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Stability Criteria for a Class of Discrete Reaction-Diffusion Equations *†

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Abstract

Dichotomy theories for difference equations are applied to obtain stability criteria for a class of discrete reaction-diffusion equations.

Discrete reaction-diffusion type partial difference equations have recently been introduced by a number of authors as models for the study of spatiotemporal chaos (see e.g. [6]). Stability criteria have also been derived for such equations which involves two-level (see [3]) as well as three-level processes (see e.g. [4]). Discretizations of the heat equation lead to several well known multi-level partial difference schemes. Thus, besides the question of existence of solutions, stability behaviors of solutions of difference schemes are also of fundamental importance, because these behaviors are related to the question of growth of numerical errors. The stability problem has been treated by several authors (see e.g. [3-5]). The techniques used to derive stability criteria in these studies include Gronwall type inequalities, Bihari type inequalities, general solutions, Laplace transforms, comparison theorems, etc. Dichotomy theory has not been utilized, however. In this paper, we intend to show that dichotomy theory is also useful in obtaining stability criteria for partial difference equations.

To this end, we will look at a reaction-diffusion equation. Let R be the set of reals and N the set of nonnegative integers. Consider a discrete reaction-diffusion equation of the form

$$u_i^{(j+1)} = a_j u_{i-1}^{(j)} + b_j u_i^{(j)} + c_j u_{i+1}^{(j)} + g_i^{(j)} + G(j, u_i^{(j)}), \tag{1}$$

where $i = 1, 2, \dots n$; $j \in N$; $\{a_j\}, \{b_j\}$ and $\{c_j\}$ are real sequences; $g = \{g_i^{(j)}\}$ is a real function defined for $i = 1, 2, \dots, n$ and $j \in N$, and G is a real function. We will also assume that side conditions

$$u_0^{(j)} = h_j \in R,\ j \in N, \tag{2}$$

$$u_{n+1}^{(j)} = q_j \in R,\ j \in N, \tag{3}$$

$$u_i^{(0)} = \tau_i \in R,\ i = 1, 2, \dots, n, \tag{4}$$

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are imposed. Let

$$\Psi = \{(i,j) \mid i = 0,1,\ldots,n+1\,;\ j \in N\}.$$

A solution of (1-4) is a discrete function $u = \{u_i^{(j)}\}_{(i,j)\in\Psi}$ which satisfies the functional relation (1) and also the side conditions (2-4). If we put $u^{(j)} = \mathrm{col}(u_1^{(j)}, u_2^{(j)}, \ldots, u_n^{(j)})$ and $\tau = \mathrm{col}(\tau_1, .., \tau_n)$, then the sequence $\{u^{(j)}\}_{j=0}^{\infty}$ will satisfy the two-term vector equation

$$u^{(j+1)} = A(j)\ u^{(j)} + f_j + F(j, u^{(j)}),\ j \in N, \tag{5}$$

subject to the initial condition

$$u^{(0)} = \tau, \tag{6}$$

where

$$A(j) = \begin{bmatrix} b_j & c_j & 0 & ... & ... & 0 \\ a_j & b_j & c_j & 0 & ... & 0 \\ 0 & a_j & b_j & c_j & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & c_j \\ 0 & ... & ... & 0 & a_j & b_j \end{bmatrix}, \tag{7}$$

$$f_j = \mathrm{col}\left(g_1^{(j)}, \ldots, g_n^{(j)}\right) + \mathrm{col}\left(a_j h_j, 0, \ldots, 0, c_j q_j\right),$$

and

$$F(j, (x_1, ..., x_n)) = \mathrm{col}(G(j, x_1), \ldots, G(j, x_n)).$$

Conversely, if $\{u^{(j)}\}_{j=0}^{\infty}$ is a solution of (5-6), then by augmenting each $u^{(j)}$ with the terms $u_0^{(j)} = h_j$ and $u_{n+1}^{(j)} = q_j$ to form $\{u_0^{(j)}, u_1^{(j)}, \ldots, u_n^{(j)}, u_{n+1}^{(j)}\}$, we see that the resulting family forms a solution of (1-4).

If $f \equiv 0$ and $F \equiv 0$, then equation (5) reduces to

$$u^{(j+1)} = A(j) u^{(j)},\ j \in N \tag{8}$$

subject to $u^{(0)} = \tau$. Let $\Phi\,(j, j_0)$ be the fundamental matrix of equation (8) defined by $\Phi\,(j, j_0) = I$ (where I is the identity matrix) for $j \le j_0$, and $\Phi\,(j, j_0) = \prod_{i=j_0}^{j-1} A\,(i)$ for $j \in N$.

