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Infinitesimal Isometric Variation of Semi-Riemannian Submanifolds *†

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Abstract

Infinitesimal isometric variations of submanifolds of a semi-Riemannian manifold are considered. We define variation vector fields of submanifolds and study their properties. It is shown that an isometric variation of a submanifold of the semi-Euclidean space is trivial if, and only if, there exist a matrix $\mathbf{a}$ and a vector $\mathbf{b}$ such that for each $x \in M$, $Z_x = \mathbf{a}f(x) + \mathbf{b}$ where $\mathbf{a}^t = -\varepsilon\mathbf{a}\varepsilon$.

1 Introduction

Let M and $\overline{M}$ be semi-Riemannian manifolds and g and $\overline{g}$ the metric tensors on M and $\overline{M}$ respectively. Suppose M is isometrically immersed in $\overline{M}$ by the immersion $f : M \to \overline{M}$ with the understanding that $f(M)$ is identified with M. In this paper all manifolds are assumed to be C^∞.

Many authors have studied the infinitesimal rigidity of Riemannian submanifolds. Variation vector field is the main tool of this subject. Our objective is to investigate the variation vector field of semi-Riemannian submanifolds.

For the smooth map $F : I \times M \to \overline{M}$, where $I = (-\varepsilon, \varepsilon)$, suppose that the submanifolds $F_s(M)$ are the semi-Riemannian submanifolds of $\overline{M}$ for all $s \in I$. For the rest of this section we will assume that the induced metrics on $F_s(M)$ is non-degenerate for all $s \in I$.

An *isometric variation* is a smooth map F such that $F_s(x) = F(s, x)$ is an isometric immersion for each $s \in I$ and $F(0, x) = f(x)$.

Let

$$g_s = (F_s)^*\overline{g}. \tag{1}$$

Then g_s is a metric tensor on M. Let $T_2(T_p(M))$ denote the tensor algebra of $(0, 2)$-tensor fields on $T_p(M)$. If we consider the following function for all $p \in M$,

$$\begin{array}{rccc} g_p : & I & \to & T_2(T_p(M)) \\ & s & \to & (g_s)_p \end{array}$$

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then, (g_p, I) is a curve on $T_2(T_p(M))$.

DEFINITION 1. The smooth map F is called an infinitesimal isometric variation of M if $g'(0) = 0$.

EXAMPLE 1. Consider the Minkowski space $\mathbb{R}^3_1$. The metric tensor $\overline{g}$ on this space is given by the following equality:

$$\overline{g}_p(v, w) = -v_1 w_1 + v_2 w_2 + v_3 w_3 \tag{2}$$

for every $v, w \in T_p(\mathbb{R}^3_1)$. Now suppose

$$M = \{(p_1, p_2) \mid |p_1| < 1,\ 0 < -p_1^2 + p_2^2 < 1\} \tag{3}$$

and

$$F : I \times M \to R^3_1,\ F = (u, v, t(1 + u^2 - v^2)),$$

where $I = (-1/2, 1/2)$, $\{u, v\}$ is the Euclidean coordinate system of $\mathbb{R}^2_1$ and $T_p(M) = S_p\{\frac{\partial}{\partial u}\big|_p, \frac{\partial}{\partial v}\big|_p\}$. Then $F_0 = f$ and rank $J(F_s) = 2$. The tensor field g_s on M defined by (1) is obtained as

$$\begin{aligned} & (g_s)_p(X_p, Y_p) \\ = \ & \langle X_p, Y_p\rangle + 4s^2[p_1^2 a_1(p) b_1(p) - p_1 p_2 a_1(p) b_2(p) \\ & -p_2 p_1 a_2(p) b_1(p) + p_2^2 a_2(p) b_2(p)], \end{aligned} \tag{4}$$

where $X_p = a_1(p)\ \frac{\partial}{\partial u}\big|_p + a_2(p)\ \frac{\partial}{\partial v}\big|_p$ and $Y_p = b_1(p)\ \frac{\partial}{\partial u}\big|_p + b_2(p)\ \frac{\partial}{\partial v}\big|_p$ for all $X_p, Y_p \in T_p(M)$. Since,

$$(g_s)_p = (-1 + 4s^2 p_1^2, -4s^2 p_1 p_2, -4s^2 p_1 p_2, 1 + 4s^2 p_2^2),$$

we have $g'(0) = 0$.

