Applied Mathematics E-Notes, 1(2001), 104-110 ©
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On Upper and Lower D-Continuous Multifunctions *

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Received 12 May 2001

Abstract

In this paper, we define upper and lower D-continuous multifunctions and obtain some of their characterizations and basic properties. Also some relationships between D-continuity and other types of continuity are given.

In 1968, Singal and Singal [9] introduced and investigated the concept of almost continuous functions. In 1981, Heldermann [2] introduced some new regularity axioms and studied the class of D-regular spaces. In 1990, Kohli [3] introduced the concept of D-continuous functions and some properties of D-continuous functions are given by him. The purpose of this paper is to extend this concept and to provide some properties of multifunctions.

A multifunction $F : X \hookrightarrow Y$ is a correspondence from X to 2^Y with $F(x)$ a nonempty subset of Y, for each $x \in X$. Let A be a subset of a topological space (X, τ). A° and $\overline{A}$ denote the interior and closure of A respectively. A subset A of X is called regular open (regular closed) [12] if, and only if, $A = (\overline{A})^\circ$ (respectively $A = \overline{(A^\circ)}$). A space (X, τ) is said to be almost regular [8] if for every regular closed set F and each point x not belonging to F, there exist disjoint open sets U and V containing F and x respectively. For a given topological space (X, τ), the collection of all sets of the form $U^+ = \{T \subseteq X : T \subseteq U\}$ ($U^- = \{T \subseteq X : T \cap U \neq \emptyset\}$) with U in τ, forms a basis (respectively subbasis) for a topology on 2^X (see [5]). This topology is called upper (respectively lower) Vietoris topology and denoted by τ_V^+ (respectively τ_V^-). We will denote such a multifunction by $F : X \hookrightarrow Y$. For a multifunction F, the upper and lower inverse set of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$ respectively, that is, $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. The graph $G(F)$ of the multifunction $F : X \hookrightarrow Y$ is strongly closed [4] if for each $(x, y) \notin G(F)$, there exist open sets U and V containing x and containing y respectively such that $(U \times \overline{V}) \cap G(F) = \emptyset$.

In [7], a multifunction $F : X \hookrightarrow Y$ is said to be (i) upper semi continuous (or u.s.c.) at a point $x \in X$ if for each open set V in Y with $F(x) \subseteq V$, there exists an open set U containing x such that $F(U) \subseteq V$; and (ii) lower semi continuous (or l.s.c.) at a point $x \in X$ if for each open set V in Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$.

In [10], a multifunction $F : X \hookrightarrow Y$ is said to be (i) upper weakly continuous (or u.w.c.) at a point $x \in X$ if for each open set V in Y with $F(x) \subseteq V$, there exists an

*Mathematics Subject Classifications: 54C10, 54C60

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open set U containing x such that $F(U) \subseteq \overline{V}$; and (ii) lower weakly continuous (or l.w.c.) at a point $x \in X$ if for each open set V in Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(z) \cap \overline{V} \neq \emptyset$ for every $z \in U$.

Let $F : X \hookrightarrow Y$ be a multi function. F is said to be upper D-continuous (briefly u.D.c.) at $x_0 \in X$, if for each open F_σ-set V with $F(x_0) \subset V$, there exists an open neighborhood U_{x_0} of x_0 such that the implication $x \in U_{x_0} \Rightarrow F(x) \subset V$ holds. F is said to be lower D-continuous (briefly l.D.c.) at $x_0 \in X$, if for each open F_σ-set V with $F(x_0) \cap V \neq \emptyset$ there exists an open neighborhood U_{x_0} of x_0 such that the implication $x \in U_{x_0} \Rightarrow F(x_0) \cap V \neq \emptyset$ holds. F is said to be D-continuous (briefly D.c.) at $x_0 \in X$, if it is both u.D.c. and l.D.c. at $x_0 \in X$. Finally, F is said to be u.D.c. (l.D.c. or D.c.) on X, if it has this property at each point $x \in X$.

