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Algebraic Equations Connected with Tangential Polygons and Their Solvability by Radicals *

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Abstract

Certain algebraic equations connected with tangential polygons are considered. By means of their geometrical interpretations, we prove that these equations are solvable by radicals.

1 Introduction

A polygon with vertices $A_1, \cdots, A_n$ will be denoted by $\mathbf{A} \equiv A_1 \cdots A_n$. The lengths of its sides will be denoted by $a_j = |A_j A_{j+1}|$ and for the interior angle at vertex A_j we write

$$\angle A_j = \angle A_{j-1} A_j A_{j+1}, \ \ A_0 \equiv A_n, \ \ j = \overline{1, n}. \tag{1}$$

We say that $\mathbf{A}$ is tangential if there exists a circle $\mathcal{C}_{\mathbf{A}}$ such that each side of $\mathbf{A}$ is a tangential line of $\mathcal{C}_{\mathbf{A}}$. We will assume throughout the article that no two of the consecutive vertices are the same in $\mathbf{A}$. The centre and the radius of $\mathcal{C}_{\mathbf{A}}$ will be denoted by C and r respectively. Also, the integral part of a real number x will be denoted by $[a]$.

Consider the angles

$$\beta_j = \mu(\angle C A_j A_{j+1}), \ j = \overline{1, n}, \tag{2}$$

where $\mu(x)$ means the "measure of x". Thus, $2\beta_j = \mu(\angle A_j)$.

Let $\mathbf{A}$ be a tangential n-gon, and let k be a positive integer such that $k \leq [(n-1)/2]$. Then the polygon $\mathbf{A}$ will be called k-tangential if any two of its consecutive sides have only one point in common, and if there holds

$$\beta_1 + \cdots + \beta_n = (n - 2k)\frac{\pi}{2}. \tag{3}$$

Consequently, a tangential polygon $\mathbf{A}$ is k-tangential if $\sum_{j=1}^{n} \varphi_j = 2k\pi$, where $\varphi_j = \mu(\angle A_j C A_{j+1})$. Indeed, from $2\beta_j = \mu(\angle A_j)$ it follows that $\beta_j + \beta_{j+1} = \pi - \varphi_j$, and using (3) twice, our assertion follows.

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It is easy to see that k cannot be greater that $(n-1)/2$ for n odd and cannot be greater than $n/2-1$ when n is even.

If $\mathbf{A}$ is k-tangential and if $t_1, \cdots, t_n$ are the lengths of its tangents, then

$$t_j = r \cot \beta_j,\ j = \overline{1, n}. \tag{4}$$

Let $t_1, \cdots, t_n$ be any given lengths of some line segments (in fact any positive numbers), and fix j between 1 and n. We will let S_j^n denote the sum of all $\binom{n}{j}$ products of the form $t_{i_1} \cdots t_{i_j}$, where $i_1, \cdots i_j$ are different elements of $\{1, \cdots, n\}$, i.e.

$$S_j^n = \sum_{1 \le i_1 < \cdots < i_j \le n} t_{i_j} \cdots t_{i_j}. \tag{5}$$

Finally we employ the symbol $P(t_1, \cdots, t_n; \beta_1, \cdots, \beta_n; r)$ to denote the tangential n-gon $\mathbf{A}$ with given lengths of its tangents $t_1, \cdots, t_n$ (the angles β_j are specified already and r is the radius of $\mathcal{C}_{\mathbf{A}}$).

2 Tangential Polygon and Corresponding Equation

Corollary 3 in the article [1] asserts that the radius r_k of $\mathcal{C}_{\mathbf{A}}$ inscribed in $P(t_1, \cdots, t_n; \beta_1, \cdots, \beta_n; r_k)$ is a root of the equation

$$S_1^n x^{n-1} - S_3^n x^{n-3} + \cdots + (-1)^\nu S_n^n = 0 \tag{6}$$

if n is odd and a root of

$$S_1^n x^{n-2} - S_3^n x^{n-4} + \cdots + (-1)^\eta S_{n-1}^n = 0 \tag{7}$$

if n is even, where $\nu = (1+3+5+\cdots+n)+1$ and $\eta = (1+3+5+\cdots+(n-1))+1$. Here we shall improve this result by the following theorem.

THEOREM 2.1. Let $t_1, \cdots, t_n$ any given lengths (in fact any positive real numbers). Then there are tangential polygons

$$P(t_1, \cdots, t_n; \beta_1^{(k)}, \cdots, \beta_n^{(k)}; r_k), \qquad k = \overline{1, [(n-1)/2]},$$

where $\beta_1^{(k)} + \cdots + \beta_n^{(k)} = (n-2k)\pi/2$.

