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On Picard-Fuchs Type Equations Related to Integrable Hamiltonian Systems *

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Abstract

The structure properties of integral submanifold imbedding mapping for a class of algebraically Liouville integrable Hamiltonian systems on cotangent phase spaces are studied in relation with Picard -Fuchs type equations. It is shown that these equations can be constructed by making use of a given a priori set of involutive invariants and proved that their solutions in the Hamilton-Jacobi separable variable case give rise to the integral submanifold imbedding mapping, which is known to be a main ingredient for Liouville-Arnold integrability by quadratures of the Hamiltonian system under consideration.

1 Introduction

We consider a completely integrable Liouville-Arnold Hamiltonian system [1, 2] on a cotangent canonically symplectic manifold $(T^*(\mathbf{R}^n), \omega^{(2)})$, $n \in \mathbf{Z}_+$, possessing exactly n functionally independent and Poisson commuting algebraic polynomial invariants $H_j : T^*(\mathbf{R}^n) \to \mathbf{R}$, $j = \overline{1,n}$. Due to the Liouville-Arnold theorem [1, 2], this Hamiltonian system can be completely integrated by quadratures in quasi-periodic functions on its integral submanifold when this submanifold is compact. This is equivalent to the statement that this compact integral submanifold is diffeomorphic to a torus $\mathbf{T}^n$, and that makes it possible to formulate the problem of integrating the system by means of searching the corresponding integral submanifold imbedding mapping $\pi_h : M_h^n \longrightarrow T^*(\mathbf{R}^n)$, where

$$M_h^n := \left\{(q,p) \in T^*(\mathbf{R}^n) : \ H_j(q,p) = h_j \in \mathbf{R}, \ j = \overline{1,n}\right\}. \tag{1}$$

Since $M_h^n \simeq \mathbf{T}^n$, and the integral submanifold (1) is invariant subject to all Hamiltonian flows $K_j : T^*(\mathbf{R}^n) \to T(T^*(\mathbf{R}^n))$, $j = \overline{1,n}$, where

$$i_{K_j}\omega^{(2)} = -dH_j, \tag{2}$$

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there exist [1, 2] corresponding "action-angle"-coordinates $(\varphi, \gamma) \in (\mathbf{T}^n_\gamma, \mathbf{R}^n)$ on the torus $\mathbf{T}^n_\gamma \backsimeq M^n_h$, specifying its imbedding $\pi_\gamma : \mathbf{T}^n_\gamma \to T^*(\mathbf{R}^n)$ by means of a set of smooth functions $\gamma \in \mathcal{D}(\mathbf{R}^n)$, where

$$\mathbf{T}^n_\gamma := \left\{(q,p) \in T^*(\mathbf{R}^n) : \ \gamma_j(H) = \gamma_j \in \mathbf{R}, \ j = \overline{1,n}\right\}. \tag{3}$$

The mapping $\gamma : \mathbf{R}^n \ni h \to \mathbf{R}^n$ induced by (3) is of great interest in many applications and was studied earlier by Picard and Fuchs subject to the corresponding differential equations it satisfies:

$$\partial\gamma_j(h)/\partial h_i = F_{ij}(\gamma; h), \tag{4}$$

where $h \in \mathbf{R}^n$ and $F_{ij} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $i, j = \overline{1,n}$, are some almost everywhere smooth functions. In the case where the right hand side of (4) is a set of algebraic functions on $\mathbf{C}^n \times \ \mathbf{C}^n \ni (\gamma; h)$, all Hamiltonian flows $K_j : T^*(\mathbf{R}^n) \to T(T^*(\mathbf{R}^n))$, $j = \overline{1,n}$, are said to be algebraically completely integrable in quadratures. Equations such as (4) were studied in [3], a recent example can also be found in [4].