THEOREM 1. Let X and Y be topological spaces. For a multifunction $F : X \hookrightarrow Y$, the following statements are equivalent: (a) F is u.D.c. (l.D.c.). (b) For every open F_σ-set V, $F^+(V)$ $(F^-(V))$ is an open set in X. (c) For every closed G_δ-set K, $F^-(K)$ $(F^+(K))$ is closed in X. (d) For each $x \in X$ and each net $\{x_\alpha\}_{\alpha\in\triangle}$ which converges to x, if V is an open F_σ-set with $F(x) \subset V$ $(F(x) \cap V \neq \emptyset)$, then there is an $\alpha_o \in \triangle$ such that for every $\alpha \geq \alpha_o$, $F(x_\alpha) \subset V$ (respectively $F(x_\alpha) \cap V \neq \emptyset$).

PROOF. (a)$\Rightarrow$(b): If V is an open F_σ-set of Y, then for each $x \in F^+(V)$, $F(x) \subset V$ and hence there is an open neighborhood U of x such that $\bigcup_{x\in U} F(x) \subset V$. Thus $F^+(V)$, being a neighborhood of each of its points, is open.

(b)$\Rightarrow$(c): Let K be a closed G_δ-set of Y. Then $Y \backslash K$ is an open F_σ-set and $F^+(Y\backslash K) = X\backslash F^-(K)$ is open. Thus $F^-(K)$ is closed in X.

(c)$\Rightarrow$(b): Let V be an open F_σ-set. Then $Y\backslash V$ is a closed G_δ-set and $F^-(Y\backslash V) = X\backslash F^+(V)$ is closed in X. Thus $F^+(V)$ is an open set in X.

(b)$\Rightarrow$(a): Let $x \in X$ and let V be an open F_σ-set containing $F(x)$. Then $F^+(V)$ is an open set containing x and $F(F^+(V)) \subset V$. Thus F is u.D.c. at x.

(b)$\Rightarrow$(d): Let $\{x_\alpha\}_{\alpha\in\triangle}$ be a net in X which converges to x and let V be an open F_σ-set containing $F(x)$. Then $F^+(V)$ is an open set containing x. Since $\{x_\alpha\}$ converges to x, there is an $\alpha_o \in \triangle$ such that for every $\alpha \geq \alpha_o$, $x_\alpha \in F^+(V)$. Thus for every $\alpha \geq \alpha_o$, $F(x_\alpha) \subset V$.

(d)$\Rightarrow$(b): Let V be an open F_σ-set of Y. To show that $F^+(V)$ is open, assume to the contrary that there is $x \in F^+(V)$ such that $F^+(V)$ is not neighborhood of x. Then there is a net $\{x_\alpha\}$ in X which converges to x and misses $F^+(V)$ frequently. Then $\{F(x_\alpha)\}$ misses V frequently, which is a contradiction.

The proof for the case where F is l.D.c. is similarly proved. The proof is complete.

As an example, let $X = \{0, 1\}, \tau = \{\emptyset, X, \{1\}\}$ and $Y = \{a, b, c\}, \vartheta = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. If we define $F : (X, \tau) \hookrightarrow (Y, \vartheta)$ with $F(0) = \{a\}$, $F(1) = \{b\}$, then F is u.D.c. (l.D.c.) but not u.s.c. (respectively l.s.c.) at $x_0 = 0$.

THEOREM 2. Let $F : (X, \tau) \hookrightarrow (Y, \vartheta)$ be a multifunction. If F is u.s.c. (l.s.c.), then F is u.D.c. (respectively l.D.c.).

PROOF. Suppose that F is u.s.c. (l.s.c.) at $x_0 \in X$. If V is an open F_σ-set in Y with $F(x_0) \subset V$ (respectively $F(x_0) \cap V \neq \emptyset$) then $F^+(V)$ (respectively $F^-(V)$) is an open set in X. Thus F is u.D.c. (respectively l.D.c.) at $x_0 \in X$. The proof is complete.

THEOREM 3. Let X be a topological space and let Y be a D-regular space [2]. If F is point compact and u.D.c. (l.D.c.), then F is u.s.c. (respectively l.s.c.).

