**Applied Mathematics E-Notes, 1(2001), 97-103 ©**
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# Optimal Boundary Control of a Nonlinear Diffusion Equation *†

Jing-xue Yin‡, Wen-mei Huang§

Received 26 April 2001

**Abstract**

We discuss the optimal boundary control governed by a nonlinear diffusion equation, and establish the existence and stability of the optimal control.

In this paper, we are concerned with the optimal boundary control governed by the following nonlinear heat conduction equation

$$\frac{\partial u}{\partial t} + \mathrm{div}\vec{J} + \lambda u = 0,\ (x,t) \in Q_T = \Theta \times (0,T), \tag{1}$$

subject to the initial value condition

$$u(x,0) = u_0(x),\ x \in \Theta, \tag{2}$$

and the boundary value condition

$$\vec{J} \cdot \vec{n} = -h(u-\alpha),\ (x,t) \in \partial\Theta \times (0,T), \tag{3}$$

where $\vec{J} = -|\nabla u|^{p-2}\nabla u$ is the heat flux, $p > 2$, $\Theta \subset R^N$ is a bounded domain with smooth boundary, $\vec{n}$ denotes the outward normal to the boundary $\partial\Theta$, $\lambda$ is a positive constant, $u_0(x)$ is a nonnegative bounded function and $h$ is the heat transfer coefficient which we take as our control. The cost functional is chosen as

$$J(h) = \frac{1}{2}\left\{\beta \int\int_{Q_T} (u - Z_d)^2 dxdt + \gamma \int_{\partial\Theta\times(0,T)} h^2 dsdt\right\},\ h \in U_M, \tag{4}$$

where $U_M$ is the admissible set, namely,

$$U_M = \{h|\ 0 \le h \le M, h \in L^\infty(\partial\Theta \times (0,T)), h \equiv 0 \text{ on } \partial\Theta \setminus \Gamma\}.$$

Here $\Gamma$ is a partial boundary of $\Theta$ with $\mathrm{mes}\,\Gamma > 0$, $Z_d$ is the desired temperature distribution, the coefficient $\beta$ and $\gamma$ are per unit costs associated with failing to achieve the

*Mathematics Subject Classifications: 35K99

†Supported by the Ministry of Science and Technology of China

‡Department of Mathematics, Jilin University, Changchu, Jilin 130012, P. R. China

§Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, P. R. China

desired temperature distribution and with imposing a heat transfer coefficient different from zero. According to different requests, $\beta$ and $\gamma$ can take different values. Then the optimal control problem of the temperature system is

$$\text{To find a } \ h^* \in U_M, \ s.t. \ J(h^*) = \inf_{h \in U_M} J(h). \tag{5}$$

Thus, the state equation (1) with the initial and boundary value condition (2), (3), together with the cost functional (4), and question (5) compose a mathematical model of the optimal boundary control of the heat transfer system. If $u > \alpha$, the system releases heat, while if $u < \alpha$, the system absorbs heat.

It was Lenhart and Wilson [1] who first studied the optimal control for such kind of system with $p = 2$, established the existence, uniqueness and stability of the optimal control, and proved that the optimal control can be formulated by $h^* = qu$, where $q$ is a nonnegative function independent of $u$. Later on, similar results were obtained by several authors, see for example [5], [6] and [7]. It is well known that for the classical heat conduction equation, i.e., the case where $p = 2$, the speed of propagation is infinite. However, for the case where $p > 2$, the state equation becomes the $p$-Laplace equation, whose solutions possess the property of finite speed of propagation of disturbances. Hence, it is more natural to consider the heat transfer system governed by the $p$-Laplace equation.

Due to the degeneracy of our equations, we are only interested in weak solutions to our problem (1)–(3) in the following sense: A nonnegative function $u \in C(0,T;L^2(\Theta)) \bigcap L^p(0,T;W^{1,p}(\Theta))$ is said to be a weak solution of the problem (1)–(3) if the following integral equality holds

$$\begin{aligned}
& \int_\Theta u(x,\tau)\varphi(x,\tau)dx + \int\int_{Q_\tau} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dxdt \\
& + \int_{\partial\Theta\times(0,\tau)} h(s,t)(u(s,t)-\alpha)\varphi(s,t)dsdt + \int\int_{Q_\tau} \lambda u(x,t)\varphi(x,t)dxdt \\
& - \int_\Theta u_0(x)\varphi(x,0)dx - \int\int_{Q_\tau} u(x,t)\varphi_t(x,t)dxdt \\
= \ & 0,
\end{aligned} \tag{6}$$

where $\varphi$ is an arbitrary test function in $C^1(\overline{Q}_T))$, $\tau \in (0,T)$ and $Q_\tau = \Theta \times (0,\tau)$.