The tensor field g_0 on the manifold M assigns a scalar product to each point of M and the index of g_0 is the same for all p. Therefore $f(M)$ is a semi-Riemannian submanifold of $\mathbb{R}^3_1$. The submanifold $F_s(M)$ is the semi-Riemannian submanifold of the space $\mathbb{R}^3_1$ for every $s \in I$ if the index of g_s is the same for all p and the following condition holds,

$$\forall Y_p \in T_p(M),\ (g_s)_p(X_p, Y_p) = 0 \Rightarrow X_p = 0.$$

In (4), if we take $Y_p = (1, 0)$ and $Y_p = (0, 1)$ then the following system of equations are obtained:

$$\begin{array}{c} (-1 + 4s^2 p_1^2) a_1(p) - 4s^2 p_1 p_2 a_2(p) = 0 \\ (-4s^2 p_1 p_2) a_1(p) + (1 + 4s^2 p_2^2) a_2(p) = 0 \end{array} \tag{5}$$

Let $\triangle$ denote the determinant of system (5), then

$$\triangle = 0 \Longleftrightarrow p_1^2 - p_2^2 = \frac{1}{4s^2}.$$

Since $\triangle \neq 0$ for each p of M, there is no non-vanishing solution of the system (5). Therefore $X_p = 0$. Hence $(g_s)_p$ is nondegenerate.

By setting $v_1 = (1,0)$, $v_2 = ((4s^2 p_1 p_2)/(-1+4s^2 p_1^2), 1)$ and $e_1 = v_1/|v_1|$, $e_2 = v_2/|v_2|$, we have

$$(g_s)_p(e_1, e_1) = -1, \tag{6}$$

$$(g_s)_p(e_2, e_2) = 1. \tag{7}$$

By (6) and (7), the index of the metric tensor g_s is 1 at each point p. It can easily be seen that the metric tensor g_s is symmetric and bilinear. Thus the manifolds $F_s(M)$ for $s \in (-1/2, 1/2)$ are the semi-Riemannian submanifolds of the space $\mathbb{R}^3_1$. Therefore F is an infinitesimal isometric variation of M.

2 Preparatory Results

Now, we consider an isometric immersion $f : M \to \overline{M}$ as before. Let Z_x be the tangent vector of the curve $\alpha : I \to \overline{M}$, $\alpha(s) = F(s,x)$ at $\alpha(0)$, for each $x \in M$, Z_x is the initial velocity of the orbit of $f(x)$ under F. The section Z is called the variation vector field of the variation of F.

LEMMA 1. Let M and $\overline{M}$ be semi-Riemannian manifolds and F be an isometric variation of M in $\overline{M}$. Let $\widetilde{f} : \overline{M} \to \mathbb{R}^n_\nu$ be an isometric immersion. Suppose that

$$\widetilde{F} = \widetilde{f} \circ F.$$

Then $\widetilde{F}$ is an infinitesimal isometric variation if, and only if, F is an infinitesimal isometric variation.

PROOF. The variation $\widetilde{F}$ gives the mapping $\widetilde{F}_s : M \to \mathbb{R}^n_\nu$, $\widetilde{F}_s(x) = \widetilde{F}(s,x)$ for all $s \in I$ and $x \in M$. Let the metric tensors $g, \overline{g}, \widetilde{g}$ be the metric tensors on the manifolds M, $\overline{M}$ and $\mathbb{R}^3_\nu$ respectively. Let us suppose

$$\widetilde{g}_s = (\widetilde{F}_s)^* \widetilde{g},$$

then

$$\begin{aligned}
& (g_p)'(0)(X_p, Y_p) = 0 \\
\Leftrightarrow\ & \lim_{h\to 0} \tfrac{1}{h}\{(\widetilde{g}_p)(h) - (\widetilde{g}_p)(0)\}(X_p, Y_p) = 0 \\
\Leftrightarrow\ & \lim_{h\to 0} \tfrac{1}{h}\{(\widetilde{g}_h)_p - (\widetilde{g}_0)_p\}(X_p, Y_p) = 0 \\
\Leftrightarrow\ & \lim_{h\to 0} \tfrac{1}{h}\{[(\widetilde{F}_h)^* \widetilde{g}]_p - [(\widetilde{F}_0)^* \widetilde{g}]_p\}(X_p, Y_p) = 0 \\
\Leftrightarrow\ & \lim_{h\to 0} \tfrac{1}{h}\{\widetilde{g}(\widetilde{f}_*((F_h)_* X_p), \widetilde{f}_*((F_h)_* Y_p)) - \widetilde{g}(\widetilde{f}_*((F_0)_* X_p), \widetilde{f}_*((F_0)_* Y_p))\} = 0 \\
\Leftrightarrow\ & \lim_{h\to 0} \tfrac{1}{h}\{\overline{g}((F_h)_* X_p), (F_h)_* Y_p) - \overline{g}(f_* X_p, f_* Y_p)\} = 0 \\
\Leftrightarrow\ & \lim_{h\to 0} \tfrac{1}{h}\{(g_h)_p) - (g_0)_p\}(X_p, Y_p) = 0 \\
\Leftrightarrow\ & \lim_{h\to 0} \tfrac{1}{h}\{g_p(h) - g_p(0)\}(X_p, Y_p) = 0 \\
\Leftrightarrow\ & g'_p(0)(X_p, Y_p) = 0 \\
\Leftrightarrow\ & [g'(0)]_p(X_p, Y_p) = 0
\end{aligned}$$

