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New Explicit and Exact Solutions for the Nizhnik-Novikov-Vesselov Equation *†

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Abstract

In this paper, several explicit and exact travelling wave solutions for the Nizhnik-Novikov-Vesselov equation are obtained by using the hyperbola function method and the Wu-elimination method, which include new singular solitary wave solutions and periodic solutions.

In several areas of physics such as condense matter physics [1], fluid mechanics [2], plasma physics [3] and optics [4], $(1+1)$ and $(2+1)$-dimensional solitons have been studied. In this paper, we will use the hyperbola function method [5] and Wu elimination method [6] to find several new exact solutions for the $(2+1)$-dimensional Nizhnik-Novikov-Vesselov (NNV) equation [7]

$$u_t = au_{xxx} + bu_{yyy} - 3av_x u - 3avu_x - 3b\omega_y u - 3b\omega u_y, \tag{1}$$

$$u_x = v_y,\ u_y = \omega_x, \tag{2}$$

where a and b are the arbitrary constants.

The $(2+1)$-dimensional NNV equation is an isotropic extension of the well known $(1+1)$-dimensional KdV equation. In order to give some new types of exact solutions of equations (1) and (2), we use hyperbola function method and Wu elimination method. By using the travelling wave transformation, equations (1) and (2) have the following formal solution:

$$u = \varphi(x,y,t) = \varphi(x),\ v = \psi(x,y,t) = \psi(\xi), \tag{3}$$

$$\omega = \theta(x,y,t),\ \xi = \lambda(x+y+kt+c). \tag{4}$$

Where λ and k are constants to be determined later and c is an arbitrary constant. Substituting equations (3) and (4) into equations (1) and (2) yields ordinary differential equations for φ, ψ, ϑ :

$$(a+b)\lambda^2\varphi^{'''} - 3a(\psi\varphi)^{'} - 3b(\theta\varphi)^{'} - k\varphi^{'} = 0, \tag{5}$$

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$$\varphi' = \psi', \tag{6}$$

$$\psi' = \theta', \tag{7}$$

Equations (6) and (7) imply that $\psi = \varphi + b_0$ and $\theta = \varphi + c_0$ where b_0 and c_0 are constants. Therefore from equations (5), (6) and (7), we have the following

$$\lambda^2\varphi''' - 6\varphi\varphi' - \lambda_0\varphi = 0, \tag{8}$$

where

$$\lambda_0 = \frac{3ab_0 + 3bc_0 + k}{a+b}$$

Thus we only need to find φ for equation (8). According to the hyperbola function method we suppose that equation (8) has the following solution

$$\varphi = A_0 + A_1 \sinh w + A_2 \cosh w + A_3 \sinh w \cosh w + A_4 \sinh^2 w, \tag{9}$$

and

$$\frac{dw}{d\xi} = \sinh w. \tag{10}$$

With the help of Mathematica or Maple, we have

$$\begin{aligned}
& \lambda^2\varphi''' - 6\varphi\varphi' - \lambda_0\varphi' \\
= \;& (24\lambda^2 A_3 - 24A_3A_4)\sinh^5 w + [24\lambda^2 A_4 - 12(A_3^2 + A_4^2)]\sinh^4 w \cosh w \\
& +(6\lambda^2 A_2 - 18A_1A_3 - 18A_3A_4)\sinh^4 w \\
& +(6\lambda^2 A_1 - 18A_1A_4 - 18A_2A_3)\sinh^3 w \cosh w \\
& +(20\lambda^2 A_3 - 12A_0A_3 - 12A_1A_2 - 18A_3A_4 - 2\lambda_0 A_3)\sinh^3 w \\
& +(8\lambda^2 A_4 - 12A_0A_4 - 6A_1^2 - 6A_2^2 - 6A_3^2 - 2\lambda_0 A_4)\sinh^2 w \cosh w \\
& +(4A_2\lambda^2 - 6A_0A_2 - 12A_1A_3 - 12A_2A_4 - \lambda_0 A_2)\sinh^2 w \\
& +(A_1\lambda^2 - 6A_0A_1 - 6A_2A_3 - A_1\lambda_0)\sinh w \cosh w \\
& +(\lambda^2 A_3 - 6A_0A_3 - 6A_1A_2 - \lambda_0 A_3)\sinh w \\
= \;& 0.
\end{aligned}$$

