Applied Mathematics E-Notes, 1(2001), 77-85 ©
Available free at mirror sites of http://math2.math.nthu.edu.tw/∼amen/

Periodic Boundary Value Problems of Impulsive Differential Equations *†

Hui Fang‡, Zhi-Cheng Wang§

Received 1 March 2001

Abstract

By means of the continuation theorem of coincidence degree theory, sufficient conditions are obtained for the existence of solutions of periodic boundary value problems involving impulsive differential equations.

In recent years, impulsive differential equations have attracted much attention since many evolution processes are subject to short term perturbations in the form of impulses. See, for instance, [1-8]. In this paper, we consider the following periodic boundary value problem of impulsive differential equation (PBVP)

$$\begin{cases} \dot{x}(t) = g(t, x(t)) + p(t),\ t \neq t_k, & k = 1, 2, \cdots, m, \\ x(t_k^+) - x(t_k) = I_k(t_k, x(t_k)), & k = 1, 2, \cdots, m, \\ x(0) = x(T), \end{cases} \tag{1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $T > 0$; $J = [0, T]$, $g \in C(J \times R, R)$, $g(0, u) = g(T, u)$ for $u \in R$; $I_k \in C(J \times R, R)$ for $k = 1, 2, \cdots, m$; $p \in C(J, R)$, and $p(0) = p(T)$.

Next we briefly state the part of Mawhin's coincidence degree theory that will be used in our study of PBVP (1). For further details, we refer the readers to [9, 10]. Let X and Y be real normed vector spaces, $L : \mathrm{dom}L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$ and $\mathrm{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero, there then exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$. It follows that $L_p = L|_{\mathrm{dom}L \cap \mathrm{Ker}P} : \mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L = \mathrm{Ker}Q$ is one-to-one and onto $\mathrm{Im}L$. We denote its inverse by K_p. If Ψ is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Psi}$ if $QN(\Psi)$ is bounded and $K_p(I - Q)N : \bar{\Psi} \to X$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $\hat{J} : \mathrm{Im}Q \to \mathrm{Ker}L$.

Mawhin's Continuation Theorem ([9, 10]). Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Psi}$. Suppose (i) For each $\lambda \in (0, 1)$, $x \in$

*Mathematics Subject Classifications: 34A37

†This work is supported by the Natural Science Foundation of Yunnan Province, P. R. China

‡Department of Systems Science and Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650051, P. R. China

§Department of Applied Mathematics, Hunan University, Changsha, Hunan 410082, P. R. China

$\partial\Psi$, $Lx \neq \lambda Nx$; (ii) $QNx \neq 0$ for each $x \in \text{Ker}L \cap \partial\Psi$; and (iii) Brouwer degree $\deg_B(\hat{J}QN, \Psi \cap \text{Ker}L, 0) \neq 0$. Then the equation $Lx = Nx$ has at least one solution in $\text{dom}L \cap \bar{\Psi}$.

Set $J' = J \setminus \{t_1, \cdots, t_m\}$. Let

$$\begin{aligned} PC_T &= \{u : J \to R |\ u \in C(J', R),\ u(t_j^-), u(t_j^+) \text{ exist for} \\ &\quad j = 1, 2, \cdots, m, \text{ and } u(0) = u(T)\} \end{aligned}$$

and

$$\begin{aligned} PC_T^1 &= \{u : J \to R |\ u \in C^1(J', R), u(t_j^+),\ u(t_j^-), u'(t_j^+), u'(t_j^-) \text{ exist,} \\ &\quad u(t_j^-) = u(t_j),\ j = 1, 2, \cdots, m, \text{ and } u(0) = u(T),\ u'(0) = u'(T)\} \end{aligned}$$

where $u(t_j^+), u(t_j^-)$ denote the right and left limits of $u(t)$ at $t = t_j$ respectively; $u'(t_j^+)$, $u'(t_j^-)$ denote the right and left limits of $u'(t)$ at $t = t_j$ respectively.

For $u(t) \in PC_T^1$, it is easy to see that the left derivative $u'_-(t_j)$ exists and is equal to $u'(t_j^-)$. In the following, $u'(t_j)$ may be understood as $u'_-(t_j)$.

