Midterm Exam 1 solution

1 (15 points) True or False. Please write down your answer on your answer sheet. T for true and F for false.

- i F. The polynomial $p(x) = x^3 - x^2 + x - 10$ has exactly two local extreme values.
- ii T. If f is not continuous at a, then f is not differentiable at a.
- iii T. The graph of $f(x) = \dfrac{\sqrt{2x^2+1} - x}{3x}$ has 2 horizontal and 1 vertical asymptotes.
- iv F. If both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} f(x)g(x)$ exist, then $\lim\limits_{x \to a} g(x)$ exists.
- v F. If f has a local maximum or minimum at c, then $f'(c) = 0$.

2 (10 points) Find the following limits.

i $\lim\limits_{t \to 0} \cos\left(\dfrac{1}{t} - \dfrac{1}{t\sqrt{1+t^2}}\right)$

Answer: Since $\cos x$ is continuous everywhere and

$$\begin{aligned}
\lim_{t \to 0}\left(\frac{1}{t} - \frac{1}{t\sqrt{1+t^2}}\right) &= \lim_{t \to 0}\left(\frac{\sqrt{1+t^2}-1}{t\sqrt{1+t^2}}\right) = \lim_{t \to 0}\left(\frac{(\sqrt{1+t^2}-1)(\sqrt{1+t^2}+1)}{t\sqrt{1+t^2}(\sqrt{1+t^2}+1)}\right) \\
&= \lim_{t \to 0}\left(\frac{t^2}{t\sqrt{1+t^2}(\sqrt{1+t^2}+1)}\right) = \lim_{t \to 0}\left(\frac{t}{\sqrt{1+t^2}(\sqrt{1+t^2}+1)}\right) \\
&= \frac{0}{2} = 0,
\end{aligned}$$

we have $\lim\limits_{t \to 0} \cos\left(\dfrac{1}{t} - \dfrac{1}{t\sqrt{1+t^2}}\right) = \cos\left(\lim\limits_{t \to 0}\left(\dfrac{1}{t} - \dfrac{1}{t\sqrt{1+t^2}}\right)\right) = \cos 0 = 1$

ii $\lim\limits_{x \to 0} \dfrac{x^2}{\sec x - 1}$

Answer:

$$\begin{aligned}
\lim_{x \to 0} \frac{x^2}{\sec x - 1} &= \lim_{x \to 0} \frac{x^2 \cos x}{1 - \cos x} = \lim_{x \to 0} \frac{x^2 \cos x(1+\cos x)}{1 - \cos^2 x} \\
&= \lim_{x \to 0}\left(\frac{x}{\sin x}\right)^2 \times \lim_{x \to 0} \cos x(1+\cos x) \\
&= 1 \times 2 = 2
\end{aligned}$$

3 (20 points) Find the following derivatives.

i Find $f'(1)$, where $f(x) = \dfrac{\sqrt[3]{x} - 2x^2}{\sqrt{x}}$.

Answer: First we simplify $f(x) = \dfrac{\sqrt[3]{x} - 2x^2}{\sqrt{x}} = x^{-1/6} - 2x^{3/2}$

Hence $f'(1) = -\frac{1}{6} - 3 = -\frac{19}{6}$

ii Find $g'(x)$, where $g(x) = \tan^2(1-x)^2$.

Answer:

$$g'(x) = 2\tan(1-x)^2 \sec^2(1-x)^2 \times 2(1-x) \times (-1) = -4(1-x)\tan(1-x)^2 \sec^2(1-x)^2$$

iii If $x^2 + xy + y^3 = -1$, find the value of y' and y'' at the point where $x = 0$.

Answer: When $x = 0$, we have $y^3 = -1$. Thus $y = -1$. Since $2x + y + xy' + 3y^2y' = 0$, we have

$$y' = \frac{-2x - y}{x + 3y^2}$$

Hence $y' = 1/3$ when $x = -1$.

$$\begin{aligned} y'' &= \frac{(-2 - y')(x + 3y^2) - (-2x - y)(1 + 6yy')}{(x + 3y^2)^2} \\ &= \frac{3(-2 - 1/3) - 1(1 - 6/3)}{9} = \frac{-7 + 1}{9} = -\frac{2}{3} \end{aligned}$$

iv Find $\dfrac{d^2y}{dx^2}$, where $y = \dfrac{\sin x}{1 - \cos x}$.

Answer: $\dfrac{dy}{dx} = \dfrac{\cos x(1 - \cos x) - \sin x(\sin x)}{(1 - \cos x)^2} = \dfrac{\cos x - 1}{(1 - \cos x)^2} = \dfrac{1}{\cos x - 1}$

$$\frac{d^2y}{dx^2} = \frac{\sin x}{(\cos x - 1)^2}$$

4 (10 points) Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x^2} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$

i Find $f'(x)$ for $x \neq 0$.

ii Find $f'(0)$.