PROOF. Suppose that V is an open set in Y with $F(x_0) \subset V$. Since Y is D-regular for every $y \in F(x_0)$, there is an open F_σ-set G_y such that $y \in G_y$ and $\overline{G_y} \subset V$. If we define the family $\Sigma = \{G_y : y \in F(x_0)\}$, then it is an open cover of $F(x_0)$ and $F(x_0) \subset \bigcup G_y \subset V$. Since F is point compact and for each $y \in F(x_0)$, G_y is an open F_σ-set, there is a finite subcover of $F(x_0)$ such that $F(x_0) \subset \bigcup_{i=1}^{n} G_{y_i} \subset V$, and if we take $\bigcup G_{y_i} = G$, then it is an open F_σ-set. Also since F is u.D.c., for $F(x_0) \subset G$, there is an open set U_{x_0} such that $x_0 \in U_{x_0}$ and the implication $x \in U_{x_0} \Rightarrow F(x) \subset G \subset V$ holds. Thus F is u.s.c. at $x_0 \in X$. The other case is similarly proved. The proof is complete.

THEOREM 4. Let X and Y be topological spaces and let $F : X \hookrightarrow Y$ be a multifunction. If the graph function $G_F : X \to X \times Y$ is u.D.c. (l.D.c.), then F is u.D.c. (respectively l.D.c.).

PROOF. Suppose G_F is u.D.c. at $x_0 \in X$. Let V be an open F_σ-set with $F(x_0) \subset V$. Then $G_F(x) \subset X \times V$ and $X \times V$ is an open F_σ-set in $X \times Y$. Since G_F is u.D.c., there is an open set U with $x_0 \in U$ such that $G_F(U) \subset X \times V$. From [6], $U \subset G_F^+(X \times V) = X \cap F^+(V) = F^+(V)$ and so F is u.D.c. at $x_0 \in X$. Suppose G_F is l.D.c. at $x_0 \in X$. Let V be an open F_σ-set with $F(x_0) \cap V \neq \emptyset$. Then

$$G_F(x_0) \cap (X \times V) = (\{x_0\} \times F(x_0)) \cap (X \times V) = \{x_0\} \times (F(x_0) \cap V) \neq \emptyset$$

and $X \times V$ is an open F_σ-set in $X \times Y$. Since G_F is l.D.c., there is an open set U with $x_0 \in U$ such that $U \subset G_F^-(X \times V)$. From [6], $U \subset G_F^-(X \times V) = X \cap F^-(V) = F^-(V)$ and so F is l.D.c. at $x_0 \in X$. The proof is complete.

Let (X, τ) be a topological space and let $\{K_\beta : \beta \in \Delta\}$ be a closed cover of X. If for any subset F of X and for the collection $\{K_\beta : \beta \in \Delta\}$ the equation $\bigcup \overline{K_\beta \bigcap F} = \overline{\bigcup (K_\beta \bigcap F)}$ holds, then the collection is called a hereditarily closure preserving closed cover of X [3].

THEOREM 5. Let X and Y be topological spaces. Then the following statements are true: (a) If $F : X \hookrightarrow Y$ is u.D.c.(l.D.c.), then the restriction multifunction $F|_A : A \hookrightarrow Y$ is u.D.c. (l.D.c.). (b) Let $F : X \hookrightarrow Y$ be a multifunction. If $\{U_\alpha : \alpha \in \triangle\}$ is an open cover of X and for each α, $F_\alpha = F|_{U_\alpha}$ is u.D.c.(l.D.c.), then F is u.D.c. (l.D.c.). (c) Let $F : X \hookrightarrow Y$ be a multifunction. If $\{K_\beta : \beta \in \triangle\}$ is a hereditarily closure preserving closed cover of X and for each $\beta \in \Delta$, $F_\beta = F|_{K_\beta}$ is u.D.c. (l.D.c.), then F is u.D.c. (respectively l.D.c.).