2 Canonical Transformations Properties

It is clear that the Picard-Fuchs equations (4) are related with the associated canonical transformation of the symplectic 2-form $\omega^{(2)} \in \Lambda^2(T^*(\mathbf{R}^n))$ in a neighborhood $U(M^n_h)$ of the integral submanifold $M^n_h \subset T^*(\mathbf{R}^n)$. More precisely, denote $\omega^{(2)}(q,p) = dpr^*\alpha^{(1)}(q;p)$, where for $(q,p) \in T^*(\mathbf{R}^n)$,

$$\alpha^{(1)}(q;p) := \sum_{j=1}^n p_j dq_j = \langle p, dq\rangle \in \Lambda^1(\mathbf{R}^n) \tag{5}$$

is the canonical Liouville 1-form on $\mathbf{R}^n$, $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbf{R}^n$, and $pr : T^*(\mathbf{R}^n) \to \mathbf{R}^n$ is the bundle projection. We define a mapping $dS_q : \mathbf{R}^n \to T^*_q(\mathbf{R}^n)$, such that on M^n_h the relationship

$$pr^*\alpha^{(1)}(q;p) + \langle t, dh\rangle = dS_q(h), \tag{6}$$

holds, where $t \in \mathbf{R}^n$ is the set of evolution parameters. From (6) one gets right away $S_q(h) = \int_{q^{(0)}}^q \langle p, dq\rangle\Big|_{M^n_h}$ for any $q, q^{(0)} \in M^n_h$. On the other hand one can define a generating function $S_\mu : \mathbf{R}^n \to \mathbf{R}$ such that

$$dS_\mu : \mathbf{R}^n \to T^*_\mu(M^n_h), \tag{7}$$

where $\mu \in M^n_h \backsimeq \otimes^n_{j=1}\mathbf{S}^1_j$ are the global separable coordinates existing on M^n_h owing to the Liouville-Arnold theorem. Thus we have the following canonical relationship

$$\langle w, d\mu\rangle + \langle t, dh\rangle = dS_\mu(h), \tag{8}$$

where $w_j := w_j(\mu_j; h) \in T^*_{\mu_j}(\mathbf{S}^1_j)$ for every $j = \overline{1,n}$. Whence

$$S_\mu(h) = \sum_{j=1}^n \int_{\mu_j^{(0)}}^{\mu_j} w(\lambda; h) d\lambda, \tag{9}$$

satisfies on $M_h^n \subset T^*(\mathbf{R}^n)$ the following relationship $dS_\mu + d\mathcal{L}_\mu = dS_q\big|_{q=q(\mu;h)}$ for some mapping $\mathcal{L}_\mu : \mathbf{R}^n \to \mathbf{R}$. As a result of (9) we get the following important expressions

$$t_i = \partial S_\mu(h)/\partial h_i, \;\; \langle p, \partial q/\partial \mu_i\rangle = w_i + \partial \mathcal{L}_\mu/\partial \mu_i, \; i = \overline{1,n}. \tag{10}$$

A construction similar to the above can be done subject to the imbedded torus $\mathbf{T}_\gamma^n \subset T^*(\mathbf{R}^n) : d\tilde{S}_q(\gamma) := \sum_{j=1}^n p_j dq_j + \sum_{i=1}^n \varphi_i d\gamma_i$, where in view of (7), $\tilde{S}_q(\gamma) := S_q(\xi(\gamma))$, $\xi(\gamma) = h$, for all $(q;\gamma) \in U(M_h^n)$. For angle coordinates $\varphi \in \mathbf{T}_\gamma^n$, we obtain from $d\tilde{S}_q(\gamma)$ that $\varphi_i = \partial \tilde{S}_q(\gamma)/\partial \gamma_i$ for all $i = \overline{1,n}$. As $\varphi_i \in \mathbf{R}/2\pi\mathbf{Z}$, $i = \overline{1,n}$, we may easily derive

$$\frac{1}{2\pi}\oint_{\sigma_j^{(h)}} d\varphi_i = \delta_{ij} = \frac{1}{2\pi}\frac{\partial}{\partial \gamma_i}\oint_{\sigma_j^{(h)}} d\tilde{S}_q(\gamma) = \frac{1}{2\pi}\frac{\partial}{\partial \gamma_i}\oint_{\sigma_j^{(h)}} \langle p, dq\rangle \tag{11}$$

