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Inequalities Involving Hadamard Products of Hermitian Matrices *†

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Abstract

We prove an inequality for Hermitian matrices, and thereby extend several inequalities involving Hadamard products of Hermitian matrices.

Let $C^{m\times n}$ denote the set of $m \times n$ complex matrices. Let H_m be the set of all nonsingular Hermitian matrices of order m. For two matrices A and B in H_m, $A > B$ ($\geq B$) or $B < A$ ($\leq A$) means $A - B$ is positive definite (respectively semidefinite). Let $\circ$ and indicate respectively the Hadamard and Kronecker products (see e.g. [3, 6]). For a positive integer n, let $\langle n\rangle = \{1, \cdots, n\}$. Let $A \in C^{m\times n}$. For nonempty index sets $\alpha \subset \langle m\rangle$ and $\beta \subset \langle n\rangle$, we denote by $A(\alpha, \beta)$ the submatrix of A lying in rows α and columns β. If $\alpha \subset \langle m\rangle \cap \langle n\rangle$, then the submatrix $A(\alpha, \alpha)$ is abbreviated by $A(\alpha)$. Let $\alpha \subset \langle m\rangle \cap \langle n\rangle$, $\alpha_1 = \langle m\rangle \setminus \alpha$ and $\alpha_2 = \langle n\rangle \setminus \alpha$. If $A(\alpha)$ is nonsingular, then

$$A/\alpha = A(\alpha_1, \alpha_2) - A(\alpha_1, \alpha)\,[A(\alpha)]^{-1}\,A(\alpha, \alpha_2)$$

is called the Schur complement of $A(\alpha)$ in A. We denote by I_n the $n \times n$ identity matrix, and by I when the order is clear.

The following result is well known (see for instance [2, Theorem 7.7.9 (a)]).

THEOREM A. Let $A, B \in H_m$ be positive definite matrices. Then

$$(A \circ B)^{-1} \leq A^{-1} \circ B^{-1}. \tag{1}$$

Wang and Zhang in [9, Theorem 1] and Zhan in [8, Theorem 2] obtained the following extension of Theorem A.

THEOREM B. Let $A, B \in H_m$ be positive definite matrices. For any positive integer n and any $C, D \in C^{m\times n}$, we have

$$(C^* \circ D^*)(A \circ B)^{-1}(C \circ D) \leq (C^* A^{-1} C) \circ (D^* B^{-1} D). \tag{2}$$

In particular, if $A = B = I$, then Theorem B becomes the following result.

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THEOREM C ([1]). For any positive integers m, n and any $C, D \in C^{m\times n}$, we have

$$(C^* \circ D^*)(C \circ D) \leq (C^*C) \circ (D^*D). \tag{3}$$

However, up to now, the equivalent conditions for equalities in (1)–(3) to hold are not known. Furthermore, the following example shows that $A > O$ and $B > O$ is not necessary for (1) to hold.

EXAMPLE 1. Let $A = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$. We have

$$A^{-1} \circ B^{-1} - (A \circ B)^{-1} = \begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix} \geq O.$$

However, $A \not\geq O$ and $B \not\geq O$.

Recently, Liu [4, Lemma 2] and Wang et al. [10, Remark 3] obtained the following extension of Theorem B.

THEOREM D. Let A,$B \in H_m$ be positive semidefinite Hermitian matrices. For any positive integer n and any $C, D \in C^{m\times n}$ that satisfy $AA^+C = C$ and $BB^+D = D$, where A^+ denotes the Moore–Penrose inverse of A, we have

$$(C^* \circ D^*)(A \circ B)^+(C \circ D) \leq (C^*A^+C) \circ (D^*B^+D). \tag{4}$$

Moreover, Wang et al. [10] showed that

$$(A \circ B)^+ \leq A^+ \circ B^+ \tag{5}$$

is not true in general.

