Applied Mathematics E-Notes, 1(2001), 65-72 ©
Available free at mirror sites of http://math2.math.nthu.edu.tw/∼amen/

On Nonresonance Impulsive Functional Differential Equations with Periodic Boundary Conditions *

Mouffak Benchohra†, Paul W. Eloe‡

Received 27 March 2001

Abstract

In this paper a fixed point theorem due to Schaefer is used to investigate the existence of solutions for first order nonresonance impulsive functional differential equations in Banach spaces with periodic boundary conditions.

1 Introduction

This paper is concerned with the existence of solutions for the nonresonance boundary value problem for functional differential equations with impulsive effects

$$y'(t) - \lambda y(t) = f(t, y_t),\ t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m, \tag{2}$$

$$y(t) = y(0),\ t \in J_0,\ My(0) - Ny(T) = 0, \tag{3}$$

where $\lambda \in R$, $f : J \times C(J_0, E) \to E$ is a given function, $J_0 = [-r, 0]$, $0 < r < \infty$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_1, ..., I_m \in C(E, E)$ are bounded, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, E a real Banach space with norm $|\cdot|$, and M and N are constant. Note that if $M = N = 1$, then (3) represents periodic boundary conditions. For notational purposes, let $t_{-1} = -r$.

For any continuous function y defined on $[-r, T] - \{t_1, \dots, t_m\}$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t + \theta),\quad \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, especially in the biological or medical domain see; the monographs of Bainov and Simeonov [2], Lakshmikantham,

*Mathematics Subject Classifications: 34A37, 34G20, 34K25.

†Department of Mathematics, University of Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie
‡Department of Mathematics, University of Dayton, Dayton, OH 45469-2316, U. S. A.

Bainov and Simeonov [10], and Samoilenko and Perestyuk [13], and the papers of Agur et al. [1], Goldbeter et al. [6].

Recently an extension to functional differential equations with impulsive effects has been done in [17] by using the coincidence degree theory. For other results on functional differential equations we refer the interested reader to the monograph of Erbe, Kong and Zhang [5], Hale [7], Henderson [8], and the survey paper of Ntouyas [12].

The fundamental tools used in the existence proofs of all above mentioned works are essentially fixed point arguments, nonlinear alternative, topological transversality [3], degree theory [11] or the monotone method combined with upper and lower solutions [4], [9].

This paper will be divided into three sections. In Section 2 we will recall briefly some basic definitions and preliminary facts which will be used throughout Section 3. In Section 3 we shall establish an existence theorem for (1)–(3). We consider the case when $\lambda \neq 0$. Note that when the impulses are absent (i.e. for $I_k \equiv 0,\ k = 1, \ldots, m$), then the problem (1)–(3) is a *nonresonance problem* since the linear part in equation (1) is invertible. Our approach is based on a fixed point theorem due to Schaefer [14] (see also, Smart [15]).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. $C(J_0, E)$ is the Banach space of all continuous functions from J_0 into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_J = \sup\{|y(t)| : t \in J\}.$$

A measurable function $y : J \to E$ is Bochner integrable if, and only if, $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [16]). $L^1(J, E)$ denotes the Banach space of functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|dt \quad \text{for all} \quad y \in L^1(J, E).$$

We introduce some notation in order to define the solution of (1)–(3). Suppose $y : [-r, T] \to E$ and each $y(t_k^-)$ and $y(t_k^+)$ exist, $k = 1, \ldots, m$. By convention, set $y(t_k^-) = y(t_k)$ for $k = 1, \ldots, m$. Let y_k denote the restriction of y to $J_k = [t_{k-1}, t_k]$ in the following sense. If $t \in (t_{k-1}, t_k]$, then $y_k(t) = y(t)$. If $t = t_{k-1}$, then $y_k(t_{k-1}) = y(t_{k-1}^+)$. Define

$$\Psi = \{y : [-r, T] \to E |\ y_k \in C(J_k, E),\ 0 \leq k \leq m+1, \text{ and } y(t) = y(0),\ t \in J_0\}.$$

Ψ is a Banach space with the norm

$$\|y\|_\Psi = \max\{\|y_k\|_k |\ k = 0, \ldots, m+1\},$$

where $||\cdot||_k$ denotes the supremum norm on J_k, $k = 0, \ldots, m+1$.

We shall also consider the set

$$\Psi^1 = \{y : [-r, T] \to E|\ y_k \in W^{1,1}(J_k, E),\ 1 \le k \le m+1, \text{ and } y(t) = y(0),\ t \in J_0\}$$

The set Ψ^1 is a Banach space with the norm

$$\|y\|_{\Psi^1} = \max\{\|y_k\|_{W^{1,1}(J_k,E)} |\ k = 1, \ldots, m+1\}.$$

A map $f : J \times C(J_0, E) \longrightarrow E$ is said to be L^1-Carathéodory if (i) $t \longmapsto f(t, u)$ is measurable for each $u \in C(J_0, E)$; (ii) $u \longmapsto f(t, u)$ is continuous for almost all $t \in J$; and (iii) for each $k > 0$, there exists $g_k \in L^1(J, R_+)$ such that $|f(t, u)| \le g_k(t)$ for all $\|u\| \le k$ and almost all $t \in J$.