PROOF. Since $\arctan x$ is continuous it is clear that for each fixed $k = \overline{1, [(n-1)/2]}$, there is r_k which satisfies

$$\sum_{j=1}^n \arctan\left(\frac{r_k}{t_j}\right) = (n-2k)\frac{\pi}{2}.$$

Thus $\beta_j^{(k)} = \arctan(r_k/t_j)$ for $j = \overline{1, n}$.

From this result it follows that the equations (6), (7) possess real roots only.

COROLLARY 2.1. The roots x_k of the equations

$$\sum_{j=1}^{(n+1)/2} (-1)^{j+1} \binom{n}{2j-1} x^{(n-2j+1)/2} = 0, \qquad n \text{ odd}, \tag{8}$$

$$\sum_{j=1}^{n/2} (-1)^{j+1} \binom{n}{2j-1} x^{(n-2j)/2} = 0, \qquad n \text{ even}, \tag{9}$$

are of the form

$$x_k = r_k^2 = \tan^2(n-2k)\frac{\pi}{2n}, \qquad k = \overline{1, [(n-1)/2]}. \tag{10}$$

Indeed, this follows by putting $t_1 = \cdots = t_n = 1$ into (6) and (7) respectively.

COROLLARY 2.2. When n is a prime number, then equation (8) is normal over the field $\mathbf{Q}$ (of rational numbers).

PROOF. If n is a prime, and if $\nu = (1+3+\cdots+n)+1$, we may assert that the equation

$$\binom{n}{1} x^{(n-1)/2} - \binom{n}{3} x^{(n-3)/2} + \cdots + (-1)^{\nu} = 0 \tag{11}$$

is irreducible over the field $\mathbf{Q}$ because the equation

$$\binom{n}{1} - \binom{n}{3} y + \cdots + (-1)^{\nu} y^{(n-1)/2} = 0$$

is irreducible over $\mathbf{Q}$ by the Eisenstein criterion. Using (10), and the straightforward formula

$$\tan(m\alpha) = \frac{\binom{m}{1}\tan\alpha - \binom{m}{3}\tan^3\alpha + \cdots}{1 - \binom{m}{2}\tan^2\alpha + \binom{m}{4}\tan^4\alpha - \cdots},$$

it is easy to see that the rootfield of the equation (11) can be generated by only one root, that is,

$$\mathbf{Q}\left(\tan^2\frac{\pi}{2n}\right) = \mathbf{Q}\left(\tan^2\frac{\pi}{2n}, \cdots, \tan^2(n-2)\frac{\pi}{2n}\right).$$

The proof is complete.

COROLLARY 2.3. The order of the Galois group of equation (8) is $(n-1)/2$ when n is a prime, and this group is solvable.

PROOF. Since $\mathbf{Q}\left(\tan^2\frac{\pi}{2n}\right)$ has the degree $(n-1)/2$ over the field $\mathbf{Q}$, the order of the Galois group of (8) is $(n-1)/2$ too. If $\mathbf{Q}_1 = \mathbf{Q}\left(\tan^2\frac{\pi}{2n}\right)$ and $\mathbf{Q}_2 = \mathbf{Q}\left(\exp\{i\pi/(2n)\}\right)$, then $\mathbf{Q}_1$ is a subfield of $\mathbf{Q}_2$. The subfield $\mathbf{Q}_1$ is the rootfield of the equation (8) and $\mathbf{Q}_2$ is the rootfield of the equation $x^{2n}-1=0$. As is well known the Galois group of the equation $x^{2n}-1=0$ is solvable and every subgroup of a solvable group is solvable.

COROLLARY 2.4. Equations (8) and (9) are solvable by radicals for any positive integer n.

PROOF. If m is a positive integer, then $\tan \pi/(2n)$ can be expressed by radicals over $\mathbf{Q}$ since $\exp\{i\pi/(2n)\}$ can be expressed by radicals over $\mathbf{Q}$.