Answer: For $x \neq 0$, we have $f'(x) = 2x \sin \dfrac{1}{x^2} - 2\dfrac{1}{x} \cos \dfrac{1}{x^2}$. For $x = 0$, we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h^2} - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h^2}$$

Since $0 \leq |\sin \frac{1}{h^2}| \leq 1$ for any $h \neq 0$, we have $0 \leq |h||\sin \frac{1}{h^2} \leq |h|$ for any nonzero h. By $\lim_{h \to 0} |h| = 0$ and the pitching theorem, we have $\lim_{h \to 0} h^2 \sin \frac{1}{h^2} = 0$. Thus $f'(0) = 0$.

5 (5 points) Prove that for all real x and y,

$$|\cos x - \cos y| \leq |x - y|$$

Answer: If $x = y$, then the equation is true. By the mean value theorem, we have

$$\cos x - \cos y = -(\sin c)(x - y)$$

for some c in between x and y. Since $|\sin c| \leq 1$ for all c, we have

$$|\cos x - \cos y| = |(\sin c)||(x - y)| \leq |x - y|$$

6 (10 points) Find equations of both lines through the points $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

Answer: Let $(a, a^2 + a)$ be a point on the curve. Since $y' = 2x + 1$ we have the tangent line to the curve at the point (a, a^2) is $y = (2a + 1)(x - a) + a^2 + a = (2a + 1)x - a^2$. Suppose $(2, -3)$ is on the tangent line, we have $-3 = 4a + 2 - a^2$. Hence we have $a = -1$, 5. Therefore the equations are $y = -x - 1$ and $y = 11x - 25$.

7 (5 points) Use a differential to estimate the value of $\sqrt[4]{16.1}$.

Answer: Let $f(x) = x^{1/4}$ and $a = 16$. The linear approximation of f at a is

$$L(x) = f(a) + f'(a)(x-a) = 2 + 1/32(x-16).$$

By substituting $x = 16.1$, we have

$$\sqrt[4]{16.1} \simeq 2 + 0.1/32 = 2\frac{1}{320} = 2.003125$$

8 (5 points) Water is poured into a conical container, vertex down, at the rate of 2 cubic feet per minute. The container is 6 feet deep and the open end is 8 feet across. How fast is the level of the water rising when the container is half full?

Answer: Let V be the volume of water, h be the hight and r be the radius. Then we know $r = 2/3h$ and

$$V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$$

Hence we have

$$\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}$$

When the container is half full, we have $h = 6/\sqrt[3]{2}$ feet and therefore

$$\frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt} = \frac{9\sqrt[3]{4}}{4\pi \times 36} \times 2 = \frac{\sqrt[3]{4}}{8\pi}$$

9 (5 points) If $1200\,\text{cm}^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Answer: Let h be the height of the box and x be the side of the base. Then we have $x^2 + 4hx = 1200$. Hence we get $h = \frac{1200-x^2}{4x}$. The volume V of the box is

$$V = x^2 h = \frac{x(1200-x^2)}{4}.$$

SInce $V' = \frac{1}{4}(1200 - 3x^2)$, we have $V' = 0$ when $x = 20$ and $V'' < 0$ for all other $x > 0$, hence V has maximum value when $x = 20$, $h = 10$. In this case, the volume is

$$V = x^2 h = 4000\text{cm}^3$$

10 (15 points) Consider the graph of $y = f(x) = 3x^4 - 8x^3 - 6x^2 + 24x - 7$.

- i (5 points) Find the intervals of increase or decrease.
- ii (5 points) Find all local maximum and minimum value of $f(x)$.
- iii (5 points) Find the intervals on which f is concave upward or downward.

Answer: $f'(x) = 12x^3 - 24x^2 - 12x + 24 = 12(x^3 - 2x^2 - x + 2) = 12(x^2-1)(x-2) = 12(x-1)(x+1)(x-2)$.

Hence f is increasing on $[-1,1] \cup [2,\infty)$ and decreasing on $(-\infty,-1] \cup [1,2]$

f has local maximum at 1 and $f(1) = 6$, f has local minimum at $-1, 2$ and the local minimum values are $f(-1) = -26$ and $f(2) = 1$

$f''(x) = 12(3x^2 - 4x - 1) = 36(x - \frac{2+\sqrt{7}}{3})(x - \frac{2-\sqrt{7}}{3})$

Hence f is concave up on $(-\infty, \frac{2-\sqrt{7}}{3}] \cup [\frac{2+\sqrt{7}}{3}, \infty)$ and concave down on $[\frac{2-\sqrt{7}}{3}, \frac{2+\sqrt{7}}{3}]$

11 (10 points) Prove that if $\lim_{x \to c} f(x) = L$, then there are positive numbers δ and B such that if $0 < |x - c| < \delta$, then $|f(x)| < B$.

Answer: Since $\lim_{x \to c} f(x) = L$, we can choose $\epsilon = 1$, then there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon = 1$. Then let $B = |L| + 1$, we have if $0 < |x - c| < \delta$, then

$$|f(x)| \leq |f(x) - L| + |L| < 1 + |L| = B$$