For $u \in PC_T$, denote its norm by $\|u\| = \sup\{|u(t)| : t \in J\}$. For $u \in PC_T^1$, denote its norm by $\|u\|_1 = \|u\| + \|u'\|$. One can easily prove that PC_T and PC_T^1 are Banach spaces.

Let $X = Y = PC_T$ and let $L : \text{dom}L = PC_T^1 \subset X \to Y$ be given by

$$Lx(t) = \begin{cases} \dot{x}(t), & t \neq t_k, \quad k = 1, 2, \cdots, m \\ x(t_k^+) - x(t_k), & t = t_k, \quad k = 1, 2, \cdots, m \end{cases}$$

It is obvious that L is a linear mapping.

THEOREM 1. Assume that $\int_0^T p(t)dt = 0$ and there exist constants $c_k, h_k > 0$ for $k = 0, 1, 2, \cdots, m$, such that the following conditions hold: (i) if $x \geq c_0$, then $g(t, x) \leq h_0$ uniformly for $t \in J$, (ii) if $|x| \geq c_0$, then $xg(t, x) > 0$ uniformly for $t \in J$, (iii) if $|x| \geq c_0$, then $xI_k(t_k, x) > 0$ for $k = 1, 2, \cdots, m$, (iv) $|I_k(t_k, x)| \leq c_k|x| + h_k$ for $k = 1, 2, \cdots, m$, and (v) $\sum_{k=1}^m c_k < 1/2$. Then PBVP (1) has at least one solution.

We remark that condition (i) in Theorem 1 can be replaced by the following condition: (a) if $x \leq -c_0$, then $g(t, x) \geq -h_0$ uniformly for $t \in J$.

Before proving Theorem 1, we need the following lemmas.

LEMMA 1. $H \subset PC_T$ is relatively compact if, and only if, every function of H is uniformly bounded on J and is equicontinuous on each J_k for $k = 0, 1, 2, \cdots, m$, where $J_0 = [0, t_1)$, $J_k = (t_k, t_{k+1})$ for $k = 1, 2, \cdots, m-1$, and $J_m = (t_m, T]$.

The proof is an immediate consequence of Ascoli-Arzela Theorem.

LEMMA 2. L is a Fredholm mapping of index zero.

PROOF. It is clear that

$$\text{Ker}L = \{u : u(t) \equiv c, c \in R\}.$$

For $y \in Y$ we define $Q : Y \to Y$ by

$$Qy = \frac{\sum\limits_{i=1}^m y(t_i) + \int_0^T y(t)dt}{T + m}.$$

We assert that $\mathrm{Im}L = \mathrm{Ker}Q$. There are two cases to consider.

Case 1. If $y \in \mathrm{Im}L$, then there exists $x \in \mathrm{dom}L \subset X$ such that $Lx = y$ i.e.

$$\begin{cases} \dot{x}(t) = y(t), & t \neq t_k,\ k = 1, 2, \cdots, m, \\ x(t_k^+) - x(t_k) = y(t_k), & k = 1, 2, \cdots, m. \end{cases}$$

Hence, we have

$$\begin{aligned} \int_0^T y(t)dt &= \int_0^{t_1} y(t)dt + \sum_{k=1}^{m-1} \int_{t_k}^{t_{k+1}} y(t)dt + \int_{t_m}^T y(t)dt \\ &= x(t_1) - x(0) + \sum_{k=1}^{m-1} [x(t_{k+1}) - x(t_k^+)] + x(T) - x(t_m^+) \\ &= x(t_1) + \sum_{k=1}^{m-1} [x(t_{k+1}) - x(t_k) - y(t_k)] - x(t_m) - y(t_m) \\ &= -\sum_{k=1}^{m} y(t_k). \end{aligned}$$

So, we have

$$\sum_{k=1}^{m} y(t_k) + \int_0^T y(t)dt = 0.$$

Therefore, $y \in \mathrm{Ker}Q$, which implies that $\mathrm{Im}L \subset \mathrm{Ker}Q$.