THEOREM 2.2. Let λ and n be given positive integers and let $t_1, \cdots, t_{\lambda n}$ be any given lengths (positive real numbers), such that

$$t_{i+jn} = t_i, \qquad i = \overline{1, n},\ j = \overline{1, \lambda - 1}. \tag{12}$$

Now, assume $\kappa = \lambda n$, and let $f_1(x), f_2(x), g_1(x), g_2(x)$ be the following polynomials:

$$\begin{aligned} f_1(x) &= S_1^\kappa x^{\kappa-1} - S_3^\kappa x^{\kappa-3} + \cdots + (-1)^{\nu_\kappa} S_\kappa^\kappa && \kappa \text{ odd} \\ f_2(x) &= S_1^\kappa x^{\kappa-2} - S_3^\kappa x^{\kappa-4} + \cdots + (-1)^{\eta_\kappa} S_{\kappa-1}^\kappa && \kappa \text{ even} \\ g_1(x) &= S_1^n x^{n-1} - S_3^n x^{n-3} + \cdots + (-1)^{\nu} S_n^n && n \text{ odd} \\ g_2(x) &= S_1^n x^{n-2} - S_3^n x^{n-4} + \cdots + (-1)^{\eta} S_{n-1}^n && n \text{ even}, \end{aligned}$$

where $\nu_\kappa = (1 + 3 + \cdots + \kappa) + 1,\ \eta_\kappa = (1 + 3 + \cdots + (\kappa - 1)) + 1$, and ν, η are defined already. Then

$$(g_1(x)\,|\,f_1(x) \text{ or } g_1(x)\,|f_2(x)) \text{ and } g_2(x)|\, f_2(x). \tag{13}$$

PROOF. As in the proof of Theorem 2.1, this theorem is almost evident since the circle $\mathcal{C}_{\mathbf{A}(1)}$ inscribed in the n-gon $\mathbf{A}(1) = A_1 \cdots A_n$ has the same radius as the circle $\mathcal{C}_{\mathbf{A}(\lambda)}$ inscribed in the λn-gon

$$\mathbf{A}(\lambda) = \underbrace{\mathbf{A}(1) \cdots \mathbf{A}(1)}_{\lambda}.$$

From

$$\lambda(\beta_1^{(k)} + \cdots + \beta_n^{(k)}) = (\lambda n - 2k)\pi/2, \qquad k = \overline{1, [(\lambda n - 1)/2]}, \tag{14}$$

we have

$$\beta_1^{(k)} + \cdots + \beta_n^{(k)} = \left(n - \frac{2k}{\lambda}\right)\frac{\pi}{2}, \qquad k = \overline{1, [(\lambda n - 1)/2]}, \tag{15}$$

and it follows that the set $\{1, \cdots, [(\lambda n - 1)/2]\}$ contains $m = [(n - 1)/2]$ numbers $k_1, \cdots, k_m$, say, that $k_j = j\lambda,\ j = \overline{1, m}$. Thus there are m tangential polygons

$$P(t_1, \cdots, t_n; \beta_1^{(k_j)}, \cdots, \beta_n^{(k_j)}; r_{k_j}), \qquad j = \overline{1, [(n - 1)/2]},$$

such that their radii are the roots of the equations

$$\begin{aligned} S_1^n x^{n-1} - S_3^n x^{n-3} + \cdots + (-1)^{\nu} S_n^n &= 0 && n \text{ odd}, \\ S_1^n x^{n-2} - S_3^n x^{n-4} + \cdots + (-1)^{\eta} S_{n-1}^n &= 0 && n \text{ even}. \end{aligned}$$

The proof is complete.

We remark that the polynomials which correspond to the angles $\left(n - \frac{2k}{\lambda}\right)\frac{\pi}{2}$ for different k may have different degrees with different parities. The following is an example.

EXAMPLE. When $\lambda = 4$ and $n = 5$, we have

$$\beta_1^{(k)} + \cdots + \beta_5^{(k)} = \left(5 - \frac{k}{2}\right)\frac{\pi}{2}, \qquad k = \overline{1,9}.$$

It follows that

$$\begin{aligned}
\tan\left(\beta_1^{(k)} + \cdots + \beta_5^{(k)}\right) &= 0, & k &= 2,6\\
\cot\left(\beta_1^{(k)} + \cdots + \beta_5^{(k)}\right) &= 0, & k &= 4,8\\
\tan\left(\beta_1^{(k)} + \cdots + \beta_5^{(k)}\right) &= 1, & k &= 1,5,9\\
\cot\left(\beta_1^{(k)} + \cdots + \beta_5^{(k)}\right) &= -1, & k &= 3,7.
\end{aligned}$$