for all X_p, $Y_p \in T_p(M)$. This completes the proof.

Now we state the following lemma for the immersed semi-Riemannian submanifold.

LEMMA 2. The variation F is an infinitesimal isometric variation if, and only if,

$$\langle D_X Z, X\rangle = 0, \text{ for all } X \in \chi(M). \tag{8}$$

Here D is the induced connection and $\langle , \rangle$ is the fibre metric on the tangent bundle $T(\overline{M})$.

Indeed, it can easily be shown by using Lemma 1. Moreover, one can be find the proof for the Riemannian submanifolds in [1] and [2].

3 Trivial Isometric Variations

Let the group of isometries of the space $\mathbb{R}^n_\nu$ be $I(\mathbb{R}^n_\nu)$ and let $\varphi(I)$ be a curve in $I(\mathbb{R}^n_\nu)$ such that $\varphi(0) = 1$. Let us define the mapping μ by the equation

$$\mu(s, x) = \varphi(s)(f(x)). \tag{9}$$

Then

$$(\mu_s)_*(X) = (\varphi(s) \circ f)_*(X) = (\varphi(s))_*(f_* X)$$

for $X \in \chi(M)$. We also have $g_s = g_0$ since

$$\begin{aligned} g_s(X, Y) &= [(\mu_s)^* \overline{g}](X, Y) = \overline{g}((\varphi(s))_*(f_* X), (\varphi(s))_*(f_* Y)) \\ &= \overline{g}(f_* X, f_* Y) = [(\mu_0)^* \overline{g}](X, Y) = g_0(X, Y) \end{aligned}$$

for all $X, Y \in \chi(M)$. Thus μ is an isometric variation.

DEFINITION 2. Let M be a semi-Riemannian manifold and F be an isometric variation of M. If the variation vector field of F coincides with the variation vector field of the isometric variation μ defined by (9) then F is said to be trivial.

THEOREM 1. A variation of a submanifold of the semi-Euclidean space is trivial if, and only if, there exits a matrix $\mathbf{a}$ and a vector $\mathbf{b}$ such that

$$Z_x = \mathbf{a} f(x) + \mathbf{b}, \text{ for each } x \in M$$

where $\mathbf{a}^t = -\varepsilon \mathbf{a} \varepsilon$.

PROOF. If $\varphi(I)$ is a curve in the space $I(\mathbb{R}^n_\nu)$, then we have,

$$\varphi(s)(f(x)) = \alpha(s) f(x) + \beta(s), \tag{10}$$

where $f(x) \in \mathbb{R}^n_\nu$, $\alpha(s)$ is a semi-orthogonal matrix and $\beta(s) \in \mathbb{R}^n_\nu$. Substituting $s = 0$ in (10), we obtain

$$f(x) = \alpha(0) f(x) + \beta(0). \tag{11}$$

The points $p_0, p_1, \ldots, p_n$ can be chosen so that the vectors $\overrightarrow{p_0 p_i}$, $(1 \le i \le n)$ form a base of the vector space $\mathbb{R}^n_\nu$ in the manifold $f(M)$. Then by using (11), we get the following equalities

$$[I - \alpha(0)](\overrightarrow{0 p_0}) = \beta(0),$$

$$[I - \alpha(0)](\overrightarrow{0 p_i}) = \beta(0), \ 1 \le i \le n.$$

Therefore,
$$[I-\alpha(0)](\overrightarrow{p_0p_i})=0. \tag{12}$$
By (12), we have $I=\alpha(0)$ and $\beta(0)=0$.