PROOF. (a) Let V be an open F_σ-set in A with $F|_A(x_0) \subset V$ $(F|_A(x_0) \cap V \neq \emptyset)$. Since F is u.D.c. (respectively l.D.c.) and $F|_A(x_0) = F(x_0) \subset V$ (respectively $F|_A(x_0) = F(x_0) \cap V \neq \emptyset$), there is an open neighborhood of x_0 such that the implication $x \in U \Rightarrow F(x) \subset V$ (respectively $F(x) \cap V \neq \emptyset$) holds. If we take $U_1 = U \cap A$, then U_1 is an open neighborhood of x_0 in A and $F|_A(U_1) \subset V$ (respectively $U_1 \subset F^-(V)$). Thus $F|_A$ is u.D.c. (respectively l.D.c.) at $x_0 \in X$.

(b) Let V be an open F_σ-set of Y. Then $F^+(V) = \bigcup\{F_\alpha^+(V) : \alpha \in \triangle\}$ $(F^-(V) = \bigcup\{F_\alpha^-(V) : \alpha \in \triangle\})$ and since for each $\alpha \in \triangle$, F_α is u.D.c.(l.D.c.) and $F_\alpha^+(V)$

(respectively $F_\alpha^-(V)$) is an open set in U_α and hence in X. Thus $F^+(V)$ (respectively $F^-(V)$) being the union of open sets is open.

(c) Let K be a closed G_δ-set of Y. Then $F^+(K) = \cup\{F_\beta^+(K) : \beta \in \triangle\}$ ($F^-(K) = \cup\{F_\beta^-(K) : \beta \in \triangle\}$) and since for each $\alpha \in \triangle$, F_β is u.D.c. (respectively l.D.c.) and $F_\beta^+(K)$ (respectively $F_\beta^-(K)$) is closed in K_β and hence in X. Also since $\{K_\beta : \beta \in \triangle\}$ is a hereditarily closure preserving closed cover of X, the collection $\{F_\beta^+(K) : \beta \in \triangle\}$ (respectively $\{F_\beta^-(K) : \beta \in \triangle\}$) is a closure preserving collection of closed sets. Thus $F^+(K)$ (respectively $F^-(K)$) is closed.

The proof is complete.

THEOREM 6. Let $F : X \hookrightarrow Y$ and $G : Y \hookrightarrow Z$ be two multifunctions. If F is u.s.c. (l.s.c.) and $G : Y \hookrightarrow Z$ is u.D.c. (respectively l.D.c.), then $G \circ F : X \hookrightarrow Z$ is a u.D.c. (respectively l.D.c.)

PROOF. Let V be an open F_σ-set in Z. Since G is u.D.c. (l.D.c.), $G^+(V)$ (respectively $G^-(V)$) is an open set in Y. Also since F is u.s.c. (respectively l.s.c.), $F^+(G^+(V)) = (G \circ F)^+(V)$ (respectively $F^-(G^-(V)) = (G \circ F)^-(V)$) is an open set in X. Thus $G \circ F$ is u.D.c. (respectively l.D.c.) The proof is complete.

THEOREM 7. Let $F : (X, \tau) \hookrightarrow (Y, \vartheta)$ be a multifunction and let Y be extremally disconnected space. If F is l.D.c. (u.D.c.), then F is l.w.c. (respectively u.w.c.).

PROOF. Let V be an open set of Y. Since Y is extremally disconnected, $\overline{V}$ is an open set of Y and so $\overline{V}$ is an open F_σ-set of Y. Also since F is u.D.c. (l.D.c.), $F^+(\overline{V})$ (respectively $F^-(\overline{V})$) is open in X. Thus F is u.w.c. (respectively l.w.c.). The proof is complete.

THEOREM 8. Let $F : (X, \tau) \hookrightarrow (Y, \vartheta)$ be a multifunction and let Y be a regular space. If F is l.w.c., then F is l.D.c.

