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$\mathcal{L}$-Classes of Inverse Semigroups *

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Abstract

In this paper, we provide some properties of inverse semigroups from those of their $\mathcal{L}$-classes.

1 Introduction

Let S be a semigroup and e be an idempotent element in S. Then a $\mathcal{H}$-class H_e is a subgroup of S. If there exists a morphism θ from S to H_e, then we can make $N^0 \times S \times N^0$ into a semigroup (here and in the sequel, N stands for the set of positive integers and N^0 the set of nonnegative integers). This is called the Bruck-Reilly extension of S determined by θ. If S is a group and θ is an endomorphism of S, then the Bruck-Reilly extension is a bisimple inverse ω-semigroup. In this case, the $\mathcal{H}$-classes are important in the study of semigroups. Furthermore, let S be a complete regular semigroup, then every $\mathcal{H}$-class is a group, and it is known that the Clifford semigroup is a semilattice [1]. On the other hand, for $\mathcal{L}$-classes and $\mathcal{R}$-classes of an inverse semigroup, fundamental properties were given by several authors [5,6,7]. In this paper, we consider the properties of inverse semigroups which are related to the structure of their $\mathcal{L}$-classes. If every $\mathcal{L}$-class of an inverse semigroup S is a semigroup, then we will show S is a Clifford semigroup. We give a characterization of inverse semigroups whose $\mathcal{L}$-classes contain a semigroup. We shall also show that if every $\mathcal{L}$-class of an inverse semigroup S is not a semigroup, then S has a chain of idempotent elements which is not well ordered.

2 Preliminaries

Let S be a semigroup. Then an equivalence $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$ if, and only if, $S^1a = S^1b$, where $S^1 = Sa \cup \{a\}$. Similarly we define the equivalence $\mathcal{R}$ by the rule that $a\mathcal{R}b$ if, and only if, $aS^1 = bS^1$. It is well known that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence. The intersection of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{H}$ and the join of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{D}$. The $\mathcal{L}$-class (resp. $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class) containing the element a will be denoted by L_a (resp. R_a, H_a, D_a). If e is

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an idempotent element of S, then H_e is a subgroup of S, and no $\mathcal{H}$-class can contain more than one idempotent.

A semigroup S is called an I-semigroup if a unary operation $a \mapsto a^{-1}$ is defined on S such that, for all $a, b \in S$,

$$(a^{-1})^{-1} = a, \ aa^{-1}a = a.$$

Completely regular semigroup is specified within an I-semigroup by

$$aa^{-1} = a^{-1}a,$$

and that a Clifford semigroup is specified by the properties

$$aa^{-1} = a^{-1}a, \ aa^{-1}bb^{-1} = bb^{-1}aa^{-1}.$$

An inverse semigroup is an I-semigroup S such that for all $a, b \in S$,

$$aa^{-1}bb^{-1} = bb^{-1}aa^{-1}.$$

PROPOSITION 2.1. The following statements are equivalent.
(1) S is an inverse semigroup.
(2) S is regular, that is for any element a in S, there exists x in S such that $axa = a$ and its idempotents commute.
(3) Every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent.
(4) Every element of S has a unique inverse.

PROPOSITION 2.2. Let S be an inverse semigroup with semilattice E of idempotents. Then the following hold.
(1) $(ab)^{-1} = b^{-1}a^{-1}$ for every $a, \ b$ in S.
(2) Both aea^{-1} and $a^{-1}ea$ are idempotent for every a in S and e in E.
(3) $a\mathcal{L}b$ if, and only if, $a^{-1}a = b^{-1}b$; $a\mathcal{R}b$ if, and only if, $aa^{-1} = bb^{-1}$.
(4) For $e, f \in E$, $e\mathcal{D}f$ if, and only if, there exists a in S such that $aa^{-1} = e$ and $a^{-1}a = f$.

The proofs of these two results can be found in [2, Theorem 5.1.1] and [2, Proposition 5.1.2] respectively.

3 Inverse semigroups

In this section, we assume that S is an inverse semigroup. If every $\mathcal{L}$-class, or, every $\mathcal{R}$-class in S is a semigroup, then we have the following.