We recall from [7] that equation (8) is said to have a dichotomy with projection matrix P if there are positive constants α and λ, and a projection matrix P such that

$$\left\|\Phi\,(j,0)\, P \Phi^{-1}\,(l,0)\right\| \le \alpha \lambda^{j-l};\ j \ge l \ge 0$$

$$\left\|\Phi\,(j,0)\,(I-P) \Phi^{-1}\,(l,0)\right\| \le \alpha \lambda^{l-j};\ l \ge j \ge 0.$$

If $\lambda = 1$, then we have an ordinary dichotomy, while if $\lambda \in (0,1)$, then we have an exponential dichotomy. We want to point out that a dichotomy is a type of conditional stability for non-autonomous difference equations. Thus, it is connected with the concepts of uniform stability and uniform asymptotic stability, see Coppel [2].

The following theorems are known.

THEOREM A ([7]). Suppose equation (8) has an ordinary dichotomy, and for all $(j,u),(j,v) \in N \times R^n$, the inequality

$$\|F(j,u) - F(j,v)\| \le \lambda(j)\|u - v\| \tag{9}$$

holds, where $\sum_{l=0}^{\infty} \|F(l,0)\| < \infty$ and $\sum_{l=0}^{\infty} \lambda(l) < \infty$. Then to each bounded solution of the equation

$$u(j+1) = A(j)u(j) + b(j), \ j \in N, \tag{10}$$

there corresponds a bounded solution of the equation

$$w(j+1) = A(j)w(j) + b(j) + F(j, w(j)), \ j \in N, \tag{11}$$

and conversely, to each bounded solution of (11), there corresponds a bounded solution of (10). Moreover, the difference between the corresponding solutions of equation (10) and equation (11) tends to zero as $j \to \infty$, provided that $\Phi(j,0)P \to 0$ as $j \to \infty$.

THEOREM B ([8]). Suppose (i) equation (8) has an exponential dichotomy, (ii) $F(j,\cdot)$, $j = 0,1,2,...$, are continuous functions such that $\sum_{j=0}^{\infty} \sup_{\|x\|\le\delta}\|F(j,x)\| < \infty$ for some $\delta > 0$, and (iii) $\Phi(j,0)P \to 0$ as $j \to \infty$. Then, to each bounded solution $u(j)$ of equation (10) there corresponds a bounded solution $v(j)$ of equation (11) such that $v(j) = u(j) + o(1)$ as $j \to \infty$.

As immediate consequences, we have the following stability results for our reaction-diffusion equations.

THEOREM 1. Assume the conditions of Theorem A hold. Then to each bounded solution of equation (8), there corresponds a bounded solution $v^{(j)}$ of equation (5), and conversely. Moreover, the difference between the corresponding solutions of equation (8) and equation (5) tends to zero as $j \to \infty$, provided that $\Phi(j,0)P \to 0$ as $j \to \infty$.

THEOREM 2. Under the conditions of Theorem B, to each bounded solution $u^{(j)}$ of equation (8), there corresponds a bounded solution $v^{(j)}$ of equation (5). Moreover, $v^{(j)} = u^{(j)} + o(1)$ as $j \to \infty$.

As another example, let us first quote a result in [1].

THEOREM C ([1], Theorem 5.6.8). Suppose there exists a constant $c > 1$ and a projection matrix P such that for all $j \ge 0$,

$$\sum_{i=0}^{j-1} \left\|\Phi(j,0)P\Phi^{-1}(i+1,0)\right\| + \sum_{i=j}^{\infty} \left\|\Phi(j,0)(I-P)\Phi^{-1}(i+1,0)\right\| \le c. \tag{12}$$

Further, suppose that for all $(j,v) \in N \times R^n$, the function $F(j,v)$ satisfies

$$\|F(j,v)\| \le \alpha\|v\|, \ \alpha < c^{-1}. \tag{13}$$

Then the following hold: i) If $v^{(j)}$ is a bounded solution of equation

$$v^{(j+1)} = A(j)\, v^{(j)} + F\left(j, v^{(j)}\right) \;,\; j \in N \tag{14}$$

such that $\left\| v^{(j)} \right\| \leq \beta$ for all $j \geq 0$. Then $v^{(j)} \to 0$ as $j \to \infty$. ii) There exists a constant $\gamma > 0$ independent of F, such that the solution $v^{(j)}$ determined by $v^{(0)} = \tau$ of equation (5) satisfies

$$\left\| v^{(j)} \right\| \leq (1 - \alpha c)^{-1} \gamma \left\| P\tau \right\|, \; j \in N.$$

As a direct consequence, we have the following result.

THEOREM 3. Assume that $f \equiv 0$ in equation (5). Then, under the hypotheses of Theorem 3, there exists a constant $\gamma > 0$ independent of F, such that the solution $v^{(j)}$, determined by $v^{(0)} = \tau$, of equation (5) satisfies

$$\left\| v^{(j)} \right\| \leq (1 - \alpha c)^{-1} \gamma \left\| P\tau \right\|, \; j \in N.$$

Further, these solutions of equation (5) converge to zero as $j \to \infty$.