for all canonical cycles $\sigma_j^{(h)} \subset M_h^n$, $j = \overline{1,n}$, constituting a basis of the one dimensional homology group $H^1(M_h^n;\mathbf{Z})$. It follows that for all $i = \overline{1,n}$, "action" variables can be found as $\gamma_i = \frac{1}{2\pi}\oint_{\sigma_i^{(h)}} \langle p, dq\rangle$. Recall now that $M_h^n \simeq \mathbf{T}_\gamma^n$ are also diffeomorphic to $\otimes_{j=1}^n \mathbf{S}_j^1$, where $\mathbf{S}_j^1$, $j = \overline{1,n}$, are some one-dimensional real circles. The evolution along any of the vector fields $K_j : T^*(\mathbf{R}^n) \to T(T^*(\mathbf{R}^n))$, $j = \overline{1,n}$, on $M_h^n \subset T^*(\mathbf{R}^n)$ is known [1, 2] to be a linear winding around the torus $\mathbf{T}_\gamma^n$. This can also be interpreted in the following manner: the independent global coordinates on circles $\mathbf{S}_j^1$, $j = \overline{1,n}$, introduced above are such that the resulting evolution undergoes a quasiperiodic motion. These coordinates may still be called Hamilton-Jacobi and are important for accomplishing the complete integrability by quadratures by solving the corresponding Picard-Fuchs type equations.

Let us denote these separable coordinates on the integral submanifold $M_h^n \simeq \otimes_{j=1}^n \mathbf{S}_j^1$ by $\mu_j \in \mathbf{S}_j^1$, $j = \overline{1,n}$, and define the corresponding imbedding mapping $\pi_h : M_h^n \to T^*(\mathbf{R}^n)$ as

$$q = q(\mu;h), \;\; p = p(\mu;h). \tag{12}$$

There exist two important cases. The first case is related to the integral submanifold $M_h^n \subset T^*(\mathbf{R}^n)$ which can be parametrized as a manifold by means of the base coordinates $q \in \mathbf{R}^n$ of the cotangent bundle $T^*(\mathbf{R}^n)$. This can be explained as follows: the canonical Liouville 1-form $\alpha^{(1)} \in \Lambda^1(\mathbf{R}^n)$, in accordance with the diagram

$$\begin{array}{llll} T^*(\,M_h^n) & \simeq & T^*(\otimes_{j=1}^n \mathbf{S}_j^1) & \xleftarrow{\pi^*} & T^*(\mathbf{R}^n) \\ pr\downarrow & & pr\downarrow & & pr\downarrow \\ M_h^n & \simeq & \otimes_{j=1}^n \mathbf{S}_j^1 & \xrightarrow{\pi} & \mathbf{R}^n \end{array} \tag{13}$$

is mapped by the imbedding mapping $\pi = pr \cdot \pi_h : M_h^n \to \mathbf{R}^n$ not depending on a set of parameters $h \in \mathbf{R}^n$, into the 1-form

$$\alpha_h^{(1)} = \pi^*\alpha^{(1)} = \sum_{j=1}^n w_j(\mu_j;h)d\mu_j, \tag{14}$$

where $(\mu, w) \in T^*(\otimes_{j=1}^n \mathbf{S}_j^1) \simeq \otimes_{j=1}^n T^*(\mathbf{S}_j^1)$. The imbedding mapping $\pi : M_h^n \to \mathbf{R}^n$ due to (14) reduces the function $\mathcal{L}_\mu : \mathbf{R}^n \to \mathbf{R}$ to zero and gives rise to the generating function $S_\mu : \mathbf{R}^n \to \mathbf{R}$ which satisfies the condition $dS_\mu = dS_q|_{q=q(\mu)}$,where as before $S_q(h) = \sum_{j=1}^n p_j dq_j + \sum_{j=1}^n t_j dh_j$ and $\det ||\partial q(\mu)/\partial\mu|| \neq 0$ almost everywhere on M_h^n for all $h \in \mathbf{R}^n$. Similar to (10), we now have

$$t_j = \partial S_\mu(h)/\partial h_j \tag{15}$$

for $j = \overline{1,n}$. From the second part of the imbedding mapping (12) we arrive, in view of (14), at the following simple result: $p_i = \sum_{j=1}^n w_j(\mu_j; h)\partial\mu_j/\partial q_i$, where $i = \overline{1,n}$ and $\det ||\partial\mu(q)/\partial q|| \neq 0$ almost everywhere on $\pi(M_h^n)$ due to the local invertibility of the imbedding mapping $\pi : M_h^n \to \mathbf{R}^n$. Thus, we can assert that the problem of complete integrability in the first case is solved iff the only imbedding mapping $\pi : M_h^n \to \mathbf{R}^n \subset T^*(\mathbf{R}^n)$ is constructed. This case was considered in detail in [6] where the corresponding Picard-Fuchs type equations were built based on an extension of the Galisot-Reeb and Francoise results [4, 5]. Namely, similar to (4), these equations are defined as follows:

$$\partial w_j(\mu_j; h)/\partial h_k = P_{kj}(\mu_j, w_j; h), \tag{16}$$

where $P_{kj} : T^*(\otimes_{j=1}^n \mathbf{S}_j^1) \times \mathbf{C}^n \to \mathbf{C}$, $k, j = \overline{1,n}$, are some algebraic functions of their arguments.