THEOREM 3.1. If every $\mathcal{L}$-class, or, every $\mathcal{R}$-class in S is a semigroup, then S is a Clifford semigroup.

PROOF. Suppose that $(a, b) \in \mathcal{L}$. Then by Proposition 2.2(3), $a^{-1}a = b^{-1}b$. Since every $\mathcal{L}$-class is a semigroup, we have $a\mathcal{L}ba$. Hence

$$a^{-1}a = (ba)^{-1}ba = a^{-1}(b^{-1}b)a = a^{-1}(a^{-1}a)a = a^{-2}a^2.$$

Also for each a in S, aa^{-1} and $a^{-1}a$ are idempotent. Since idempotents commute, so we obtain

$$\begin{aligned} aa^{-1} &= aa^{-1}aa^{-1} = aa^{-2}a^2a^{-1} = \left(aa^{-1}\right)\left(a^{-1}a\right)\left(aa^{-1}\right) \\ &= \left(a^{-1}a\right)\left(aa^{-1}\right)\left(aa^{-1}\right)\left(a^{-1}a\right) = a^{-1}a^2a^{-2}a = a^{-1}aa^{-1}a = a^{-1}a. \end{aligned}$$

Thus S is a Clifford semigroup. A similar argument can be applied to the case where each $\mathcal{R}$-class in S is a semigroup. The proof is complete.

Next if there is a $\mathcal{L}$-class L_a in S such that L_a is a semigroup, then we have the following characterization.

THEOREM 3.2. Let a be an element of S. Then the following conditions are equivalent.

(1) L_a is a semigroup.
(2) For every element b in L_a, $b^{-1}b = b^{-2}b^2$.
(3) For every element c in $R_{a^{-1}}$, $cc^{-1} = c^2c^{-2}$.
(4) $R_{a^{-1}}$ is a semigroup.

PROOF. To see that (1) implies (2), let b be an element in L_a. Then $a^{-1}a = b^{-1}b$. Since L_a is a semigroup, $b\mathcal{L}ab$, thus

$$b^{-1}b = (ab)^{-1}ab = b^{-1}(a^{-1}a)b = b^{-1}(b^{-1}b)b = b^{-2}b^2.$$

(2) $\Rightarrow$ (3): Let c be an element in $R_{a^{-1}}$. Then since $a^{-1} \in R_{a^{-1}}$,

$$a^{-1}a = a^{-1}\left(a^{-1}\right)^{-1} = cc^{-1} = \left(c^{-1}\right)^{-1}c^{-1}.$$

It follows that $a\mathcal{L}c^{-1}$, so $c^{-1} \in L_a$. By (2), this shows that

$$cc^{-1} = \left(c^{-1}\right)^{-1}c^{-1} = \left(c^{-1}\right)^{-2}\left(c^{-1}\right)^2 = c^2c^{-2}.$$

(3) $\Rightarrow$ (4): Let $g,\ h \in R_{a^{-1}}$. Then $g\mathcal{R}h$. Since $\mathcal{R}$ is a left congruence, it follows that $g^2\mathcal{R}gh$ and by our assumption, $g\mathcal{R}g^2$ and hence that $g\mathcal{R}gh$. Thus $gh \in R_g = R_{a^{-1}}$.

(4) $\Rightarrow$ (1): Let $x, y \in L_a$, then $x^{-1}x = a^{-1}a = y^{-1}y$. Hence

$$x^{-1}(x^{-1})^{-1} = a^{-1}(a^{-1})^{-1} = y^{-1}(y^{-1})^{-1}.$$

This shows that $x^{-1}, y^{-1} \in R_{a^{-1}}$. Since $R_{a^{-1}}$ is a semigroup, we see that $y^{-1}x^{-1} = (xy)^{-1} \in R_{a^{-1}}$. Thus

$$(xy)^{-1}xy = (xy)^{-1}((xy)^{-1})^{-1} = a^{-1}(a^{-1})^{-1} = a^{-1}a$$

It follows that $xy \in L_a$, so L_a is a semigroup.

THEOREM 3.3. Suppose that a is an element in S such that L_a is a semigroup. Then the following hold.