The main results of this paper are as follows.

THEOREM 1. Assume that $Z_d \in L^2(Q_T)$, $u_0 \in C^1(\overline{\Theta})$ and satisfies the following compatibility condition

$$|\nabla u_0|^{p-2}\nabla u_0 \cdot \vec{n} = -hu_0, \quad x \in \partial\Theta.$$

Then there exists an optimal control $h^* \in U_M$ which minimizes the cost functional $J(h)$ defined by (4).

As for the stability of the optimal control $h$, we have

THEOREM 2. Suppose $u = u(h)$ and $u_\varepsilon = u(h+\varepsilon l)$ are solutions of problem (1)–(3), corresponding to $h \in U_M$, $h + \varepsilon l \in U_M$ respectively. Then

$$\|u_\varepsilon - u\|_{L^2(Q_T)} = O(\varepsilon), \ \varepsilon \to 0.$$

We need several lemmas which will be used in the proof of our main results.

LEMMA 1. For any $u$, $v \in L^p(0,T;W^{1,p}(\Theta))$, the following inequality holds:

$$\int\int_{Q_T} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v)dxdt \geq 0.$$

Indeed, the above inequality follows easily from the convexity of $\Phi(X) = |X|^p$.

LEMMA 2 (Feng [3]). Let $B_0$, $B$ and $B_1$ be reflexive Banach spaces which satisfy $B_0 \overset{C}{\hookrightarrow} B \hookrightarrow B_1$, where $\hookrightarrow$ denotes imbedding and $\overset{C}{\hookrightarrow}$ denotes compact imbedding. Then we have

$$L^{r_0}(0,T;B_0) \cap \{\phi | \phi_t \in L^{r_1}(0,T;B_1)\} \overset{C}{\hookrightarrow} L^{r_0}(0,T;B);$$

$$L^{\infty}(0,T;B_0) \cap \{\phi | \phi_t \in L^{r_2}(0,T;B_1)\} \overset{C}{\hookrightarrow} C(0,T;B);$$

$$L^{r_0}(0,T;B) \cap \{\phi | \phi_t \in L^{r_0}(0,T;B)\} \overset{C}{\hookrightarrow} C(0,T;B).$$

Here $1 \leq r_0,\ r_1 \leq \infty$, and $1 < r_2 \leq \infty$.

LEMMA 3. Under the assumption in Theorem 1, there exists a unique solution $u_h$ of the problem (1)–(3) for any $h \in U_M$.

PROOF. For the existence of solution $u_h$, we refer to [2]. The uniqueness can also be proved in a rather standard way as follows. Let $u_1$, $u_2$ be solutions of the problem (1)–(3). From the definition of a solution of (1)-(3), we have

$$\begin{aligned} & \int\int_{Q_\tau} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla\varphi dxdt - \int\int_{Q_\tau} (u_1-u_2)\varphi_t dxdt \\ & + \int_{\partial\Theta\times(0,\tau)} h(u_1-u_2)\varphi dsdt + \int\int_{Q_\tau} \lambda(u_1-u_2)\varphi dxdt \\ = \ & \int_\Theta (u_2(x,\tau) - u_1(x,\tau))\varphi(x,\tau)dx, \end{aligned}$$

for all $\tau \in (0,T)$. Choosing $\varphi = u_1 - u_2$, we obtain

$$\begin{aligned} & \int\int_{Q_\tau} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2) \cdot \nabla(u_1-u_2)dxdt \\ & + \int_{\partial\Theta\times(0,\tau)} h(u_1-u_2)^2 dsdt + \int\int_{Q_\tau} \lambda(u_1-u_2)^2 dxdt \\ = \ & \int\int_{Q_\tau} (u_1-u_2)(u_1-u_2)_t dxdt - \int_\Theta (u_1(x,\tau) - u_2(x,\tau))^2 dx, \end{aligned}$$

for all $\tau \in (0,T)$. Noticing that

$$\int\int_{Q_\tau} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2) \cdot \nabla(u_1-u_2) \geq 0,$$

we get

$$\int_{\partial\Theta\times(0,\tau)} h(u_1-u_2)^2 dsdt + \int\int_{Q_\tau} \lambda(u_1-u_2)^2 dxdt \le -\frac{1}{2}\int_\Theta (u_1-u_2)^2(x,\tau)dx \le 0,$$

which implies that $u_1(x,t) = u_2(x,t)$, a.e. $(x,t) \in Q_T$.

We are now in a position to present and prove our main results.