We now define a solution of problem (1)–(3). A function $y \in \Psi \cap \Psi^1$ is said to be a solution of (1)–(3) if y satisfies the equation $y'(t) - \lambda y(t) = f(t, y_t)$ a.e. on $J - \{t_1, \ldots t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \ldots, m,\ \ y(t) = y(0)$ for all $t \in J_0$, and $My(0) - Ny(T) = 0$.

Our main result is based on the following:

LEMMA 1 (See also [15], p. 29). Let S be a convex subset of a normed linear space X and assume $0 \in S$. Let $K : S \to S$ be a completely continuous operator, and let

$$\Phi(K) = \{y \in S : y = \mu K(y) \ \text{ for some } \ 0 < \mu < 1\}.$$

Then either $\Phi(K)$ is unbounded or K has a fixed point.

We now consider the following "linear problem" (4), (2), (3), where (4) is the equation

$$y'(t) - \lambda y(t) = g(t), \quad t \ne t_k,\ k = 1, \ldots, m, \tag{4}$$

where $g \in L^1(J_k, E)$, $k = 1, \ldots, m$. For short, we shall refer to (4), (2), (3) as (LP). Note that (LP) is not really a linear problem since the impulsive functions are not necessarily linear. However, if I_k, $k = 1, \ldots, m$, are linear, then (LP) is a linear impulsive problem.

We state and prove the following auxiliary result. Eloe and Henderson [4] have constructed the analogous Green's function for the problem (1), (2), (3) in the case of n-dimensional systems. The proof here gives an alternate development. In the development we require that N be a nonzero constant, although the conclusion of the lemma is valid in the case $N = 0$.

LEMMA 2. $y \in \Psi^1$ is a solution of (LP), if and only if $y \in \Psi$ is a solution of the following impulsive integral equation

$$y(t) = \begin{cases} y(0) & t \in J_0 \\ \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)) & t \in J \end{cases}, \tag{5}$$

where

$$H(t,s) = (M - Ne^{\lambda T})^{-1} \begin{cases} Me^{-\lambda(s-t)} & 0 \le s \le t \le T \\ Ne^{\lambda T}e^{-\lambda(s-t)} & 0 \le t < s \le T \end{cases}. \tag{6}$$

PROOF. We prove only one of the implications. Suppose that $y \in \Psi^1$ is a solution of (LP). Then

$$y' - \lambda y = g(t), \quad t \neq t_k,$$

i.e.,

$$(e^{-\lambda t} y(t))' = e^{-\lambda t} g(t), \quad t \neq t_k. \tag{7}$$

Assume that $t_k < t \leq t_{k+1},\ k = 0, ..., m$. By integration of (7) we obtain

$$e^{-\lambda t_{i+1}} y(t_{i+1}) - e^{-\lambda t_i^+} y(t_i^+) = \int_{t_i}^{t_{i+1}} e^{-\lambda s} g(s) ds, \quad i = 0, \ldots, k-1.$$

Adding appropriate terms, we obtain

$$e^{-\lambda t} y(t) - y(0) = \sum_{0 < t_k < t} e^{-\lambda t_k} (y(t_k^+) - y(t_k)) + \int_0^t e^{-\lambda s} g(s) ds. \tag{8}$$

Thus,

$$y(T) = e^{\lambda T} \left[y(0) + \sum_{k=1}^{m} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^T e^{-\lambda s} g(s) ds \right].$$

Substitute this expression into (3) to obtain

$$y(0) = (M - Ne^{\lambda T})^{-1} N e^{\lambda T} \left[\sum_{k=1}^{m} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^T e^{-\lambda s} g(s) ds \right]. \tag{9}$$

Substitute (9) into (8) to obtain

$$\begin{aligned} e^{-\lambda t} y(t) &= (M - Ne^{\lambda T})^{-1} N e^{\lambda T} \left[\sum_{k=1}^{m} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^T e^{-\lambda s} g(s) ds \right] \\ &\quad + \sum_{0 < t_k < t} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^t e^{-\lambda s} g(s) ds. \end{aligned} \tag{10}$$