Denote the corresponding polynomials by $p_j(x)$, $j = 1,2,3,4$. Then we have

$$\begin{aligned}
p_1(x) &= x^4 - S_2^5 x^2 + S_4^5,\\
p_2(x) &= S_1^5 x^4 - S_3^5 x^2 + S_5^5,\\
p_3(x) &= x^5 - S_1^5 x^4 - S_2^5 x^3 + S_3^5 x^2 + S_4^5 x - S_5^5,\\
p_4(x) &= x^5 + S_1^5 x^4 - S_2^5 x^3 - S_3^5 x^2 + S_4^5 x + S_5^5,
\end{aligned}$$

and consequently

$$\begin{aligned}
p_3(x)p_4(x) &= x^{10} - ((S_1^5)^2 + 2S_2^5)x^8 + (2S_4^5 + 2S_1^5 S_3^5 + (S_2^5)^2)x^6\\
&\quad -(2S_1^5 S_5^5 + 2S_2^5 S_4^5 + (S_3^5)^2)x^4 + (2S_3^5 S_5^5 + (S_4^5)^2)x^2 - (S_5^5)^2,
\end{aligned}$$

and

$$4\prod_{j=1}^{4} p_j(x) = S_1^{20} x^{18} - S_3^{20} x^{16} + S_5^{20} x^{14} - \cdots - S_{19}^{20}.$$

We remark that

$$2p_1(x)p_2(x) = S_1^{10} x^8 - S_3^{10} x^6 + S_5^{10} x^4 - S_7^{10} x^2 + S_9^{10}.$$

Therefore the following relations

$$p_2(x) | S_1^{20} x^{18} - S_3^{20} x^{16} + \cdots - S_{19}^{20}$$

and

$$2p_1(x)p_2(x) | S_1^{20} x^{18} - S_3^{20} x^{16} + \cdots - S_{19}^{20},$$

are easily seen.

COROLLARY 2.5. If $n \leq 10$, then both equations

$$S_1^\kappa x^{\kappa-1} - S_3^\kappa x^{\kappa-3} + \cdots + (-1)^{\nu_\kappa} S_\kappa^\kappa = 0, \qquad \kappa \text{ odd}, \tag{16}$$

and

$$S_1^\kappa x^{\kappa-2} - S_3^\kappa x^{\kappa-4} + \cdots + (-1)^{\eta_\kappa} S_{\kappa-1}^\kappa = 0, \qquad \kappa \text{ even}, \tag{17}$$

are solvable by radicals for all positive integers λ. (Here $\kappa = \lambda n$).

PROOF. From (15) it follows that

$$\begin{aligned} \cot(\lambda\psi_k) &= 0, & \kappa \text{ odd}, \\ \tan(\lambda\psi_k) &= 0, & \kappa \text{ even}, \end{aligned}$$

where $\psi_k = \beta_1^{(k)} + \cdots + \beta_n^{(k)}$. Transforming these two equations into sums of terms of the form $\tan(m\beta_j^{(k)})$ where m is a positive integer, then by means of

$$\tan(m\alpha) = \frac{1}{\cot(m\alpha)} = \frac{\binom{m}{1}\tan\alpha - \binom{m}{3}\tan^3\alpha + \cdots}{1 - \binom{m}{2}\tan^2\alpha + \binom{m}{4}\tan^4\alpha - \cdots}$$

and by replacing $\tan\beta_j^{(k)}$ with x/t_j, one obtains algebraic equations in x which are solvable by radicals (here x denotes the radius of $\mathcal{C}_{\mathbf{A}}$). Indeed, when $n \le 10$, each of the equations

$$S_1^n x^{n-1} - S_3^n x^{n-3} + \cdots + (-1)^\nu S_n^n = 0, \qquad n \text{ odd}, \tag{18}$$

or

$$S_1^n x^{n-2} - S_3^n x^{n-4} + \cdots + (-1)^\eta S_{n-1}^n = 0, \qquad n \text{ even}. \tag{19}$$

is solvable by radicals, and $\tan\left(n - \frac{2k}{\lambda}\right)\frac{\pi}{2}$ can be explicitly expressed by radicals over $\mathbf{Q}(t_1, \cdots, t_n)$ according to Theorem 2.1 and its Corollaries.

COROLLARY 2.6. If equations (18) and (19) are solvable by radicals, then equations (16) and (17) are solvable by radicals as well.

We remark that it is quite possible that (16) and (17) are solvable by radicals for any n positive integer. But this remains to be proved.

References

[1] M. Radić, Some relations and properties concerning tangential polygons, Math. Commun. **4**(1999), 197-206.

[2] M. Radić, On algebraic equations concerning semi-tangential polygons, Math. Commun., 6(2001), 73-81.