(1) $a^{-1}a$ is the largest idempotent in the $\mathcal{D}$-class D_a. This shows that each $\mathcal{D}$-class contains at most one $\mathcal{L}$-class which is semigroup.
(2) For any element $b \in L_a$, each b^nb^{-n} $(n \in N)$ is idempotent and

$$bb^{-1} \geq b^2b^{-2} \geq \dots \geq b^nb^{-n} \geq \dots$$

PROOF. To see that (1) holds, let f be an idempotent in D_a. Then there exists an element b in S such that $b^{-1}b = a^{-1}a$ and $bb^{-1} = f$. This shows that b is contained in L_a, so by Theorem 3.2(2), $b^{-1}b = b^{-2}b^2$. Since idempotents commute,

$$f = bb^{-1} = bb^{-1}bb^{-1} = bb^{-2}b^{-2}b^{-1} = f\left(b^{-1}b\right)f = \left(b^{-1}b\right)f = \left(a^{-1}a\right)f.$$

This implies that $a^{-1}a \geq f$. Therefore $a^{-1}a$ is the largest idempotent in $\mathcal{D}_a$. Next we assume that there exists $\mathcal{L}$-class L_b in the $\mathcal{D}$-class D_a which is a semigroup. Then $a^{-1}a$ and $b^{-1}b$ are the largest idempotent elements in D_a, hence $a^{-1}a = b^{-1}b$ and $L_a = L_b$.

To see that (2) holds, suppose that $b \in L_a$. Then $b^n b^{-n}$, $n \in N$, are idempotent elements and by Theorem 3.2(2),

$$a^{-1}a = b^{-1}b = b^{-2}b^2 = \ldots = b^{-n}b^n.$$

It follows that $b^{-1}b = a^{-1}a \geq bb^{-1}$ by (1). Therefore we obtain $bb^{-1} = \left(bb^{-1}\right)\left(b^{-1}b\right)$ and

$$b^2b^{-2} = b\left(bb^{-1}\right)b^{-1} = b\left(bb^{-1}\right)\left(b^{-1}b\right)b^{-1} = \left(b^2b^{-2}\right)\left(bb^{-1}\right).$$

This implies that $bb^{-1} \geq b^2b^{-2}$. Next we assume that $b^kb^{-k} \geq b^{k+1}b^{-(k+1)}$ for $k \geq 1$. Then

$$\begin{aligned}
b^{k+2}b^{-(k+2)} &= b\left(b^{k+1}b^{-(k+1)}\right)b^{-1} = b\left(b^{k+1}b^{-(k+1)}\right)\left(b^kb^{-k}\right)b^{-1} \\
&= b^{k+2}b^{-1}\left(b^{-k}b^k\right)b^{-(k+1)} = b^{k+2}b^{-1}\left(b^{-(k+1)}b^{k+1}\right)b^{-(k+1)} \\
&= \left(b^{k+2}b^{-(k+2)}\right)\left(b^{k+1}b^{-(k+1)}\right)
\end{aligned}$$

This shows that $b^{k+1}b^{-(k+1)} \geq b^{k+2}b^{-(k+2)}$, so an inductive argument leads to $bb^{-1} \geq b^2b^{-2} \geq \ldots \geq b^nb^{-n} \geq \ldots$. The proof is complete.

COROLLARY 3.4. Let a be an element in a periodic inverse semigroup S and assume that the $\mathcal{L}$-class L_a is a semigroup. Then L_a is group.

PROOF. Since S is periodic, L_a becomes a periodic semigroup. Let b be an element in L_a. Then there exists a positive integer n such that b^n is idempotent. It follows that $b^n = a^{-1}a$ since $a^{-1}a$ is the largest idempotent in D_a and $b^n \in L_a$, hence $b^nb^{-n} = a^{-1}a$. Now Theorem 3.3 (2) shows that

$$a^{-1}a \geq bb^{-1} \geq b^2b^{-2} \geq \cdots \geq b^nb^{-n}.$$

Therefore we have that $b \in H_{a^{-1}a}$, and so $L_a \subseteq H_{a^{-1}a} \subseteq L_a$. Thus $L_a = H_{a^{-1}a}$ is a group. The proof is complete.