First consider Theorem 1. Without loss of generality, we assume that $\alpha = 0$. Let $\{h_n\}$ be a sequence in $U_M$ for which

$$\lim_{n\to\infty} J(h_n) = \inf_{h\in U_M} J(h).$$

By Lemma 3, for each $n$, we can define $u_n = u(h_n)$ as the solution of the problem (1)–(3) with $h = h_n$, namely, $u_n$ satisfies the following integral equality

$$\begin{aligned} & \int_\Theta u_n(x,\tau)\varphi(x,\tau)dx + \int\int_{Q_\tau} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi dxdt \\ & + \int_{\partial\Theta\times(0,\tau)} h_n(s,t)u_n(s,t)\varphi(s,t)dsdt + \int\int_{Q_\tau} \lambda u_n(x,t)\varphi(x,t)dxdt \\ & - \int_\Theta u_0(x)\varphi(x,0)dx - \int\int_{Q_\tau} u_n(x,t)\varphi_t(x,t)dxdt \\ = & \ 0 \end{aligned} \tag{7}$$

Using the regularity results in [4] for the $p$-Laplace equation, we see that $u_{nt} \in L^2(Q_T)$ and satisfies the following estimate

$$\int\int_{Q_\tau} |u_{nt}(x,t)|^2 dxdt \le C,\ \tau \in (0,T), \tag{8}$$

where $C$ is a positive constant independent of $n$. By virtue of this and the definition of weak solutions, after an approximation process, we may always choose $u_n$, or $\psi u_n$ for some smooth function $\psi$, as a test function in (7). First, take $u_n$ as the test function in (7) and obtain

$$\begin{aligned} & \frac{1}{2}\int_\Theta \{u_n^2(x,\tau) - u_0^2(x)\}dx + \int\int_{Q_\tau} |\nabla u_n|^p dxdt \\ & + \int_0^\tau \int_{\partial\Theta} h_n u_n^2 dxdt + \int\int_{Q_\tau} \lambda u_n^2 dxdt \\ = & \ 0. \end{aligned} \tag{9}$$

Noticing that the last three terms in (8) are nonnegative, we have

$$\int_\Theta u_n^2(x,\tau)dx \le \int_\Theta u_0^2(x)dx,\ \tau \in (0,T), \tag{10}$$

$$\int\int_{Q_\tau} |\nabla u_n(x,t)|^p dxdt \le \frac{1}{2}\int_\Theta u_0^2(x)dx,\ \tau \in (0,T). \tag{11}$$

From Lemma 2, there exists a subsequence of $\{u_n\}$, denoted also by $\{u_n\}$, $u^* \in C(0,T;L^2(\Theta)) \cap L^p(0,T;W^{1,p}(\Theta))$ and $w \in L^{p/(p-1)}(Q_T)$, which satisfy $u_n \to u^*$ a.e. $Q_T$, $u_{nt} \rightharpoonup u_t^*$ in $L^2(Q_T)$, $\nabla u_n \rightharpoonup \nabla u^*$ in $L^p(Q_T)$, and $|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup w$ in $L^{p/(p-1)}(Q_T)$. We claim that $w = |\nabla u^*|^{p-2}\nabla u^*$. Indeed, in view of

$$\int\int_{Q_T} u^*\varphi_t dxdt - \int\int_{Q_T} w_i\varphi_{x_i} dxdt - \int\int_{Q_T} \lambda u^*\varphi dxdt = 0, \tag{12}$$

for $\varphi \in C_0^\infty(Q_T)$, we need only to show the following

$$\int\int_{Q_T} |\nabla u^*|^{p-2}\nabla u^* \cdot \nabla\varphi dxdt = \int\int_{Q_T} w_i\varphi_{x_i} dxdt,\ \varphi \in C_0^\infty(Q_T). \tag{13}$$

Actually, for any $v \in L^p(0,T;W^{1,p}(\Theta)) \cap C(0,T;L^2(\Theta))$, $\psi \in C_0^\infty(Q_T)$, $0 \le \psi \le 1$, supp$\psi \subset \Theta$, we have

$$\int\int_{Q_T} \psi(|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u_n - v)dxdt \ge 0. \tag{14}$$

Choosing $\varphi = \psi u_n$ in (7), we obtain

$$\begin{aligned} &\int\int_{Q_T} \lambda\psi u_n^2 dxdt + \int\int_{Q_T} \psi|\nabla u_n|^p dxdt \\ = &\ \frac{1}{2}\int\int_{Q_T} \psi_t u_n^2 dxdt - \int\int_{Q_T} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi dxdt. \end{aligned}$$