Let λ be the variation which is defined by the curve φ. Let us denote the variation vector field of λ by Z_1. The orbit of the point $f(x)$ in the variation λ is the curve
$$s\to\alpha(s)f(x)+\beta(s).$$
Thus we have,
$$(Z_1)_x=\alpha'(0)f(x)+\beta'(0).$$
Since $\alpha(s)$ is a semi-orthogonal matrix,
$$(\alpha(s))^t=\varepsilon(\alpha(s))^{-1}\varepsilon, \tag{13}$$
or simply, $\alpha^t=\varepsilon\alpha^{-1}\varepsilon$. We have
$$(\alpha^t)'=\varepsilon(\alpha^{-1})'\varepsilon. \tag{14}$$
Since α is a semi-orthogonal matrix, $|\det\alpha|=1$. Thus if we let $\alpha^{-1}=\widetilde{\alpha}$, then $\alpha\widetilde{\alpha}=I$. Now
$$\alpha\widetilde{\alpha}=I\Rightarrow\alpha'\widetilde{\alpha}+\alpha\widetilde{\alpha}'=0\Rightarrow\alpha'(0)\widetilde{\alpha}(0)+\alpha(0)\widetilde{\alpha}'(0)=0.$$
Since $\alpha(0)=I$ and $\alpha\widetilde{\alpha}=I$, then $\widetilde{\alpha}(0)=I$. From this, it follows that
$$\widetilde{\alpha}'(0)=-\alpha'(0). \tag{15}$$
By (14), we find
$$(\alpha'(0))^t=\varepsilon(\alpha^{-1}(0))'\varepsilon.$$
Using $\alpha^{-1}(0)=\widetilde{\alpha}(0)$ and considering the equality (15) one can obtain
$$(\alpha'(0))^t=-\varepsilon(\alpha'(0))\varepsilon.$$
Let us denote $\mathbf{a}=\alpha'(0)$ and $\mathbf{b}=\beta(0)$ then
$$(Z_1)_x=\mathbf{a}f(x)+\mathbf{b}.$$
Consider the variation F of M in $\mathbb{R}^n_\nu$. If F is trivial then the variation vector field of F coincides with the variation vector field of λ. Therefore the variation field of F is in the form
$$Z_x=\mathbf{a}f(x)+\mathbf{b}.$$
Conversely, suppose that the variation vector field of F is in the form
$$Z_x=\mathbf{a}f(x)+\mathbf{b}$$
such that $\mathbf{a}$ is a matrix satisfying the equality $\mathbf{a}^t=-\varepsilon\mathbf{a}\varepsilon$. i.e., $\mathbf{a}$ is an element of the Lie algebra on $O_\nu(n)$. The matrix $\mathbf{a}$ is in $O_\nu(n)$ if, and only if, $\exp(s\mathbf{a})$ is in $O_\nu(n)$ such that $|s|$ is sufficiently small [3]. In this case the deformation F takes the form
$$F(s,x)=\exp(s\mathbf{a})f(x)+s\mathbf{b}.$$

Let us take $\varphi(s) = \exp(s\mathbf{a})$. Then $\varphi(0) = I$, and $\varphi(s)$ is an isometry for all s. The proof is complete.

Now we present the following example for illustrating the previous theorem.

EXAMPLE 2. Let M be a unit disc in $\mathbb{R}^2_1$ and $f : M \to \mathbb{R}^3_1$ be an immersion. Consider the following isometric variation of M.

$$\mu : I \times M \to \mathbb{R}^3_1,\ \mu(s, x) = \varphi(s)f(x),$$

where $\varphi(s)$ is an isometry which is determined by the equality $[\varphi(s)]u = \alpha(s)u$. If we take $\alpha(s) \in O_1(3)$ to be

$$\alpha(s) = \begin{bmatrix} \cosh s & 0 & \sinh s \\ 0 & 1 & 0 \\ \sinh s & 0 & \cosh s \end{bmatrix}.$$

Then,

$$[\varphi(s)]u = (u_1 \cosh s + u_3 \sinh s, u_2, u_1 \sinh s + u_3 \cosh s).$$

Let us denote the variation vector field of μ by Z_1. It is easily seen that $Z_1 = (0, 0, u)$.

Now we consider the variation F of M given by

$$F : I \times M \to \mathbb{R}^3_1,\ F = (u, v, tu),\ I = (-\frac{1}{2}, \frac{1}{2}).$$

Then F is an infinitesimal isometric variation of M. Let Z be a variation vector field of F. Then $Z = (0, 0, u)$. Since Z_1 and Z coincides, F is a trivial deformation.

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