Now, we will apply our previous results to equation (8) in the case where $a_j \equiv a$, $b_j \equiv b$ and $c_j \equiv c$, and $ac \neq 0$. Then the matrices $A(j)$ defined by (7) are all equal to a constant matrix A. Furthermore, the eigenvalues of A are given by $\lambda_k(A) = b + 2\sigma\sqrt{ac}\cos(k\pi/(n+1))\,,\; k = 1, 2, ..., n$, where σ is the sign of a, and the corresponding eigenvectors are given by

$$\operatorname{col}\left(\sin\frac{k\pi}{n+1}, \left(\frac{c}{a}\right)^{-1/2} \sin\frac{2k\pi}{n+1}, ..., \left(\frac{c}{a}\right)^{-(n-1)/2} \sin\frac{nk\pi}{n+1} \right), \; k = 1, ..., n.$$

In view of the eigenvalues, the spectral radius $\rho(A)$ of A is equal to $|b| + 2\sqrt{ac}\cos(\pi/(n+1))$ when $ac > 0$, and $\rho(A) = \sqrt{b^2 - 4ac\cos^2(\pi/(n+1))}$ when $ac < 0$ (see [3]). Hence, in view of the inequality $2\sqrt{ac} \leq |a| + |c|$, it follows that $\rho(A) \leq |a| + |b| + |c|$. As a consequence, $\Phi(j, 0) = A^j \to 0$ as $j \to \infty$, when $|a| + |b| + |c| < 1$.

COROLLARY 1. Assume that (i) $A(j) = A$ is a constant matrix with diagonal elements a, superdiagonal elements c and subdiagonal elements a such that $ac \neq 0$ and $|a| + |b| + |c| < 1$, and (ii) for all $(j, u), (j, v) \in N \times R^n$, $\|F(j, u) - F(j, v)\| \leq \lambda(j) \|u - v\|$, where $\sum\limits_{l=0}^{\infty} \|F(l, 0)\| < \infty$ and $\sum\limits_{l=0}^{\infty} \lambda(l) < \infty$. Then, to each bounded solution $u^{(j)}$ of (8), there corresponds a bounded solution $v^{(j)}$ of equation (5), and conversely. Furthermore, $v^{(j)} = u^{(j)} + o(1)$ as $j \to \infty$.

THEOREM 4. Assume that (i) $A(j) = A$ is a constant matrix with diagonal elements a, superdiagonal elements c and subdiagonal elements a such that $ac \neq 0$. If (ii) and (iii) of Theorem B hold and r of the eigenvalues of A are of modulus less than 1, and $n - r$ of its eigenvalues are of modulus greater than 1, then to each bounded solution $u^{(j)}$ of equation (8) there corresponds a bounded solution $v^{(j)}$ of equation (5), such that $v^{(j)} = u^{(j)} + o(1)$ as $j \to \infty$.

Indeed, the conditions imposed on A imply that equation (8) has an exponential dichotomy with projection matrix P, which has rank r. Thus there is a r-dimensional subspace of solutions of equation (8) tending to zero uniformly and exponentially as

$j \to \infty$ [1, Section 5.8]. To each solution of equation (8) belonging to this subspace, there then corresponds a bounded solution of equation (5).

It is well known that a linear system of difference equations with constant coefficient matrix has an exponential dichotomy if, and only if, none of the eigenvalues of its coefficient matrix is of modulus equal to 1. In the case of nonconstant coefficient matrix this result is not valid (see [9]). However, we have the following result when $\{A(j)\}$ is uniformly bounded.

THEOREM 5. Assume that (I) $\|A(j)\| \leq M$ for all $j \in N$, (II) $A(j)$ has l eigenvalues of modulus less than $1-\varepsilon_1 = \beta_1$, and $n-l$ eigenvalues of modulus $1+\varepsilon_1 = \beta_2$, $\varepsilon_1 > 0$, for each $j \in N$, (III) There exists sufficiently small positive $\delta = \delta(\varepsilon_1, M)$, such that for sufficiently large $T \in N$, $\sup_{j \geq T} \|A(j+1) - A(j)\| < \delta$, and (IV) The conditions (ii) and (iii) of Theorem B hold. Then, to each bounded solution $u^{(j)}$ of equation (8) there corresponds a bounded solution $v^{(j)}$ of equation (5), such that $v^{(j)} = u^{(j)} + o(1)$ as $j \to \infty$, provided that $\Phi(j,0)P \to 0$ as $j \to \infty$.

Indeed, conditions (I), (II) and (III) imply that equation (8) has an exponential dichotomy (see [9]). Thus, by (IV) and Theorem B, our result follows.

Although there are other direct consequences of dichotomy theories, we believe the previous examples are sufficient to illustrate how they can be obtained.

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