It follows from (14) that

$$\begin{aligned} &\frac{1}{2}\int\int_{Q_T} \psi_t u_n^2 dxdt - \int\int_{Q_T} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi dxdt \\ &- \int\int_{Q_T} \psi|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dxdt - \int\int_{Q_T} \lambda\psi u_n^2 dxdt \\ &- \int\int_{Q_T} \psi|\nabla v|^{p-2}\nabla v \cdot \nabla(u_n - v)dxdt \\ \ge &\ 0 \end{aligned} \tag{15}$$

Letting $n \to \infty$ in (15), we get

$$\begin{aligned} &\frac{1}{2}\int\int_{Q_T} \psi_t {u^*}^2 dxdt - \int\int_{Q_T} u^* w_i\psi_{x_i} dxdt - \int\int_{Q_T} \psi w_i v_{x_i} dxdt \\ &- \int\int_{Q_T} \lambda\psi {u^*}^2 dxdt - \int\int_{Q_T} \psi|\nabla v|^{p-2}\nabla v \cdot \nabla(u^* - v)dxdt \\ \ge &\ 0 \end{aligned} \tag{16}$$

Take $\psi u^*$ as a test function in (12) to obtain

$$\begin{aligned} &\frac{1}{2}\int\int_{Q_T} {u^*}^2\psi_t dxdt - \int\int_{Q_T} w_i\psi_{x_i} u^* dxdt \\ &- \int\int_{Q_T} w_i\psi u^*_{x_i} dxdt - \int\int_{Q_T} \lambda {u^*}^2\psi dxdt \\ = &\ 0 \end{aligned} \tag{17}$$

Using (16), we have

$$\int\int_{Q_T} \psi(w_i - |\nabla v|^{p-2} v_{x_i})(u^*_{x_i} - v_{x_i}) dxdt \geq 0. \tag{18}$$

Choosing $v = u^* - \theta\varphi$ in (18), where $\theta \geq 0$, $\varphi \in C_0^\infty(Q_T)$, we get

$$\int\int_{Q_T} \psi(w_i - |\nabla(u^* - \theta\varphi)|^{p-2}(u^* - \theta\varphi)_{x_i})\varphi_{x_i} dxdt \geq 0.$$

Letting $\theta \to 0$, we have

$$\int\int_{Q_T} \psi(w_i - |\nabla u^*|^{p-2} u^*_{x_i})\varphi_{x_i} dxdt \geq 0, \ \varphi \in C_0^\infty(Q_T).$$

Obviously, if we let $\theta \leq 0$, we can get another inequality which has reverse direction. Therefore, we can choose a function $\psi$, with $\mathrm{supp}\varphi \subset \mathrm{supp}\psi$, and $\psi = 1$ on $\mathrm{supp}\varphi$, such that (13) is true, which implies $w = |\nabla u^*|^{p-2}\nabla u^*$.

By

$$h_n \overset{*}{\rightharpoonup} h^* \text{ in } L^\infty((0,T) \times \partial\Theta),$$

and the continuity of the mapping from $H^1(\Theta)$ to $L^2(\partial\Theta)$, we have

$$u_n \to u^* \text{ in } L^2((0,T); L^2(\partial\Theta)),$$

and let $n \to \infty$ in (7), we see that $u^*$ is a weak solution of the problem (1)–(3) with $h^*$ as the heat transfer coefficient.

At last, by the lower semicontinuity of the cost functional and using the weak convergencies derived above, we see that $h^*$ is an optimal control. The proof of Theorem 1 is complete.

We now turn to the proof of Theorem 1. From the definition of a solution to our problem (1)–(3), we see that $u_\varepsilon$ and $u$ satisfy the integral equality (7). Choosing $\varphi = u_\varepsilon - u$ in (7), we have

$$\begin{aligned} & \frac{1}{2}\int_\Theta (u_\varepsilon(x,T) - u(x,T))^2 dx + \int\int_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u) dxdt \\ & - \int\int_{Q_T} |\nabla u|^{p-2}\nabla u \cdot \nabla(u_\varepsilon - u) dxdt + \int\int_{Q_T} \lambda(u_\varepsilon - u)^2 dxdt \\ & + \int_0^T \int_{\partial\Theta} h(u_\varepsilon - u)^2 dsdt + \int_0^T \int_{\partial\Theta} \varepsilon l u_\varepsilon(u_\varepsilon - u) dsdt \\ = \ & 0. \end{aligned}$$

Using

$$\int\int_{Q_T} (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_\varepsilon - u) dxdt \geq 0,$$

we get

$$\int\int_{Q_T} \lambda(u_\varepsilon - u)^2 dxdt \leq \varepsilon l \int_0^T \int_{\partial\Theta} u_\varepsilon(u - u_\varepsilon) dsdt \leq C\varepsilon.$$

The proof is complete.

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