PROOF. Let F be l.w.c. at $x_0 \in X$ and let V be an open F_σ-set in Y with $F(x_0) \cap V \neq \emptyset$. Since Y is a regular space, for each $y \in F(x_0) \cap V$, there is an open set G_y such that $y \in G_y \subset \overline{G_y} \subset V$. Thus $F(x_0) \cap G_y \neq \emptyset$. Also since F is l.w.c., there is an open neighborhood U of x_0 such that the implication $x \in U \Rightarrow F(x) \cap \overline{G_y} \neq \emptyset$ holds. Hence $F(U) \cap \overline{G_y} \subset F(U) \cap V \neq \emptyset$ and F is l.D.c. at $x_0 \in X$. The proof is complete.

THEOREM 9. Let $F : (X, \tau) \hookrightarrow (Y, \vartheta)$ be a multifunction and let Y be a regular space. If the family $\overline{\Sigma} = \{\overline{T} : T \in \vartheta\}$ has the local finite property and F is u.w.c., then F is u.D.c.

PROOF. Let V be an open F_σ-set in Y with $F(x_0) \subset V$. Since Y is regular, for each $y \in F(x_0)$, there is an open set G_y such that $y \in G_y \subset \overline{G_y} \subset V$. So $F(x_0) \subset \bigcup_{y \in F(x_0)} G_y \subset \bigcup \overline{G_y} \subset V$. If we take $V_1 = \bigcup_{y \in F(x_0)} G_y$, then since F is u.w.c. at $x_0 \in X$, for $F(x_0) \subset V_1$, there is an open neighborhood U of x_0 such that $F(U) \subset \overline{V_1}$. Also since $\overline{\Sigma} = \{\overline{G_y} | G_y \in \vartheta\}$ has the local finite property $\overline{V_1} = \overline{\cup G_y} \subset \cup \overline{\overline{G_y}} = \cup \overline{G_y} \subset V$, $F^+(V)$ is open in X. Thus F is u.D.c. at $x_0 \in X$. The proof is complete.

Now we give a multifunction F which is u.D.c. (l.D.c.) but not u.w.c. (respectively l.w.c.). Let $X = \{0, 1\}, \tau = \{\emptyset, X, \{1\}\}$ and $Y = \{a, b, c\}$, $\vartheta = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. If we define $F : (X, \tau) \hookrightarrow (Y, \vartheta)$ with $F(0) = \{a\}$, $F(1) = \{b\}$, then F is u.D.c. (l.D.c.) but not u.w.c. (respectively l.w.c.) at $x_0 = 0$.

THEOREM 10. Let $F : X \hookrightarrow Y$ be a quotient multifunction. Then a multifunction $G : Y \hookrightarrow Z$ is u.D.c. if, and only if, $G \circ F$ is u.D.c.

PROOF. Since quotient map is u.D.c., from Theorem 6, $G \circ F$ is u.D.c. Conversely, let V be an open F_σ-set of Z. Then $(G \circ F)^+(V) = F^+(G^+(V))$ is open in X. Since F is a quotient map, $G^+(V)$ is open in Y, and so G is u.D.c. The proof is complete.

THEOREM 11. Suppose for each $\alpha \in \triangle$, $F_\alpha : X_\alpha \hookrightarrow Y_\alpha$ is a multifunction and let $F : \Pi X_\alpha \hookrightarrow \Pi Y_\alpha$ be a multifunction defined by $F((x_\alpha)) = (F_\alpha(x_\alpha))$ for each point (x_α) in ΠX_α. If F is u.D.c. (l.D.c.), then for each $\alpha \in \triangle$, F_α is u.D.c. (respectively l.D.c.).

PROOF. Let G_{α_o} be a closed G_δ-set of Y_{α_o}. Then $G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} Y_\alpha$ is a closed G_δ-set of ΠY_α. Since F is u.D.c. (l.D.c.), $F^-(G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} Y_\alpha) = F^-(G_{\alpha_o}) \times \prod_{\alpha \neq \alpha_o} X_\alpha$ (respectively $F^+(G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} Y_\alpha) = F^+(G_{\alpha_o}) \times \prod_{\alpha \neq \alpha_o} X_\alpha$) is closed in ΠX_α. Consequently $F^-_{\alpha_o}(G_{\alpha_o})$ (resepctively $F^+(G_{\alpha_o})$) is closed in X_{α_o} and so G_{α_o} is u.D.c. (respectively l.D.c.). The proof is complete.