In the second case where the integral submanifold $M_h^n \subset T^*(\mathbf{R}^n)$ cannot be imbedded almost everywhere into the base space $\mathbf{R}^n \subset T^*(\mathbf{R}^n)$, the relationship like (14) does not take place, and we are forced to consider the usual canonical transformation from $T^*(\mathbf{R}^n)$ to $T^*(\mathbf{R}^n)$ based on a mapping $d\mathcal{L}_q : \otimes_{j=1}^n \mathbf{S}_j^1 \to T^*(\mathbf{R}^n)$, where $\mathcal{L}_q : \otimes_{j=1}^n \mathbf{S}_j^1 \to \mathbf{R}$ enjoys for all $\mu \in \otimes_{j=1}^n \mathbf{S}_j^1 \backsimeq M_h^n \ni q$ the following relationship : $pr^*\alpha^{(1)}(q;p) = \sum_{j=1}^n w_j\ d\mu_j + d\mathcal{L}_q(\mu)$. In this case we can derive for any $\mu \in \otimes_{j=1}^n \mathbf{S}_j^1$ the hereditary generating function $\mathcal{L}_\mu : \mathbf{R}^n \to T^*(\otimes_{j=1}^n \mathbf{S}_j^1)$ introduced before as

$$\sum_{j=1}^n w_j(\mu_j; h)\ d\mu_j + \sum_{j=1}^n t_j dh_j + d\mathcal{L}_\mu = d\mathcal{L}_q|_{q=q(\mu;h)}\,, \tag{17}$$

satisfying evidently the following canonical transformation condition:

$$dS_{q(\mu;h)}(h) = \sum_{j=1}^n w_j(\mu_j; h)\ d\mu_j + \sum_{j=1}^n t_j dh_j + d\mathcal{L}_\mu(h), \tag{18}$$

for almost all $\mu \in \otimes_{j=1}^n \mathbf{S}_j^1$ and $h \in \mathbf{R}^n$. Based on (18) and (17) one can derive the following relationships:

$$\partial\mathcal{L}_\mu(h)/\partial h_j = \langle p, \partial q/\partial h_j\rangle|_{M_h^n} \tag{19}$$

for all $j = \overline{1,2}$, $\mu \in \otimes_{j=1}^n \mathbf{S}_j^1$ and $h \in \mathbf{R}^n$. Whence the following important analytical results

$$\begin{aligned} &t_s = \textstyle\sum_{j=1}^n \int_{\mu_j^{(0)}}^{\mu_j} (\partial w_j(\lambda; h)/\partial h_s) d\lambda, \\ &\textstyle\sum_{j=1}^n p_j(\mu; h)(\partial q_j/\partial\mu_s) = w_s + \partial\mathcal{L}_\mu(h)/\partial\mu_s, \end{aligned} \tag{20}$$

hold for all $s = \overline{1,2}$ and $\mu, \mu^{(0)} \in \otimes_{j=1}^n \mathbf{S}_j^1$ with parameters $h \in \mathbf{R}^n$ being fixed. Thereby we have found a natural generalization of the relationships for p-variables subject to the extended integral submanifold imbedding mapping $\pi_h : M_h^n \rightarrow T^*(\mathbf{R}^n)$ in the form (12).

Assume now the functions $w_j : \mathbf{C}\times\mathbf{C}^n\rightarrow\mathbf{C}$, $j = \overline{1,n}$, satisfy the Picard-Fuchs equations (16) and also the following [3, 5] algebraic conditions:

$$w_j^{n_j} + \sum_{k=0}^{n_j-1} c_{j,k}(\lambda; h) w_j^k = 0, \tag{21}$$

where $c_{j,k} : \mathbf{C}\times\mathbf{C}^n\rightarrow\mathbf{C}$, $k = \overline{0, n_j - 1}$, $j = \overline{1,n}$, are some polynomials in $\lambda \in \mathbf{C}$. In view of the Riemann theorem [7, 8], each algebraic curve of (21) is known to be topologically equivalent to some Riemannian surface $\Gamma_h^{(j)}$ of genus $g_j \in \mathbf{Z}_+$, $j = \overline{1,n}$. Thereby, one can realize the local diffeomorphism $\rho : M_h^n \rightarrow \otimes_{j=1}^n \Gamma_h^{(j)}$, by mapping the homology group basis cycles $\sigma_j^{(h)} \subset M_h^n$, $j = \overline{1,n}$, into the homology subgroup $H_1(\otimes_{j=1}^n \Gamma_h^{(j)}; \mathbf{Z})$ basis cycles $\sigma_j(\Gamma_h) \subset \Gamma_h^{(j)}$, $j = \overline{1,n}$, satisfying the following relationships: $\rho(\ \sigma_j^{(h)}) = \sum_{k=1}^n n_{jk}\ \sigma_k(\Gamma_h)$, where $n_{jk} \in \mathbf{Z}$, $k = \overline{1,j}$ and $j = \overline{1,n}$, are some fixed integers. Based now on (17) one can write down, for instance, action-variables expressions as follows:

$$\gamma_i = \frac{1}{2\pi} \sum_{j=1}^n n_{ij} \oint_{\sigma_j(\Gamma_h)} w_j(\lambda; h) d\lambda, \tag{22}$$

where $i = \overline{1,n}$. Subject to the evolution on $M_h^n \subset T^*(\mathbf{R}^n)$ one can easily obtain from (19) that

$$dt_i = \sum_{j=1}^n (\partial w_j(\mu_j; h)/\partial h_i) d\mu_j \tag{23}$$

at $dh_i = 0$ for all $i = \overline{1,n}$, giving rise to a global τ-parametrization of the set of circles $\otimes_{j=1}^n \mathbf{S}_j^1 \subset \otimes_{j=1}^n \Gamma_h^{(j)}$. That is, one can define some inverse algebraic functions to Abelian type integrals (22) as $\mu = \mu(\tau; h)$, where as before, $\tau = (t_1, t_2, ..., t_n) \in \mathbf{R}^n$ is a vector of evolution parameters. Recall now the expressions (12) for the integral submanifold mapping $\pi_h : M_h^n \rightarrow T^*(\mathbf{R}^n)$, one can at last write down the "quadratures" mappings for the evolutions on $M_h^n \subset T^*(\mathbf{R}^n)$ as follows: $q = q(\mu(\tau; h)) = \tilde{q}(\tau; h)$, $p = p(\mu(\tau; h)) = \tilde{p}(\tau; h)$, where obviously, a vector $(\tilde{q}, \tilde{p}) \in T^*(\mathbf{R}^n)$ is quasiperiodic in each variable $t_i \in \tau$, $i = \overline{1,n}$.

THEOREM 1. Every completely integrable Hamiltonian system admitting an algebraic submanifold $M_h^n \subset T^*(\mathbf{R}^n)$ possesses a separable canonical transformation (18) which is described by differential algebraic Picard-Fuchs type equations whose solutions are algebraic curves (21).

Therefore, the main ingredient of the scheme of integrability by quadratures is finding the Picard-Fuchs type equations (16) corresponding to the integral submanifold imbedding mapping (12) which depends in general on $\mathbf{R}^n \ni h$-parameters, and then integrating them to curves (21) carrying separable variables.

Similar to the differential-geometric approach developed in [6], one can find 1-forms $h_j^{(1)} \in \Lambda^1(T^*(\mathbf{R}^n))$, $j = \overline{1,n}$, enjoying the following identity on $T^*(\mathbf{R}^n) : \omega^{(2)}(q,p) :=$

$\sum_{j=1}^{n} dp_j \wedge dq_j = \sum_{j=1}^{n} dH_j \wedge h_j^{(1)}$. The 1-forms $h_j^{(1)} \in \Lambda^1(T^*(\mathbf{R}^n))$, $j = \overline{1,n}$, possess the following important properties when they are pulled back to the integral submanifold (1): $\pi_h^* h_j^{(1)} := \bar{h}_j^{(1)} = dt_j$, where $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, and $\pi_{h*} d/dt_j = K_j \cdot \pi_h$ for all $j = \overline{1,n}$. The above expressions combined with (23) give rise easily to the following set of relationships

$$\bar{h}_j^{(1)} = \sum_{j=1}^{n} (\partial w_j(\mu_j; h)/\partial h_i) d\mu_j \tag{24}$$