Now employ (10) to obtain

$$\begin{aligned} e^{-\lambda t} y(t) &= (M - Ne^{\lambda T})^{-1} N e^{\lambda T} \Bigg[\sum_{0 < t_k < t} e^{-\lambda t_k} I_k(y(t_k)) + \sum_{t \leq t_k < T} e^{-\lambda t_k} I_k(y(t_k)) \\ &\quad + \int_0^T e^{-\lambda s} g(s) ds + (M - Ne^{\lambda T})(Ne^{\lambda T})^{-1} \sum_{0 < t_k < t} e^{-\lambda t_k} I_k(y(t_k)) \\ &\quad + (M - Ne^{\lambda T})(Ne^{\lambda T})^{-1} \int_0^t e^{-\lambda s} g(s) ds \Bigg] \\ &= (M - Ne^{\lambda T})^{-1} N e^{\lambda T} \Bigg[MN^{-1} e^{-\lambda T} \sum_{0 < t_k < t} e^{-\lambda t_k} I_k(y(t_k)) \\ &\quad + \sum_{t \leq t_k < T} e^{-\lambda t_k} I_k(y(t_k)) + MN^{-1} e^{-\lambda T} \int_0^t e^{-\lambda s} g(s) ds + \int_t^T e^{-\lambda s} g(s) ds \Bigg]. \end{aligned}$$

Thus

$$\begin{aligned} y(t) &= (M - Ne^{-\lambda T})^{-1}\Big[M\int_0^t e^{-\lambda(s-t)}g(s)ds + N\int_t^T e^{-\lambda(s-t-T)}g(s)ds \\ &\quad + M\sum_{0<t_k<t} e^{-\lambda(t_k-t)}I_k(y(t_k)) + N\sum_{t\le t_k<T} e^{-\lambda(t_k-t-T)}I_k(y(t_k))\Big] \\ &= \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)). \end{aligned}$$

3 Main Result

We are now in a position to state and prove our existence result for the problem (1)–(3). For the study of this problem we first list the following hypotheses:

(H1) $f : J \times C(J_0, E) \longrightarrow E$ is an L^1-Carathéodory map;

(H2) there exist constants c_k such that $|I_k(y)| \le c_k,\ k = 1, ..., m$ for each $y \in E$;

(H3) there exists $m \in L^1(J, R)$ such that

$$|f(t, y_t)| \le m(t) \quad \text{for almost all } t \in J \text{ and all } y \in \Psi;$$

(H4) for each bounded $B \subset \Psi$ and $t \in J$ the set

$$\left\{\int_0^T H(t,s)f(s,y_s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)) : y \in B\right\}$$

is relatively compact in E.

REMARK. (i) If the dimension of E is finite then (H4) is trivially satisfied. (ii) Condition (H4) is satisfied if for each $t \in J$ the map $C(J_0, E) \to E : u \longmapsto f(t, u)$ sends bounded sets into relatively compact sets.

THEOREM 1. Assume that hypotheses (H1)-(H4) hold. Then the problem (1)–(3) has at least one solution on J_1.

PROOF. Transform the problem into a fixed point problem. Consider the operator, $K : \Psi \to \Psi$ defined by:

$$(Ky)(t) = \begin{cases} y(0) & t \in J_0 \\ \int_0^T H(t,s)f(s,y_s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)) & t \in J \end{cases}.$$

Then clearly from Lemma 2 the fixed points of K are solutions to (1)–(3). We shall show that K satisfies the assumptions of Lemma 1. The proof will be given in several steps.

Step 1: K maps bounded sets into bounded sets in Ψ. Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in \Psi : \|y\|_\Psi \leq q\}$ one has $\|Ky\|_\Psi \leq \ell$. Let $y \in B_q$, then for each $t \in J$, we have

$$(Ky)(t) = \int_0^T H(t,s)f(s,y_s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)).$$

By (H1) we have for each $t \in J$,

$$\begin{aligned} |(Ky)(t)| &\leq \int_0^T |H(t,s)||f(s,y_s)|ds + \sum_{k=1}^m |H(t,t_k)||I_k(y(t_k))| \\ &\leq \int_0^T |H(t,s)||g_q(s)|ds + \sum_{k=1}^m |H(t,t_k)| \sup\{|I_k(y)| : \|y\|_\Psi \leq q\}. \end{aligned}$$

Then for each $h \in K(B_q)$ we have

$$\begin{aligned} \|h\|_\Psi &\leq \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T |g_q(s)|ds + \sum_{k=1}^m \sup_{t\in J} |H(t,t_k)| \sup\{|I_k(y)| : \|y\|_\Psi \leq q\} \\ &= \ell. \end{aligned}$$

Step 2: K maps bounded sets into equicontinuous sets of Ψ. Indeed, let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$ and B_q be a bounded set of Ψ as in Step 1. Let $y \in B_q$. Then

$$\begin{aligned} &|(Ky)(\tau_2) - (Ky)(\tau_1)| \\ \leq &\int_0^T |H(\tau_2,s) - H(\tau_1,s)||g_q(s)|ds + \sum_{k=1}^m |H(\tau_2,t_k) - H(\tau_1,t_k)|c_k. \end{aligned}$$

As $\tau_2 \to \tau_1$ the right-hand side of the above inequality tends to zero.