Motivated by the works of [4], [10] and our Example, in this note, we first prove an inequality for nonsingular Hermitian matrices, and then we obtain a condition on A, B for which inequality (2) holds. Furthermore, necessary and sufficient conditions under which our inequalities become equalities are presented.

THEOREM 1. Let $\alpha \subset \langle m\rangle$, $\alpha' = \langle m\rangle \setminus \alpha$, $\beta \subset \langle n\rangle$ and $\beta' = \langle n\rangle \setminus \alpha$. If $A \in H_m$ and $A(\alpha) > O$, then

$$(C^*AC)(\beta') \geq [C(\alpha',\ \beta')]^*\, [A^{-1}(\alpha')]^{-1} C(\alpha', \beta') \tag{6}$$

for all $C \in C^{m\times n}$, and the equality holds in (6) if, and only if,

$$A(\alpha)C(\alpha, \beta') + A(\alpha, \alpha')C(\alpha', \beta') = O. \tag{7}$$

PROOF. It is easy to see that there exist permutation matrices P and R such that

$$PAP^T = \begin{pmatrix} A(\alpha) & A(\alpha, \alpha') \\ [A(\alpha, \alpha')]^* & A(\alpha') \end{pmatrix},$$

$$PCR = \begin{pmatrix} C(\alpha, \beta) & C(\alpha, \beta') \\ C(\alpha', \beta) & C(\alpha', \beta') \end{pmatrix}$$

and

$$R^T(C^*AC)R = \begin{pmatrix} (A^*AC)(\beta) & (C^*AC)(\beta,\beta') \\ [(C^*AC)(\beta,\beta')]^* & (C^*AC)(\beta') \end{pmatrix}. \tag{8}$$

Let

$$Q = \begin{pmatrix} I & -[A(\alpha)]^{-1} A(\alpha,\alpha') \\ O & I \end{pmatrix},$$

then

$$Q^*PAP^TQ = \begin{pmatrix} A(\alpha) & O \\ O & A/\alpha \end{pmatrix} \tag{9}$$

and

$$Q^{-1}PCR = \begin{pmatrix} * & X \\ C(\alpha',\beta) & C(\alpha',\beta') \end{pmatrix}, \tag{10}$$

where $X = C(\alpha,\ \beta') + [A(\alpha)]^{-1} A(\alpha,\ \alpha')C(\alpha',\ \beta')$ and $*$ denotes a block irrelevant to our discussions. It follows from [2, p.18] that

$$A/\alpha = \left((A/\alpha)^{-1}\right)^{-1} = [A^{-1}(\alpha')]^{-1}. \tag{11}$$

Note that $R^T(C^*AC)R = (Q^{-1}PCR)^*(Q^*PAP^TQ)(Q^{-1}PCR)$, by (8), (9), (10) and (11), we then have

$$\begin{aligned}(C^*AC)(\beta') &= \begin{pmatrix} X^* & [C(\alpha',\beta')]^* \end{pmatrix} \begin{pmatrix} A(\alpha) & O \\ O & A/\alpha \end{pmatrix} \begin{pmatrix} X \\ C(\alpha',\beta') \end{pmatrix} \\ &= X^*A(\alpha)X + [C(\alpha',\beta')]^*(A/\alpha)C(\alpha',\beta') \\ &= X^*A(\alpha)X + [C(\alpha',\beta')]^*[A^{-1}(\alpha')]^{-1}C(\alpha',\beta').\end{aligned}$$

This implies that (6) holds and also that equality holds in (6) if, and only if, $X^*A(\alpha)X = O$, i.e., $X = O$, or equivalently, we have (7) (as $A(\alpha) > O$). The proof is complete.

We remark that in Theorem 1, if we assume $A(\alpha) < O$ instead of $A(\alpha) > O$, then (6) becomes

$$(C^*AC)(\beta') \le [C(\alpha',\beta')]^*[A^{-1}(\alpha')]^{-1}C(\alpha',\beta')$$

for all $C \in C^{m\times n}$, and equality holds if, and only if, (7) holds.

