) Compute the following derivatives.

i $\frac{d}{dx}[(\cos x)^{\sin x}]$

Answer:

$$\begin{aligned}\frac{d}{dx}[(\cos x)^{\sin x}] = \frac{d}{dx}e^{\sin x \ln \cos x} &= e^{\sin x \ln \cos x}\left(\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right) \\ &= (\cos x)^{\sin x}\left(\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right)\end{aligned}$$

ii $\frac{d}{dx}\arctan(2^x)$

Answer:

$$\frac{d}{dx}\arctan(2^x) = \frac{1}{1 + 2^{2x}} \times \ln 2 \times 2^x = \frac{\ln 2(2^x)}{1 + 2^{2x}}$$

iii $\frac{d}{dx}[\ln(\csc x)]$

Answer:

$$\frac{d}{dx}[\ln(\csc x)] = \frac{1}{\csc x}(-\csc x \cot x) = -\cot x$$

iv Find $g'(0)$, where $g(x) = \dfrac{\sqrt{x^2+4}\, e^x}{(x+1)^5}$ (Hint: Use logarithmic differentiation.)

Answer:

$$\ln g(x) = \frac{1}{2}\ln(x^2+4) + x - 5\ln(x+1)$$

Therefore

$$g'(x) = g(x)\left(\frac{x}{x^2+4} + 1 - \frac{5}{x+1}\right),$$

and

$$g'(0) = g(0)(-4) = -8$$

v Find $h'(x)$, where $h(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \frac{z}{\sqrt{z+1}}\, dz$.

Answer:

$$h'(x) = \frac{x^2}{\sqrt{x^2+1}} \times 2x - \frac{\sqrt{x}}{\sqrt{\sqrt{x}+1}} \times \frac{1}{2\sqrt{x}} = \frac{2x^3}{\sqrt{x^2+1}} - \frac{1}{2\sqrt{\sqrt{x}+1}}$$

4 (25 points) Compute the following integrals.

i Find $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\, dx$

Answer:

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\, dx = \int_1^2 \frac{u-1}{2\sqrt{u}}\, du = \left(\frac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\Big|_1^2 = \frac{2\sqrt{2}}{3} - \sqrt{2} - \frac{1}{3} + 1 = \frac{2}{3} - \frac{\sqrt{2}}{3}$$

ii Find $\displaystyle\int \frac{e^x}{e^x+1}\, dx$

Answer:

$$\int \frac{e^x}{e^x+1}\, dx = \int \frac{1}{u}\, du = \ln(e^x+1) + C$$

iii Find $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\, dx$

$$\int_{-1}^1 \sqrt{1-x^2}\, dx = -\int_{\pi}^0 \sqrt{1-\cos^2\theta}\sin\theta\, d\theta = \int_0^{\pi} \sin^2\theta\, d\theta = \int_0^{\pi} \frac{-\cos 2\theta + 1}{2}\, d\theta = \frac{\pi}{2}$$

iv Find $\displaystyle\int_0^1 (x^2-1)(x^3-3x+2)^4\, dx$

Answer:

$$\int_0^1 (x^2-1)(x^3-3x+2)^4\, dx = \int_2^0 \frac{1}{3}u^4\, du = -\frac{u^5}{15}\Big|_0^2 = -\frac{32}{15}$$

v Find $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\, dx$

$$\int \frac{x}{\sqrt{1-x^4}}\, dx = \int \frac{1}{2\sqrt{1-u^2}}\, du = \frac{1}{2}\arcsin x^2 + C$$

5 Let Ω the region enclosed by curves $y = x^2$ and $y = 2x$.

i (2 points) Find the area of Ω.

Answer: The area A is $\displaystyle\int_0^2 (2x - x^2)\,dx = x^2 - \frac{1}{3}x^3\Big|_0^2 = \frac{4}{3}$

ii (6 points) Find the volume of the solid generated by revolving Ω about the x-axis and the y-axis.