Setting the coefficients of $\sinh^j w \cosh^i w$, $i = 0, 1$ and $j = 0, 1, 2, 3, 4, 5$ to zero, we have

$$24\lambda^2 A_3 - 24A_3A_4 = 0, \tag{11}$$

$$24\lambda^2 A_4 - 12(A_3^2 + A_4^2) = 0, \tag{12}$$

$$6\lambda^2 A_3 - 18A_1A_3 - 18A_2A_4 = 0, \tag{13}$$

$$6\lambda^2 A_1 - 18A_1A_4 - 18A_2A_3 = 0, \tag{14}$$

$$20\lambda^2 A_3 - 12A_0A_3 - 12A_1A_2 - 18A_3A_4 - 2\lambda_0 A_3 = 0, \tag{15}$$

$$8\lambda^2 A_4 - 12A_0A_4 - 6A_1^2 - 6A_2^2 - 6A_3^2 - 2\lambda_0 A_4 = 0, \tag{16}$$

$$4A_2\lambda^2 - 6A_0A_2 - 12A_1A_3 - 12A_2A_4 - \lambda_0 A_2 = 0, \tag{17}$$

$$A_1\lambda^2 - 6A_0A_1 - 6A_2A_3 - A_1\lambda_0 = 0, \tag{18}$$

$$A_3\lambda^2 - 6A_0A_3 - 6A_1A_2 - \lambda_0 A_3 = 0. \tag{19}$$

Using Wu's algebraic elimination method to solve the system of overdetermined equations (11)-(19) with respect to the unknown numbers A_0, A_1, A_2, A_3, A_4, λ, λ_0 yields the following cases:

Case 1: $A_0 = A_1 = A_2 = 0, A_3 = \pm\lambda^2, A_4 = \lambda^2, \lambda_0 = \lambda^2$.

Case 2: $A_0 = A_1 = A_2 = A_3 = 0, A_4 = 2\lambda^2, \lambda_0 = 4\lambda^2$.

Let equations (8) and (19) combine with the following

$$\frac{dw}{d\xi} = \cosh w, \tag{20}$$

then we have

Case 3: $A_0 = A_1 = A_3 = 0, A_4 = 2\lambda^2, \lambda_0 = 8\lambda^2$.

By integrating the equation $dw/d\xi = \sinh w$ and taking the integration constant to be zero, we have

$$\sinh w = -\mathrm{csch}\,\xi, \tag{21}$$

and

$$\cosh w = -\coth\xi. \tag{22}$$

Also from (20), we obtain

$$\sinh w = -\cot\xi, \tag{23}$$

and

$$\cosh w = \csc\xi. \tag{24}$$

According to equations (9), (22)–(24) and cases 1-3 we can obtain the following singular solitary wave solutions (I-II) and periodic solutions (III) for the NNV equations.

(I) $u_1(x, y, t) = \lambda^2(\mathrm{csch}^2\xi \pm \mathrm{csch}\,\xi\coth\xi)$, where $\xi = \lambda(x + y + ((a + b)\lambda^2 - 3ab_0 - 3bc_0)t + c)$

(II) $u_2(x, y, t) = 2\lambda^2\mathrm{csch}^2\xi$, where $\xi = \lambda(x + y + (4(a + b)\lambda^2 - 3ab_0 - 3bc_0)t + c)$.

(III) $u_3(x, y, t) = 2\lambda^2\cot^2\xi$, where $\xi = \lambda(x + y + (8(a + b)\lambda^2 - 3ab_0 - 3bc_0)t + c)$.

In summary, by using the hyperbola function method and with the aid of Mathematica and Wu-elimination method, we obtain more solitary wave solutions and periodic solutions for the NNV equation. The method can also be applied to solve other systems of nonlinear equation, such as coupled KdV equations, coupled scalar field equations and so on.

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