As a special case, let $A \in H_m$ be positive definite in Theorem 1. Then by (9), A/α is positive definite and

$$\begin{aligned} Q^{-1}PA^{-1}P^T(Q^*)^{-1} &= (Q^*PAP^TQ)^{-1} = \begin{pmatrix} A(\alpha) & O \\ O & A/\alpha \end{pmatrix}^{-1} \\ &= \begin{pmatrix} [A(\alpha)]^{-1} & O \\ O & (A/\alpha)^{-1} \end{pmatrix},\end{aligned}$$

and hence

$$\begin{aligned} & PA^{-1}P^T \\ =\ & Q\begin{pmatrix} [A(\alpha)]^{-1} & O \\ O & (A/\alpha)^{-1} \end{pmatrix} Q^* \\ =\ & \begin{pmatrix} A(\alpha)^{-1} + A(\alpha)^{-1}A(\alpha,\alpha')(A/\alpha)^{-1}A(\alpha,\alpha')^*\left[A(\alpha)^{-1}\right]^* & * \\ * & * \end{pmatrix} \end{aligned} \tag{12}$$

This implies that

$$A^{-1}(\alpha) = A(\alpha)^{-1} + A(\alpha)^{-1}A(\alpha,\alpha')(A/\alpha)^{-1}A(\alpha,\alpha')^* \left[A(\alpha)^{-1}\right]^* \geq A(\alpha)^{-1}.$$

Summarizing, Theorem 1 contains the known result that the inequality $A^{-1}(\alpha) \geq A(\alpha)^{-1}$ holds for any $n \times n$ positive definite matrix A and $\alpha \subseteq \langle n \rangle$.

LEMMA 1. Let $\gamma = \{j(m+1)+1 :\ j = 0,1,\cdots,\ m-1\}$ and $\delta = \{j(n+1)+1 :\ j = 0,1,\cdots,\ n-1\}$. Then $A \circ B = (A \quad B)(\gamma,\delta)$ for any $A, B \in C^{m\times n}$.

The proof follows by a direct computation and is skipped.

THEOREM 2. Let m, n be given positive integers. Let $\gamma = \{j(m+1)+1 :\ j = 0,1,\cdots,\ m-1\}$ and $\delta = \{j(n+1)+1 :\ j = 0,1,\cdots,\ n-1\}$. Also let $\gamma' = \langle m^2 \rangle \setminus \gamma$. Let A, B be $m \times m$ nonsingular Hermitian matrices that satisfy $\left(A^{-1} \quad B^{-1}\right)(\gamma') > O$. Then for any positive integer n and any $C, D \in C^{m\times n}$, the inequality (2) holds. Furthermore, equality holds in (2) if, and only if,

$$(A^{-1} \quad B^{-1})(\gamma')(C \quad D)(\gamma',\delta) + (A^{-1} \quad B^{-1})(\gamma',\gamma)(C \circ D) = O.$$

PROOF. The fact that $(A \quad B)^{-1} = A^{-1} \quad B^{-1} \in H_{m^2}$ follows from elementary properties of the Hadamard product. Replacing α' by γ, β' by δ, A by $(A \quad B)^{-1}$ and C by $C \quad D$ in Theorem 1 respectively, we have that

$$\begin{aligned} & \left[(C \quad D)^*(A^{-1} \quad B^{-1})(C \quad D)\right](\delta) \\ \geq\ & \left[(C \quad D)(\gamma,\delta)\right]^* \left[(A \quad B)(\gamma)\right]^{-1} (C \quad D)(\gamma,\delta) \end{aligned} \qquad (13)$$

and also that equality holds in (13) if, and only if,

$$(A^{-1} \quad B^{-1})(\gamma')(C \quad D)(\gamma',\delta) + (A^{-1} \quad B^{-1})(\gamma',\gamma)(C \quad D)(\gamma,\delta) = O. \qquad (14)$$