THEOREM 12. Let $F : X \to \Pi X_\alpha$ be a multifunction into a product space. If F is u.D.C. (l.D.c), then each $\alpha \in \triangle$, $P_\alpha \circ F$ is u.D.c. (respectively l.D.c.).

PROOF. Let G_{α_o} be an open F_σ-set of X_{α_o}. Then, $(P_{\alpha_o} \circ F)^+(G_{\alpha_o}) = F^+(P^+_{\alpha_o}(G_{\alpha_o}))$ $= F^+(G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$ (respectively $(P_{\alpha_o} \circ F)^-(G_{\alpha_o}) = F^-(P^-_{\alpha_o}(G_{\alpha_o})) = F^-(G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$). Since F is u.D.c. (respectively l.D.c.) and since $G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha$ is an open F_σ-set, $F^+(G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$ (respectively $F^-(G_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$) is open in X. Thus $P_\alpha \circ F$ is u.D.c. (respectively l.D.c.). The proof is complete.

THEOREM 13. The set of all points of X for which $F : X \hookrightarrow Y$ is not u.D.c. is identical to the union of the boundaries of the inverse image of open F_σ-sets of Y.

PROOF. Suppose F is not u.D.c. at a point $x \in X$. Then there exists an open F_σ-set V containing $F(x)$ such that for every open set U containing x, $F(U) \not\subseteq V$. Thus for every open set U containing x, $U \cap (X \backslash F^+(V)) \neq \emptyset$. Therefore, x cannot be an interior point of $F^+(V)$. Hence x is a boundary point of $F^+(V)$. Now, let x belong to the boundary of $F^+(V)$ for some open F_σ-set of Y (that is $x \in F^+(V)$ but $x \notin [F^+(V)]^o$). Then $F(x) \subset V$. If F is u.D.c. at x, then there is an open set U containing x such that $F(U) \subset V$. Thus $x \in U \subset F^+(V)$, and so x is an interior point of $F^+(V)$. This is contrary to the fact that x belongs to the boundary of $F^+(V)$. Hence F is not u.D.c. at x. The proof is complete.

THEOREM 14. A u.D.c. image of a connected space is connected for a multifunction F.

PROOF. Let $F : X \hookrightarrow Y$ be a u.D.c. multifunction from a connected space X onto a space Y. Suppose Y is not connected and let $Y = A \cup B$ be a partition of Y. Then both A and B are open and closed subsets of Y. Since F is u.D.c., $F^+(A)$ and $F^+(B)$ are open subsets of X. In view of the fact that $F^+(A)$ and $F^+(B)$ are disjoint, $X = F^+(A) \cup F^+(B)$ is a partition of X. This is contrary to the connectedness of X. The proof is complete.

THEOREM 15. Let $F : X \to Y$ be u.D.c. If every pair of distinct points of Y are contained in disjoint open sets such that one of them may be chosen to be an F_σ-set. Then F has strongly closed graph.

PROOF. Suppose $(x, y) \notin G(F)$. Then $y \notin F(x)$. By the hypothesis on Y, there are disjoint open sets V_1 and V_2 containing $F(x)$ and y respectively, and V_1 is an F_σ-set. Since F is u.D.c., $F^+(V_1)$ is open. Thus $\underline{U} = F^+(V_1)$ is an open set containing x and $F(U) \subset V_1 \subset Y \backslash \overline{V_2}$. Consequently, $U \times \overline{V}$ does not contain any points of $G(F)$, and so $G(F)$ is strongly closed in $X \times Y$. The proof is complete.

Let (X, τ) be a topological space. Then X is said to be a D-normal space if for every distinct closed subsets K and F of X, there are two open F_σ-sets U and V such that $K \subseteq U$, $F \subseteq V$ and $U \cap V = \emptyset$.