Case 2. If $y \in \mathrm{Ker}Q$, then $\sum_{i=1}^m y(t_i) + \int_0^T y(t)dt = 0$. Define the function $x(t)$ by

$$x(t) = \begin{cases} a_1 + \int_{t_1}^t y(s)ds, & t \in [0, t_1]; \\ a_{i+1} + \int_{t_{i+1}}^t y(s)ds, & t \in (t_i, t_{i+1}], \quad i = 1, 2, \cdots, m-1; \\ a_{m+1} + \int_T^t y(s)ds, & t \in (t_m, T]; \end{cases}$$

where $a_1, ..., a_{m+1}$ are given by the following system of equations

$$\left\{ \begin{array}{rcl} \sum\limits_{i=1}^{m} (1 + t_i - t_{i-1})a_i + (T - t_m)a_{m+1} &=& -\sum\limits_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_{t_{i+1}}^t y(s)dsdt \\ && -\int_{t_m}^T \int_T^t y(s)dsdt, \\ a_1 - a_2 &=& -y(t_1) - \int_{t_1}^{t_2} y(s)ds, \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ a_i - a_{i+1} &=& -y(t_i) - \int_{t_i}^{t_{i+1}} y(s)ds, \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ a_{m-1} - a_m &=& -y(t_{m-1}) - \int_{t_{m-1}}^{t_m} y(s)ds, \\ a_m - a_{m+1} &=& -y(t_m) - \int_{t_m}^T y(s)ds. \end{array} \right. \tag{2}$$

Let Δ denote the determinant of coefficients of system (2). It is easy to see that $\Delta = (m+T)(-1)^m \neq 0$. Therefore, system (2) has the unique solution $(a_1, \cdots, a_{m+1})$.

It is easy to verify that $x(t) \in PC_T^1$. Next we prove that $Lx = y$. Clearly, we have

$$x'(t) = y(t),\ t \neq t_k,\ k = 1, 2, \cdots, m.$$

By (2), $x(t_i) = a_i$ for $i = 1, 2, \cdots, m$, and

$$\begin{aligned} x(t_i^+) &= a_{i+1} + \int_{t_{i+1}}^{t_i} y(s)ds,\ i = 1, 2, \cdots, m-1, \\ x(t_m^+) &= a_{m+1} + \int_T^{t_m} y(s)ds, \end{aligned}$$

we have

$$x(t_i^+) - x(t_i) = a_{i+1} + \int_{t_{i+1}}^{t_i} y(s)ds - a_i = y(t_i),\ i = 1, 2, \cdots, m-1,$$

and

$$x(t_m^+) - x(t_m) = a_{m+1} + \int_T^{t_m} y(s)ds - a_m = y(t_m).$$

This proves that $Lx = y$. Hence, we have $\mathrm{Ker}Q \subset \mathrm{Im}L$.

Combining Case 1 and Case 2, we have $\mathrm{Im}L = \mathrm{Ker}Q$.

It is easy to verify that $Q^2y = Qy$ for $y \in Y$. Therefore Q is a continuous projector.

Clearly, $\mathrm{Im}L$ is a closed subspace of Y, and $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$.

Since $\dim\mathrm{Ker}L = 1$ and $\mathrm{codim}\mathrm{Im}L = \dim\mathrm{Im}Q = 1$, L is a Fredholm mapping of index zero. The proof is complete.

Define $P = Q : PC_T \to PC_T$ by

$$Qy = \frac{\sum_{i=1}^{m} y(t_i) + \int_0^T y(t)dt}{T+m},\ y \in PC_T.$$

From the proof of Lemma 2, we know that $P = Q$ are continuous projectors such that $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$ and $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$. Consequently, the restriction L_P of L to $\mathrm{dom}L \cap \mathrm{Ker}P$ is one-to-one and onto $\mathrm{Im}L$, so that its inverse $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ is defined. Further, we have the following result.

LEMMA 3. $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ is a compact mapping.

PROOF. For any $y \in \mathrm{Im}L$, set

$$z(t) = \begin{cases} a_1 + \int_{t_1}^t y(s)ds, & t \in [0, t_1]; \\ a_{i+1} + \int_{t_{i+1}}^t y(s)ds, & t \in (t_i, t_{i+1}], \quad i = 1, 2, \cdots, m-1; \\ a_{m+1} + \int_T^t y(s)ds, & t \in (t_m, T]. \end{cases}$$

where $a_1, ..., a_{m+1}$ are given by (2). It follows from the first equation of (2) that $z(t) \in \mathrm{Ker}P$. Thus, we have $K_P y(t) = z(t)$. Therefore, by Lemma 1 and the definition of K_P, it is easy to see that K_P is a compact mapping. The proof is complete.