at $dh_j = 0$ on $M_h^n \backsimeq \otimes_{j=1}^n \mathbf{S}_j^1 \subset \otimes_{j=1}^n \Gamma_h^{(j)}$ for all $j = \overline{1,n}$. Since we are interested in the integral submanifold imbedding mapping (12) being locally diffeomorphic in a neighborhood $U(M_h^n) \subset T^*(\mathbf{R}^n)$, the Jacobian $\det ||\partial q(\mu; h)/\partial \mu|| \neq 0$ almost every where in $U(M_h^n)$. On the other hand, as was proved in [4], the set of 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, can be represented in $U(M_h^n)$ as

$$\bar{h}_j^{(1)} = \sum_{k=1}^{n} \bar{h}_{jk}^{(1)}(q,p) dq_k \Big|_{M_h^n}, \tag{25}$$

where $\bar{h}_{jk}^{(1)} : T^*(\mathbf{R}^n) \to \mathbf{R}$, $k, j = \overline{1,n}$, are some algebraic expressions of their arguments. Thereby, one easily finds from (25) and (24) that

$$\partial w_i(\mu_i; h)/\partial h_j = \sum_{k=1}^{n} \bar{h}_{jk}^{(1)}(q(\mu; h), p(\mu; h))(\partial q_k(\mu; h)/\partial \mu_i) \tag{26}$$

for all $i, j = \overline{1,n}$. Subject to the p-variables in (26) we must, in view of (20), use the expressions

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} p_j(\mu; h)(\partial q_j/\partial \mu_s) = w_s + \partial \mathcal{L}_\mu(h)/\partial \mu_s, \\ &\partial \mathcal{L}_\mu(h)/\partial h_j = \langle p, \partial q/\partial h_j \rangle|_{M_h^n}, \end{aligned} \tag{27}$$

being true for $s = \overline{1,n}$ and all $\mu \in \otimes_{j=1}^n \mathbf{S}_j$, $h \in \mathbf{R}^n$ in the neighborhood $U(M_h^n) \subset T^*(\mathbf{R}^n)$ chosen before. Thereby, we arrive at the following

$$\partial w_i(\mu_i; h)/\partial h_j = \bar{P}_{ji}(\mu, w; h), \tag{28}$$

where the expressions $\bar{P}_{ji}(\mu, w; h) := \sum_{k=1}^{n} \bar{h}_{jk}^{(1)}(q(\mu; h), p(\mu; h)) \partial q_k/\partial \mu_i)$, $i, j = \overline{1,n}$, depend only on $\Gamma_h^{(i)} \ni (\mu_i, w_i)$-variables for each $i \in \{\overline{1,n}\}$ and all $j = \overline{1,n}$. This condition can be written down as follows:

$$\partial \bar{P}_{ji}(\mu, w; h)/\partial \mu_k = 0, \tag{29}$$

for $j, i \neq k \in \{\overline{1,n}\}$ at almost all $\mu \in \otimes_{j=1}^n \mathbf{S}_j^1$.

THEOREM 2. Let there be given a completely integrable Hamiltonian system on the coadjoint manifold $T^*(\mathbf{R}^n)$ whose integral submanifold $M_h^n \subset T^*(R^n)$ is described by the Picard-Fuchs type algebraic equations (28). The corresponding imbedding mapping $\pi_h : M_h^n \to T^*(R^n)$ defined in (12) is a solution of a compatibility condition subject to the differential-algebraic relationships (29) on the canonical transformations generating function (17).

To show that the scheme described above really leads to an algorithmic procedure for constructing the Picard-Fuchs type equations (28) and the corresponding integral submanifold imbedding mapping $\pi_h : M_h^n \to T^*(\mathbf{R}^n)$ in the form (12), we apply it to a so called truncated Focker-Plank Hamiltonian system on the canonically symplectic cotangent space $T^*(\mathbf{R}^n)$.