THEOREM 16. Let F and G be u.D.c. and point closed multifunctions from a space X to a D-normal space Y. Then the set $A = \{x | F(x) \cap G(x) \neq \emptyset\}$ is closed in X.

PROOF. Let $x \in X \backslash A$. Then $F(x) \cap G(x) = \emptyset$ and so by the hypothesis on Y, there are disjoint open F_σ-sets U and V containing $F(x)$ and $G(x)$ respectively. Since F and G are u.D.c., the sets $F^+(U)$ and $G^+(V)$ are open and contain x. Let $H = F^+(U) \cap G^+(V)$. Then H is an open set containing x and $H \cap A = \emptyset$. Thus A is closed in X. The proof is complete.

As a corollary, the set of fixed points of a u.D.c. self map of a D-normal space is closed.

THEOREM 17. Let $F : X \hookrightarrow Y$ be u.D.c., $F(x) \neq F(y)$ for each distinct pair $x, y \in X$ and point closed from a topological space X to a D-normal space Y. Then X is Hausdorff.

PROOF. Let x and y be any two distinct points in X. Then $F(x) \cap F(y) = \emptyset$. Since Y is D-normal, there are disjoint open F_σ-sets U and V containing $F(x)$ and $F(y)$ respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint open sets containing x and y respectively. Thus X is Hausdorff. The proof is complete.

Let (X, τ) be a topological space. Since the intersection of two open F_σ-sets is an open F_σ-set, the collection of all open F_σ-subsets of (X, τ) is a base for a topology τ^* on X. It is immediate that a space (X, τ) is D-regular if, and only if, $\tau^* = \tau$ [3]. The following example shows that a D-regular space may not be first countable.

EXAMPLE. Let X be the set of positive integers. Let $N(n, E)$ denote the number of integers in a set $E \subset X$ which are less than or equal to n. We describe the Appert's topology on X by declaring open any set which excludes the integer 1, or any set E containing 1 for which $\lim_{n\to\infty} N(n, E) = 1$. Then the Appert space is completely normal, completely regular and hence from [2] D-regular. However, it is not first countable.

THEOREM 18. Let (X, τ) be a topological space. Then the following statements are equivalent: (a) (X, τ) is a D-regular space. (b) Every u.D.c. and point compact multifunction F from a topological space Y into (X, τ) is u.s.c. (c) The identity mapping I_X from (X, τ^*) onto (X, τ) is continuous.

PROOF. (a)$\Rightarrow$(b): Let $F : (Y, \vartheta) \hookrightarrow (X, \tau)$ be a u.D.c. multifunction and let V be an open set in X with $F(x) \subset V$. Then since F is point compact and (X, τ) is D-regular, there is an open F_σ-set V_1 such that $F(x) \subset V_1 \subset V$. Since F is u.D.c.,there exits an open set U containing x such that $F(U) \subset V_1 \subset V$. Thus F is u.s.c. at x.

(b)⇒(c): Let $I_X : (X, \tau^*) \hookrightarrow (X, \tau)$ be the identity mapping. Let $F(x) \subset V$ and V be an open F_σ-set in X. Then $I_X^+(V) = V$ is an open F_σ-set and $I_X^+(V) \in \tau^*$. Thus I_X is u.D.c. at x. From (b), I_X is u.s.c. at x.

(c)⇒(a): Let V an open set in (X, τ) with $x \in V$. From (c), $I_X : (X, \tau^*) \hookrightarrow (X, \tau)$ is u.s.c. and, for $I_X(x) = x \subset V$, there is an open F_σ-set U in (X, τ^*) such that $I_X(U) \subset V$ and $x \in U = I_X(U) \subset V$. Thus (X, τ) is D-regular space. The proof is complete.

In [1], a space X is said to be sequential if a subset U of X is open if, and only if, every sequence converging to a point in U is eventually in U.

THEOREM 19. Let $F : X \to Y$ be a u.D. continuous function from a sequential space X into a countably compact Hausdorff space Y. If Y has a neighborhood base of closed G_δ-sets then F is upper continuous.

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