Answer: Denote the volume of the solid generated by revolving Ω about the x-axis and the y-axis by V_x and V_y respectively. Then

$$V_x = \int_0^2 \pi(4x^2 - x^4)\,dx = (\frac{32}{3} - \frac{32}{5})\pi = \frac{64}{15}\pi$$

$$V_y = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi(\frac{16}{3} - \frac{16}{4}) = \frac{8}{3}\pi$$

iii (2 points) Locate the centroid of Ω.

Answer: By Pappus's theorem, we have

$$\bar{y} = \frac{V_x}{2\pi A} = \frac{8}{5}, \quad \bar{x} = \frac{V_y}{2\pi A} = 1.$$

6 (5 points) The base of a solid is the region enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the volume of the solid given that each cross section perpendicular to the x-axis is an isosceles triangle with base in the region and altitude equal to one-half the base.

Answer:

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\frac{2b}{a}\sqrt{a^2 - x^2}\right)\left(\frac{b}{a}\sqrt{a^2 - x^2}\right) = \frac{b^2}{a^2}(a^2 - x^2)$$

$$V = \int_{-a}^{a} A(x)\,dx = 2\int_0^a A(x)\,dx = \frac{2b^2}{x^2}\int_0^a (a^2 - x^2) = \frac{4}{3}ab^2$$

7 (5 points) Simplify the following expression

$$\tan(\operatorname{arcsec} x).$$

Answer: Let $\theta = \operatorname{arcsec} x$. Then $\theta \in (0, \pi/2)$ if $x > 0$ and $\theta \in (-\pi/2, x)$ if $x < 0$. Therefore

$$\tan^2(\operatorname{arcsec} x) = \tan^2\theta = \sec^2\theta - 1 = x^2 - 1$$

Hence $\tan(\operatorname{arcsec} x) = \sqrt{x^2 - 1}$ if $x > 0$ and $\tan(\operatorname{arcsec} x) = -\sqrt{x^2 - 1}$ if $x < 0$.

8 (10 pints) Let $L(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$ for $x > 0$. Prove that

i $L(a) + L(b) = L(ab)$, for all $a > 0$ and $b > 0$.

Answer: Let $t = au$. Then $dt = adu$ and

$$\int_a^{ab} \frac{1}{t}\, dt = \int_1^b \frac{1}{au}\, adu = \int_1^b \frac{1}{u}\, du = L(b)$$

Therefore we have

$$L(a) + L(b) = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt = \int_1^{ab} \frac{1}{t}\, dt = L(ab)$$

ii $L(a^n) = nL(a)$, for $a > 0$ and $n \in \mathbb{R}$.

Answer: Let Let $t = u^n$. Then $dt = nu^{n-1}du$ and

$$L(a^n) = \int_1^{a^n} \frac{1}{t}\, dt = \int_1^a \frac{1}{u^n} nu^{n-1}\, du = n \int_1^a \frac{1}{u}\, dt = nL(a)$$

9 (10 points) Let f be a continuous function on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$. Prove that if $\displaystyle\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ on $[a, b]$.

Answer: Suppose $f(x) \neq 0$ on $[a, b]$, then there exists $c \in [a, b]$, such that $f(c) > 0$. Since $f(x)$ is continuous on $[a, b]$, then for $\epsilon = f(c)/2 > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $|x - c| < \delta$ and $x \in [a, b]$. Let $[k, l] = [c - \delta/2, c + \delta/2] \cap [a, b]$. Then for all $x \in [k, l]$, we have

$$f(x) > f(c) - \epsilon = 2\epsilon - \epsilon = \epsilon.$$

Since $f(x) > 0$, we have

$$\int_a^b f(x)\, dx \geq \int_k^l f(x)\, dx > \int_k^l \epsilon\, dx = \epsilon(l - k) > 0$$

It contradicts to the fact $\int_b^a f(x)\, dx = 0$. Hence $f(x) = 0$ on $[a, b]$.