LEMMA 4. If the conditions of Theorem 1 hold, then there exists a constant $M > 0$, such that every solution $x(t)$ of the problem

$$Lx = \lambda Nx,\ \lambda \in (0,1)$$

satisfies $\|x\| \leq M$.

PROOF. Let $Lx = \lambda Nx$ for $x(t) \in X$, i.e.

$$\begin{cases} \dot{x}(t) = \lambda g(t,x(t)) + \lambda p(t),\ t \neq t_k, & k = 1,2,\cdots,m, \\ x(t_k^+) - x(t_k) = \lambda I_k(t_k, x(t_k)), & k = 1,2,\cdots,m. \end{cases} \tag{3}$$

Then we have

$$\begin{aligned} \int_0^T \dot{x}(t)dt &= \int_0^{t_1} \dot{x}(t)dt + \sum_{k=1}^{m-1} \int_{t_k}^{t_{k+1}} \dot{x}(t)dt + \int_{t_m}^T \dot{x}(t)dt \\ &= x(t_1) - x(0) + \sum_{k=1}^{m-1} [x(t_{k+1}) - x(t_k^+)] + x(T) - x(t_m^+) \\ &= x(t_1) + \sum_{k=1}^{m-1} [x(t_{k+1}) - x(t_k) - \lambda I_k(t_k, x(t_k))] - x(t_m) - \lambda I_m(t_m, x(t_m)) \\ &= -\lambda \sum_{k=1}^{m} I_k(t_k, x(t_k)) \\ &= \lambda \int_0^T g(t,x(t))dt, \end{aligned}$$

which implies that

$$\int_0^T g(t,x(t))dt + \sum_{k=1}^{m} I_k(t_k, x(t_k)) = 0 \tag{4}$$

Set $E_1 = \{t \in [0,T] : x(t) \geq c_0\}$, $E_2 = \{t \in [0,T] : x(t) \leq -c_0\}$, and $E_3 = \{t \in [0,T] : |x(t)| < c_0\}$. From (4), we have

$$\begin{aligned} \int_{E_2} g(t,x(t))dt &= -\int_{E_1} g(t,x(t))dt - \int_{E_3} g(t,x(t))dt - \sum_{k=1}^{m} I_k(t_k, x(t_k)) \\ &\geq -h_0 T - \gamma T - \sum_{k=1}^{m} c_k \|x\| - \sum_{k=1}^{m} h_k, \end{aligned}$$

where

$$\gamma = \max_{t \in [0,T], |x| \leq c_0} |g(t,x)|.$$

Therefore, we have

$$\int_0^T |g(t,x(t))|dt = \int_{E_1} |g(t,x(t))|dt + \int_{E_2} |g(t,x(t))|dt + \int_{E_3} |g(t,x(t))|dt$$

$$\begin{aligned} &= \int_{E_1} |g(t,x(t))|dt - \int_{E_2} g(t,x(t))dt + \int_{E_3} |g(t,x(t))|dt \\ &\leq h_0T + (h_0T + \gamma T + \sum_{k=1}^{m} h_k) + \sum_{k=1}^{m} c_k\|x\| + \gamma T. \end{aligned}$$

That is

$$\int_0^T |g(t,x(t))|dt \leq M_1 + \sum_{k=1}^{m} c_k\|x\| \tag{5}$$

where

$$M_1 = 2h_0T + 2\gamma T + \sum_{k=1}^{m} h_k.$$

Next we make the following claims.

Claim 1. There exists $\tau \in [0,T]$, such that $|x(\tau)| \leq c_0$. Indeed, assume to the contrary that $|x(t)| > c_0$ for any $t \in [0,T]$. Without loss of generality, assume that $x(0) > 0$. Then, by $x(t) \in C[0,t_1]$, we have $x(t) > 0$ for any $t \in [0,t_1]$. By (3) and condition (iii) of Theorem 1, we know that $x(t_1^+) > 0$. Therefore, we have $x(t) > 0$ for any $t \in [t_1,t_2]$. By using simple induction, we can prove that $x(t) > 0$ for any $t \in [0,T]$. So, we have $x(t) > c_0$ for any $t \in [0,T]$. Hence, by conditions (ii) and (iii) of Theorem 1, we have $g(t,x(t)) > 0$ for any $t \in [0,T]$ and $I_k(t_k,x(t_k)) > 0$ for $k = 1,2,\cdots,m$. This is contrary to (4). The proof of Claim 1 is complete.