Step 3: $K : \Psi \to \Psi$ is continuous. Indeed, let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ψ. Then there is an integer q such that $\|y_n\|_\Psi \leq q$ for all $n = 0, 1, 2, ...$ and $\|y\|_\Psi \leq q$, so $y_n \in B_q$ and $y \in B_q$. We have then by the dominated convergence theorem

$$\begin{aligned} \|Ky_n - Ky\|_\Psi &\leq \sup_{t\in J}\left[\int_0^T |H(t,s)||f(s,y_{ns}) - f(s,y_s)|ds \right. \\ &\qquad \left. + \sum_{k=1}^m |H(t,t_k)||I_k(y_n(t_k)) - I_k(y(t_k))|\right] \\ &\to 0. \end{aligned}$$

Thus K is continuous.

As a consequence of Steps 1 to 3 and (H4) together with the Arzela-Ascoli theorem we can conclude that $K : \Psi \to \Psi$ is completely continuous.

Step 4: The set

$$\Phi(K) := \{y \in \Psi : y = \mu K(y), \quad \textit{for some} \quad 0 < \mu < 1\}$$

is bounded. Indeed, let $y \in \Phi(K)$. Then $y = \mu K(y)$ for some $0 < \mu < 1$. Thus for each $t \in J$

$$y(t) = \mu \int_0^T H(t,s) f(s, y_s) ds + \mu \sum_{k=1}^{m} H(t, t_k) I_k(y(t_k)).$$

This implies by (H2)-(H3) that for each $t \in J$ we have

$$\begin{aligned} |y(t)| &\leq \int_0^T |H(t,s) f(s, y_s)| ds + \sum_{k=1}^{m} |H(t, t_k) I_k(y(t_k))| \\ &\leq \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T m(s) ds + \sum_{k=1}^{m} \sup_{t\in J} |H(t, t_k)| \, c_k \\ &= b, \end{aligned}$$

where b is independent of y. This shows that $\Phi(K)$ is bounded.

Set $X := \Psi$. As a consequence of Lemma 1 we deduce that K has a fixed point which is a solution of (1)–(3). The proof is complete.

Clearly, hypothesis (H3) is a strong hypothesis. Now that the alternative method due to Schaefer [14] has been established, standard hypotheses to obtain a priori bounds on solutions can be applied. For example, since $H(t,s) \leq Me^{|\mu|T}$ for some positive constant M, if $|f(t,x)| \leq g(t)|x|$ on $[0,T] \times R$, then a standard Gronwall inequality can be applied to obtain a priori bounds on solutions.

References

[1] Z. Agur, L. Cojocaru, G. Mazur, R.M. Anderson, and Y.L. Danon, Pulse mass measles vaccination across age cohorts, Proc. Nat. Acad. Sci. USA 90(1993), 11698-11702.

[2] D. D. Bainov, and P.S. Simeonov, Systems with Impulse Effect, Ellis Horwood Ltd., Chichister, 1989.

[3] J. Dugundji and A. Granas, Fixed Point Theory, Monografie Mat. PWN, Warsaw, 1982.

[4] P. W. Eloe and J. Henderson, A boundary value problem for a system of ordinary differential equations with impulse, Rocky Mountain J. Math. 27(1997), 785-799.

[5] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation Theory for Functional Differential Equations, Pure and Applied Mathematics, 1994.

[6] A. Goldbeter, Y. X. Li, and G. Dupont, Pulsatile signalling in intercellular communication: Experimental and theoretical aspects. Mathematics applied to Biology and Medicine, Werz. Pub. Winnipeg, Canada (1993), 429-439.

[7] J. K. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.

[8] J. Henderson, Boundary Value Problems for Functional Differenial Equations, World Scientific, Singapore, 1995.

[9] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, Boston, 1985.

[10] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[11] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems, in "Regional Conference Series in Math.", Vol. 40, Amer. Math. Soc., Providence, RI, 1979.

[12] S. K. Ntouyas, Initial and boundary value problems for functional differential equations via the topological transversality method: A survey, Bull. Greek Math. Soc. 40(1998), 3-41.

[13] A. M. Samoilenko, and N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[14] H. Schaefer, Uber die Methode der a priori Schranken, Math. Ann. 129(1955), 415-416.

[15] D. R. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, 1974.

[16] K. Yosida, Functional Analysis, 6th ed., Springer-Verlag, Berlin, 1980.

[17] Y. J. Dong, Periodic boundary value problems for functional differential equations with impulses, J. Math. Anal. Appl. 210(1997), 170-181.