By elementary properties of the Hadamard product and Lemma 1, we obtain

$$(A \quad B)(\gamma) = A \circ B,\ (C \quad D)(\gamma,\delta) = C \circ D \qquad (15)$$

and

$$\begin{aligned} & \left[(C^* \quad D^*)(A^{-1} \quad B^{-1})(C \quad D)\right](\delta) \\ =\ & \left[(C^*A^{-1}C) \quad (D^*B^{-1}D)\right](\delta) \\ =\ & (C^*A^{-1}C) \circ (D^*B^{-1}D). \end{aligned} \qquad (16)$$

Combining (13)-(16), the theorem follows.

COROLLARY 1. Let m, n be positive integers, and let γ, γ', δ have the same meanings as in Theorem 2. (i) Let $A, B \in H_m$ satisfy $\left(A^{-1} \quad B^{-1}\right)(\gamma') > O$. Then the inequality (1) holds. Furthermore, equality holds in (1) if, and only if, $(A \quad B)(\gamma') \oplus (A \circ B) = P^T(A \quad B)P$ for some permutation matrix P. (ii) For any $C, D \in C^{m\times n}$, the inequality (3) holds. Furthermore, equality holds in (3) if, and only if, $(C \quad D)(\gamma',\delta) = O$.

PROOF. Let $C = D = I_m$ in Theorem 2. Then the inequality (1) holds. Furthermore, equality holds in (1) if, and only if, $(A^{-1} \quad B^{-1})(\gamma',\ \gamma) = O$. Noting $A^{-1} \quad B^{-1} \in H_{m^2}$, we have $(A^{-1} \quad B^{-1})(\gamma,\ \gamma') = O$. Hence

$$P^T(A^{-1} \quad B^{-1})P = (A^{-1} \quad B^{-1})(\gamma) \oplus (A^{-1} \quad B^{-1})(\gamma')$$

for some permutation matrix P. By Lemma 1, we see that (i) holds. And (ii) follows by choosing $M = N = I$ in Theorem 2. The proof is complete.

We remark that if $A > O$ and $B > O$, then A and B automatically satisfy the assumptions of Theorem 2 and Corollary 1(i), and hence Corollary 1(i) and Theorem 2 extend respectively Theorem A and Theorem B. We also recover and complete the result of Theorem C in Corollary 1(ii).

Consider the matrices A, B in Example 1. Since

$$(A^{-1} \quad B^{-1})(\gamma') = (A^{-1} \quad B^{-1})(2,3) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

by Corollary 1(i) we have inequality (1). Furthermore, by Theorem 2, the inequality (2) also holds, for any positive integer n and any $C, D \in C^{2\times n}$. Note, however, that we cannot apply Theorems A and B to draw the same conclusions, as $A \not\geq O$ and $B \not\geq O$.

Instead of $\left(A^{-1} \quad B^{-1}\right)(\gamma') > O$, if $\left(A^{-1} \quad B^{-1}\right)(\gamma') < O$ holds in the hypotheses of Theorem 2 or Corollary 1, then the inequalities in the conclusions are reversed.

In the literature, many of the inequalities involving Hadamard products are obtained under the assumption that the matrices involved are positive definite (or positive semidefinite) (see [4], [5], [8], [9], [10]). In this paper we obtain some of these inequalities under weaker assumptions. We only require positive definiteness of certain principal submatrix of some nonsingular Hermitian matrix. However, there are still some other known matrix inequalities involving Hadamard products, such as

$$(A^{-1} \circ B^{-1})/\alpha \geq [(A \circ B)/\alpha]^{-1}$$

and

$$(A^s \circ B^s)^{1/s} \geq (A^r \circ B^r)^{1/r}$$

(for nonzero integers s, r, $s > r$), that are not considered in this paper. These inequalities are known to be valid when the matrices A, B are both positive definite. It would be of interest to find out whether they still hold under weaker assumptions.

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