Corollary 3.4 shows that there is no finite inverse semigroup with a $\mathcal{L}$-class which is a semigroup but not a group.

EXAMPLE 3.5. Let $\mathcal{I}_X$ be a symmetric inverse semigroup, that is, the set of all partial one to one maps of a set X. Let $X = \{1, 2\}$. Then the following two $\mathcal{L}$-classes

$$\left\{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right\} \quad \text{and} \quad \{\emptyset\}$$

are groups.

EXAMPLE 3.6. Let

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ 1 & 2 & 3 & \cdots & n+1 & \cdots \end{pmatrix}$$

and let $S = \{\sigma^{-m}\sigma^{n} \mid m, n \in N^0\}$. Then S has a $\mathcal{L}$-class which is a semigroup but not a group. Note that S is an inverse semigroup and isomorphic to bicyclic semigroup $N^0 \times N^0$. Thus $\mathcal{L}$-class $L_{\sigma\sigma^{-1}} \cong L_{(0,0)} = \{(m, 0) \in N^0 \times N^0 \mid m \in N^0\}$ is a semigroup but not a group.

For any subset T in an inverse semigroup S, we define $R(T) = \{x \in S \mid Tx \subseteq T\}$. Clearly $R(T)$ is a semigroup. Let a be an element of S. Then since $a^{-1}a$ is a right identity in $R(L_a)$, $a^{-1}a \in R(L_a)$

THEOREM 3.7. The following statements hold.

(1) For any $y \in R(L_a)$, $y^{-1}y \in R(L_a)$.

(2) $a^{-1}a$ is the smallest idempotent element in $R(L_a)$.

(3) There is an unique idempotent element in $R(L_a)$ if, and only if, $R(L_a) \subseteq L_a$.

(4) L_a is a semigroup if, and only if, $L_a \subseteq R(L_a)$.

(5) L_a is a group if, and only if, $D_a \subseteq R(L_a)$.

PROOF. (1): Let $y \in R(L_a)$. Then since $L_a y \subseteq L_a$, $(by)^{-1}(by) = a^{-1}a = b^{-1}b$ for any element $b \in L_a$. Hence for any $b \in L_a$,

$$\begin{aligned} \left(by^{-1}y\right)^{-1} by^{-1}y &= y^{-1}yb^{-1}by^{-1}y \\ &= y^{-1}y\left(y^{-1}b^{-1}by\right)y^{-1}y = y^{-1}b^{-1}by = b^{-1}b = a^{-1}a. \end{aligned}$$

It follows that $L_a y^{-1}y \subseteq L_a$. Thus $y^{-1}y \in R(L_a)$.

(2): Let f be an idempotent element in $R(L_a)$. Then since $af \in L_a$, $a^{-1}a = (af)^{-1}af = f\left(a^{-1}a\right)$. This implies that $a^{-1}a \leq f$, so $a^{-1}a$ is the smallest idempotent element in $R(L_a)$.

(3): Assume that there is an unique idempotent element in $R(L_a)$. Then by (2), $a^{-1}a$ is the unique idempotent in $R(L_a)$. Let $y \in R(L_a)$. Then by (1), $y^{-1}y \in R(L_a)$. Since $y^{-1}y$ is an idempotent element, we have that $y^{-1}y = a^{-1}a$, so $y \in L_a$. The converse is clear since L_a contains the unique idempotent element.

(4): Assume that L_a is a semigroup. Then $L_a L_a \subseteq L_a$, so $L_a \subseteq R(L_a)$. Conversely since $L_a \subseteq R(L_a)$, $L_a L_a \subseteq L_a$, hence L_a is a semigroup.

(5): Assume that L_a is a group. Then $L_a = R_a = D_a$. Thus by (4), $D_a = L_a \subseteq R(L_a)$. Conversely since $L_a L_a \subseteq L_a D_a \subseteq L_a R(L_a) \subseteq L_a$, L_a become a semigroup. Further, by Theorem 3.3(1), D_a contains an unique idempotent element. This shows that $D_a = R_a = L_a = H_a$. Hence L_a is a group.