Claim 2. $|x(t)| \leq |x(\tau)| + \int_0^T |\dot{x}(t)|dt + \sum_{k=1}^{m} |I_k(t_k,x(t_k))|$ for any $t \in [0,T]$. Indeed, without loss of generality, we assume that $t \in (t_k,t_{k+1}]$ and $\tau \in (t_{k+q},t_{k+q+1}]$, $0 \leq q \leq m-k$. Then we have

$$\begin{aligned} & x(t) - x(\tau) \\ &= x(t) - x(t_{k+1}) + \sum_{j=1}^{q}[x(t_{k+j}) - x(t_{k+j+1})] + x(t_{k+q+1}) - x(\tau) \\ &= \int_{t_{k+1}}^{t} \dot{x}(t)dt + \sum_{j=1}^{q}[x(t_{k+j}^+) - x(t_{k+j+1}) - \lambda I_{k+j}(t_{k+j},x(t_{k+j}))] + \int_{\tau}^{t_{k+q+1}} \dot{x}(t)dt \\ &= \int_{t_{k+1}}^{t} \dot{x}(t)dt - \sum_{j=1}^{q}\int_{t_{k+j}}^{t_{k+j+1}} \dot{x}(t)dt - \lambda\sum_{j=1}^{q} I_{k+j}(t_{k+j},x(t_{k+j})) + \int_{\tau}^{t_{k+q+1}} \dot{x}(t)dt \\ &= \int_{\tau}^{t} \dot{x}(t)dt - \lambda\sum_{j=1}^{q} I_{k+j}(t_{k+j},x(t_{k+j})). \end{aligned}$$

Hence, we have

$$|x(t)| \leq |x(\tau)| + \int_0^T |\dot{x}(t)|dt + \sum_{k=1}^{m} |I_k(t_k,x(t_k))|.$$

The proof of Claim 2 is complete.

On the other hand, by (3) and (5), we obtain

$$\int_0^T |\dot{x}(t)|dt \le \lambda \int_0^T |g(t,x(t))|dt + \lambda \int_0^T |p(t)|dt \le M_1 + \sum_{k=1}^m c_k\|x\| + \int_0^T |p(t)|dt.$$

Therefore, we have

$$|x(t)| \le c_0 + M_1 + \sum_{k=1}^m c_k\|x\| + \int_0^T |p(t)|dt + \sum_{k=1}^m c_k\|x\| + \sum_{k=1}^m h_k$$

Hence, we have

$$\|x\| \le \frac{1}{1-2\sum\limits_{k=1}^m c_k}\left[c_0 + M_1 + \sum_{k=1}^m h_k + \int_0^T |p(t)|dt\right]. \tag{6}$$

The proof is complete by taking M to be the right hand side of (6).

We now turn to the proof of Theorem 1. Let $r > \max\{M, c_0\}$ with M as given in Lemma 4 and let $\Psi = \{x \in PC_T : \|x\| < r\}$. Define a map $N : X \to Y$ by

$$Nx(t) = \begin{cases} g(t,x(t)) + p(t), & t \ne t_k, \quad k = 1,2,\cdots,m, \\ I_k(t_k, x(t_k)), & t = t_k, \quad k = 1,2,\cdots,m. \end{cases}$$

We assert that N is continuous. Indeed, assume that $x_n, x \in PC_T$ such that $x_n \to x$. Then there exists $H > 0$ such that $\|x_n\| \le H$ and $\|x\| \le H$. Given any $\varepsilon > 0$, it follows from the uniform continuity of $g, I_1, ..., I_m$ on $[0,T] \times \{x \in R : |x| \le H\}$ that there exists $0 < \delta < \varepsilon$ such that $|g(t,x_1) - g(t,x_2)| < \varepsilon$ and $|I_k(t,x_1) - I_k(t,x_2)| < \varepsilon$ for any $x_1, x_2 \in \{x \in R^n : |x| \le H\}$ with $|x_1 - x_2| < \delta$. Again, for $\delta > 0$, there exists a positive integer K such that, for $n \ge K$, $\|x_n - x\| \le \delta$. Since