Consider the following dynamical system on the canonically symplectic phase space $T^*(\mathbf{R}^2)$:

$$\left.\begin{array}{l} dq_1/dt = p_1 + \alpha(q_1+p_2)(q_2+p_1), \\ dq_2/dt = p_2, \\ dp_1/dt = -(q_1+p_2) - \alpha[q_2p_1 + 1/2(p_1^2+p_2^2+q_2^2)], \\ dp_2/dt = -(q_2+p_1), \end{array}\right\} = K_1(q,p), \tag{30}$$

where $K_1 : T^*(\mathbf{R}^2) \to T(T^*(\mathbf{R}^2))$ is the corresponding vector field on $T^*(\mathbf{R}^2) \ni (q,p)$, and $t \in \mathbf{R}$ is an evolution parameter, called a truncated four-dimensional Focker-Plank flow. It is easy to verify that the functions $H_j : T^*(\mathbf{R}^2) \to \mathbf{R}$, $j = \overline{1,2}$, where

$$\begin{array}{l} H_1 = 1/2(p_1^2+p_2^2+q_1^2) + q_1p_2 + \alpha(q_1+p_2)[q_2p_1 + 1/2(p_1^2+p_2^2+q_2^2)], \\ H_2 = 1/2(p_1^2+p_2^2+q_2^2) + q_2p_1, \end{array} \tag{31}$$

are functionally independent invariants with respect to the flow (30). Moreover, the invariant (31) is the Hamiltonian function for (30), that is, the relationship $i_{K_1}\omega^{(2)} = -dH_1$ holds on $T^*(\mathbf{R}^2)$, where the symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(\mathbf{R}^2))$ is given as follows: $\omega^{(2)} := d(pr^*\alpha^{(1)}) = \sum_{j=1}^2 dp_j \wedge dq_j$, with $\alpha^{(1)} \in \Lambda^1(\mathbf{R}^2)$ to be the canonical Liouville form on $\mathbf{R}^2 : \alpha^{(1)}(q;p) = \sum_{j=1}^2 p_j dq_j$ for any $(q,p) \in T^*(\mathbf{R}^2) \simeq \Lambda^1(\mathbf{R}^2)$.

The invariants (31) commute with each other subject to the associated Poisson bracket on $T^*(\mathbf{R}^2) : \{H_1, H_2\} = 0$. Thereby, in view of the abelian Liouville-Arnold theorem [1, 2], the dynamical system (30) is completely integrable by quadratures on $T^*(\mathbf{R}^2)$, and we can apply our scheme to the commuting invariants (31) subject to the symplectic structure $\omega^{(2)} \in \Lambda^2(\mathbf{R}^2)$. One easily states that $\omega^{(2)} = \sum_{i=1}^2 dH_i \wedge h_i^{(1)}$, where the corresponding 1-forms $\pi_h^* h_i^{(1)} := \bar{h}_i^{(1)} \in \Lambda^1(M_h^2)$, $i = \overline{1,2}$, are given as

$$\begin{array}{l} \bar{h}_1^{(1)} = \frac{p_2 dq_1 - (p_1+q_2)dq_2}{p_1p_2 - (p_1+q_2)(q_1+p_2) - \alpha h_2(p_1+q_2)}, \\ \bar{h}_2^{(1)} = \frac{-[(q_1+p_2)(1+\alpha p_2)+\alpha h_2]dq_1 + (p_1+\alpha[h_2+(q_2+p_1)(\ q_1+p_2)])dq_2}{p_1p_2 - (q_2+p_1)(\alpha h_2+q_1+p_2)}, \end{array} \tag{32}$$

and an invariant submanifold $M_h^2 \subset T^*(\mathbf{R}^2)$ is defined as

$$M_h^2 := \left\{(q,p) \in T^*(\mathbf{R}^2) : \ H_i(q,p) = h_i \in \mathbf{R}, i = \overline{1,2}\right\}$$

for some parameters $h \in \mathbf{R}^2$. Based now on expressions (32) and (18), we can easily construct functions $\bar{P}_{ij}(w;h)$, $i,j = \overline{1,2}$, in (28), defined on $T^*(M_h^2) \simeq T^*(\otimes_{j=1}^2 \mathbf{S}_j^1)$ subject to the integral submanifold imbedding mapping $\pi_h : M_h^2 \to T^*(\mathbf{R}^2)$ in coordinates $\mu \in \otimes_{j=1}^2 \mathbf{S}_j^1 \subset \otimes_{j=1}^2 \Gamma_h^{(j)}$, which we will not write down in detail due to their cumbersome form. Having applied then the criterion (29), we arrive at the following compatibility relationships subject to the mappings $q : (\otimes_{j=1}^{2}\mathbf{S}_j^1) \times \mathbf{R}^2 \to \mathbf{R}^2$ and $p : (\otimes_{j=1}^{2}\mathbf{S}_j^1) \times \mathbf{R}^2 \to T_q^*(\mathbf{R}^2)$:

$$\begin{array}{l} \partial q_1/\partial\mu_1 - \partial q_2/\partial\mu_2 = 0, \\ w_1\partial\mathcal{L}_\mu/\partial w_1 - w_2\partial\mathcal{L}_\mu/\partial w_2 = 0, \\ \partial^2 q_1/\partial\mu_2\partial h_2 + \partial^2 w_2/\partial\mu_2\partial h_2 = 0, \\ w_1\partial w_1/\partial h_1 - w_2\partial w_2/\partial h_2 = 0, \\ \partial w_1/\partial h_1(\partial q_1/\partial h_1) = \partial w_2/\partial h_1(\partial q_2/\partial h_1), \\ \partial(w_1\partial w_1/\partial h_2)/\partial h_2 - \alpha^2\partial q_1/\partial\mu_1 = 0, \end{array} \tag{33}$$

and so on, subject to variables $\mu \in \otimes_{j=1}^{2}\mathbf{S}_j^1$ and $h \in \mathbf{R}^2$. Solving all equations like (33), one can find right away that the expressions

$$\begin{array}{l} p_1 = w_1, \ \ p_2 = w_2, \ q_1 = c_1 + \mu_1 - w_2(\mu_2; h), \\ q_2 = c_2 + \mu_2 - w_1(\mu_1; h), \ \mathcal{L}_\mu(h) = -w_1 w_2, \end{array} \tag{34}$$

where $c_j(h_1, h_2) \in \mathbf{R}^1$, $j = \overline{1,2}$, are constant, hold on $T^*(M_h^2)$, giving rise to the following Picard-Fuchs type equations in the form (28):

$$\begin{array}{l} \partial w_1(\mu_1; h)/\partial h_1 = 1/w_1, \\ \partial w_1(\mu_1; h)/\partial h_2 = \alpha^2 h_2/w_1, \\ \partial w_2(\mu_2; h)/\partial h_1 = 0, \\ \partial w_2(\mu_2; h)/\partial h_2 = 1/w_2. \end{array} \tag{35}$$

The Picard-Fuchs equations (35) can be easily integrated by quadratures as follows:

$$w_1^2 + k_1(\mu_1) - \alpha^2 h_2 - 2h_1 = 0, w_2^2 + k_2(\mu_2) - 2h_2 = 0, \tag{36}$$

where $k_j : \mathbf{S}_j^1 \to \mathbf{C}$, $j = \overline{1,2}$, are still unknown functions. To determine these functions, it is necessary to substitute (34) into expressions (31), and make use of (36) to reach the following results: $k_1 = \mu_1^2$, $k_2 = \mu_2^2$ under the condition that $c_1 = -\alpha h_2$, $c_2 = 0$. Thereby, we have constructed the corresponding algebraic curves $\Gamma_h^{(j)}$, $j = \overline{1,2}$, in the form (21):

$$\Gamma_h^{(1)} := \{(\lambda, w_1) : w_1^2 + \lambda^2 - \alpha^2 h_2^2 - 2h_1) = 0\}, \ \Gamma_h^{(2)} := \{(\lambda, w_2) : w_2^2 + \lambda^2 - 2h_2 = 0\}, \tag{37}$$

where $(\lambda, w_j) \in \mathbf{C}\times\mathbf{C}$, $j = \overline{1,2}$, and $h \in \mathbf{R}^2$ are arbitrary parameters. Making use now expressions (37) and (34), we can construct in explicit form the integral submanifold imbedding mapping $\pi_h : M_h^2 \to T^*(\mathbf{R}^2)$ for the flow (30):

$$\begin{array}{l} q_1 = \mu_1 - \sqrt{2h_2 - \mu_2^2} - \alpha h_2^2, \ p_1 = w_1(\mu_1; h), \\ q_2 = \mu_2 - \sqrt{2h_1 - \alpha^2 h_2^2 - \mu_1^2}, \ \ p_2 = w_2(\mu_2; h), \end{array} \tag{38}$$

where $(\mu, w) \in \otimes_{j=1}^{2}\Gamma_h^{(j)}$. As was mentioned before, the formulas in (38) together with the explicit expressions (20) make it possible right away to find solutions to the truncated Focker-Plank flow (30) by quadratures, thereby completing its integrability.

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