COROLLARY 3.8. Let a be an element in the inverse semigroup S. Then the following statements hold.

(1) There exists an unique idempotent element in $R(L_a)$ if, and only if, $a^{-1}a$ is a right identity element of $R(L_a)$.

(2) L_a is a semigroup if, and only if, $L_a = R(L_a) \cap D_a$.

PROOF. (1): Let y be an element in $R(L_a)$. Then since $R(L_a) \subseteq L_a$, $ya^{-1}a = yy^{-1}y = y$. Thus $a^{-1}a$ is a right identity element of $R(L_a)$. Conversely let f be an idempotent element in $R(L_a)$, then $a^{-1}a \leq f$. Further since $a^{-1}a$ is a right identity element in $R(L_a)$, $fa^{-1}a = f$. It follows that $f = a^{-1}a$.

(2): Let $z \in R(L_a) \cap D_a$, then $a^{-1}a \leq z^{-1}z$ and $z^{-1}z \leq a^{-1}a$ by Theorem 3.3(1) and Theorem 3.6(2). Hence $z^{-1}z = a^{-1}a$ which implies $z \in L_a$, that is, we have equality. The converse is clear from Theorem 3.6(4).

Finally, we consider an inverse semigroup S which satisfies the condition that each $\mathcal{L}$-class is not a semigroup. We have the following.

THEOREM 3.9. Let S be an inverse semigroup such that each $\mathcal{L}$-class is not a semigroup. Then there is a chain of idempotents in S which is not well ordered.

PROOF. Let e_λ be an idempotent element in S. Then since L_{e_λ} is not a semigroup, there is an idempotent e_μ such that $L_{e_\lambda}L_{e_\mu} \cap L_{e_\mu} \neq \emptyset$ and $e_\lambda \neq e_\mu$. Let c be an element in $L_{e_\lambda}L_{e_\mu} \cap L_{e_\mu}$. Then there exist $a,\ b \in L_{e_\lambda}$ such that $c = ab$. Since e_λ is a right identity of L_{e_λ}, it follows that $b = be_\lambda$. Hence we have

$$Se_\lambda = Sc = (Sa)b = (Se_\lambda)be_\lambda = (Se_\lambda be_\lambda)e_\lambda = Se_\mu e_\lambda.$$

This shows that $e_\lambda e_\mu$ is an element in L_{e_λ}, so Proposition 2.1(3) implies that $e_\mu e_\lambda = e_\mu$, which means that $e_\lambda > e_\mu$. We can apply this process repeatedly to obtain a chain of idempotents in S. We claim that this chain is not well-ordered. Indeed, if there is a minimal element in the chain, then the $\mathcal{L}$-class which contains the minimal element must become a semigroup, but this is contrary to our assumption.

EXAMPLE 3.10. Let S be the inverse semigroup generated by the following set

$$\left\{ \begin{pmatrix} \ell & \ell+1 & \cdots & \ell+n & \cdots \\ \ell+1 & \ell+2 & \cdots & \ell+n+1 & \cdots \end{pmatrix} \,\middle|\, \ell \in Z,\ n \in N^0 \right\}$$

and its inverse. Then each $\mathcal{L}$-class is

$$L_m = \{\sigma \in S \mid \mathrm{Im}\sigma = (m, m+1, m+2, \cdots)\}.$$

Clearly L_m is not semigroup.

References

[1] A. H. Clifford, Semigroups admitting relative inverses, Ann. of Math., 42(1941), 1037–1049.

[2] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Vol. II. Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I. 1967.

[3] J. M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, 1995.

[4] J. M. Howie and B. M. Schein, Anti-uniform semilattice, Bull. Australian Math. Soc., 1(1969), 263–268.

[5] A. E. Liber, On the theory of generalized groups (in Russian), Doklady Akad. Nauk. SSSR, 97(1954), 25–28.

[6] B. M. Schein, Representations of generalized groups, Izv. Uchebn. Zaved. Mathematika, 3(28)(1962), 164–176.

[7] V. V. Vagner, Generalized groups (in Russian), Doklady Akad. Nauk. SSSR, 84(1952), 1119–1122.