$$Nx_n(t) - Nx(t) = \begin{cases} g(t,x_n(t)) - g(t,x(t)), & t \ne t_k, \\ I_k(t_k, x_n(t_k)) - I_k(t_k, x(t_k)), & t = t_k, \end{cases}$$

for $k = 1,2,\cdots,m$, we have

$$|Nx_n(t) - Nx(t)| \le \varepsilon, \ n \ge K, t \in [0,T].$$

Hence, we have

$$\|Nx_n - Nx\| \le \varepsilon, \ n \ge K.$$

This proves that N is continuous on PC_T. Therefore, PBVP (1) is equivalent to the operator equation

$$Lx = Nx, \ x \in \mathrm{dom}L.$$

We now apply Mawhin's Continuation Theorem for Ψ. In view of Lemma 3, N is L-compact on $\bar{\Psi}$. By Lemma 4, it is easy to see that $Lx \ne \lambda Nx$ for any $(x, \lambda) \in$

$(\text{dom}L \cap \partial\Psi) \times (0,1)$. Note that for $x \in \partial\Psi \cap \text{Ker}L = \partial\Psi \cap R$, we must have $x = r$ or $x = -r$. Therefore, for such an x,

$$QNx = \frac{\sum_{i=1}^{m} I_i(t_i, x) + \int_0^T g(t,x)dt}{T+m} \neq 0.$$

Let

$$\Phi(x,\mu) = \mu x + (1-\mu)\frac{\sum_{i=1}^{m} I_i(t_i, x) + \int_0^T g(t,x)dt}{T+m}, \quad \mu \in [0,1].$$

Then, for any $x \in \partial\Psi \cap \text{Ker}L, \mu \in [0,1]$, we have

$$x\Phi(x,\mu) = \mu x^2 + (1-\mu)\frac{\sum_{i=1}^{m} xI_i(t_i, x) + \int_0^T xg(t,x)dt}{T+m} > 0, \quad \mu \in [0,1],$$

which implies that $\Phi(x,\mu)$ is a homotopy. Therefore, by the property of invariance under a homotopy of coincidence degree, we have

$$\begin{aligned}
\deg_B(\hat{J}QN, \Psi \cap \text{Ker}L, 0) &= \deg_B(\Phi(x,0), \Psi \cap \text{Ker}L, 0) \\
&= \deg_B(\Phi(x,1), \Psi \cap \text{Ker}L, 0) \\
&= \deg_B(x, \Psi \cap \text{Ker}L, 0) = 1,
\end{aligned}$$

where the isomorphism $\hat{J}$ of $\text{Im}Q$ onto $\text{Ker}L$ is the identity mapping, since $\text{Im}Q = \text{Ker}L$. It follows from Mawhin's Continuation Theorem that $Lx = Nx$ has at least one solution in $\text{dom}L \cap \bar{\Psi}$. Therefore, PBVP (1) has at least one solution. The proof is complete.

References

[1] D. D. Bainov and P. S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Longman, Harlow, 1993.

[2] D. D. Bainov and P. S. Simeonov, Systems with Impulse Effect, Stability Theory and Applications, Ellis Horwood, Chichster, 1989.

[3] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[4] X. Liu, Nonlinear boundary value problems for first order impulsive integro-differential equations, Appl. Anal. 36(1990), 119-130.

[5] Y. Li and Q. D. Zhou, Periodic solutions to ordinary differential equations with impulses, Sci. China Ser. A 36(1993), 778-790.

[6] A. S. Vatsala and Y. Sun, Periodic boundary value problems of impulsive differential equations, Appl. Anal. 44(1992), 145-158.

[7] J. J. Nieto, Basic theory for nonresonance impulsive periodic problems of first order, J. Math. Anal. Appl. 205(1997), 423-433.

[8] Y. J. Dong, Periodic boundary value problems for functional differential equations with impulses, J. Math. Anal. Appl. 210(1997), 170-181.

[9] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems. CBMS, Vol 40. RI: Amer Math Soc Providence, 1979.

[10] R. E. Gaines and J. L. Mawhin, Lecture Notes in Mathematics, Vol. 586, Springer-Verlag, Berlin, 1997.