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Quadratic Systems with Homoclinic Cycles Described by Quintic Curves *

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Abstract

We find quadratic systems with homoclinic cycles described by quintic curves.

The determination of homoclinic bifurcations of quadratic systems is not known in general [1,5,6,7]. In [2-4], cubic or quartic homoclinic cycles are found. In this paper, we present quadratic systems with homoclinic cycles which are described by quintic curves.

Consider the quadratic system

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P and Q are second order bivariate polynomials with real coefficients. If the system (1) has a homoclinic cycle, then without loss of any generality, we may suppose that (1) the homoclinic cycle passes through the origin, which is a hyperbolic saddle; (2) the stable and unstable manifold of the origin are tangent to the lines $x^2 - y^2 = 0$ and the homoclinic orbit is located in the region $D = \{(x,y)||y| < |x|, x > 0\}$; and (3) one of the infinite singular points is 'in the y-axis direction'.

Under these assumptions, the corresponding normal form is

$$\frac{dx}{dt} = \bar{P}(x,y),\ \frac{dy}{dt} = \bar{Q}(x,y), \tag{2}$$

where $\bar{P}(x,y) = cx + y + a_1x^2 + a_2xy$, $\bar{Q}(x,y) = x + cy - x^2 - b_2xy - b_1y^2$, $|c| < 1$ and a_1, a_2, b_1, b_2 are real. It is easy to see that the quintic algebraic homoclinic cycle through the origin of the system must take the following form:

$$F(x,y) = x^2 - y^2 + F_3 + F_4 + F_5 = 0, \tag{3}$$

where F_3, F_4 and F_5 are homogeneous polynomials of degrees 3, 4 and 5 respectively.

By Batins' formula, we are able to find, by means of the software Mathematica, an invariant quintic curve of the above system.

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THEOREM 1. If the coefficients in the polynomials $\bar{P}(x,y)$ and $\bar{Q}(x,y)$ satisfy

$$\begin{aligned}
a_1 &= -\frac{150c+10c^3+225c\alpha+93c^3\alpha-6c^5\alpha}{5(75+29c^2)},\\
a_2 &= -\alpha,\\
b_1 &= 2\alpha,\\
b_2 &= \frac{575c+25c^3+425c\alpha+214c^3\alpha-15c^5\alpha}{5(75+29c^2)},
\end{aligned}$$

where

$$\alpha=\frac{5[-1125-270c^2+43c^4+(75+29c^2)\sqrt{225-18c^2+c^4}]}{2(-5625-2925c^2-259c^4+21c^6)}.$$

Then (2) has an invariant quintic curve described by

$$F(x,y)=x^2-y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{40}x^4+a_{31}x^3y+a_{50}x^5=0, \tag{4}$$

which satisfies

$$F_x\bar{P}+F_y\bar{Q}=(2c+Ax+By)F, \tag{5}$$

where $A=5a_1$, $B=5a_2$, $a_{12}=\alpha$,

$$\begin{aligned}
a_{30} &= \frac{-250-30c^2-625\alpha-225c^2\alpha+18c^4\alpha}{5(75+29c^2)},\\
a_{21} &= -\frac{2(100c+25c\alpha+27c^3\alpha)}{5(75+29c^2)},\\
a_{31} &= \frac{2c\alpha(375+5c^2+55c^2\alpha-3c^4\alpha)}{15(75+29c^2)},\\
a_{40} &= \left\{27500c^4-4800c^6+180c^8+3515625\alpha+1968750c^2\alpha\right.\\
&\quad \left.-5000c^4\alpha+71250c^6\alpha+5391c^8\alpha-108c^{10}\alpha\right\}\\
&\quad /\left\{5(75+29c^2)(5625+2925c^2+259c^4-21c^6)\right\},\\
a_{50} &= -\left\{\alpha(-3093750c^2-427500c^4-8900c^6+2500c^8-30c^{10}\right.\\
&\quad +10546875\alpha+6187500c^2\alpha-701250c^4\alpha-325750c^6\alpha\\
&\quad \left.+42317c^8\alpha-1614c^{10}\alpha+18c^{12}\alpha)\right\}\\
&\quad /\left\{75(75+29c^2)(5625+2925c^2+259c^4-21c^6)\right\}.
\end{aligned}$$

Next, note that when $c<0$, if we let $x=\bar{x}$, $y=-\bar{y}$, $t=-\bar{t}$, and $c=-\bar{c}$ in (2), then the resulting system has the same form. Therefore, we will restrict our attention to the cases where $c=0$ or $0<c<1$ in the following discussions.

From now on, we will suppose that the coefficients a_1,a_2,b_1,b_2 and α satisfy the conditions of Theorem 1. To emphasize its dependence on the parameter c, the system (2) will also be denoted by $E(c)$.

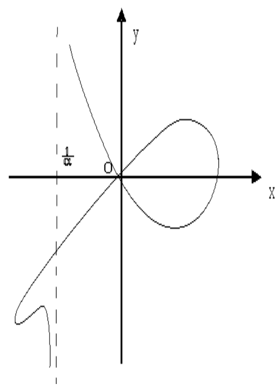
The curve (4) can be rewritten as

$$F(x,y)=(a_{12}x-1)y^2+(a_{21}+a_{31}x)x^2y+x^2(1+a_{30}x+a_{40}x^2+a_{50}x^3)=0. \tag{6}$$

Solving y from the equation $F(x,y)=0$, we get

$$\begin{aligned} y_\pm &= \Big\{-(a_{21}+a_{31}x)x^2 \pm x((a_{13}^2-4a_{12}a_{50})x^4 \\ &\quad +2(a_{21}a_{31}-2a_{12}a_{40}+2a_{50})x^3+(a_{21}^2-4a_{12}a_{30}+4a_{40})x^2 \\ &\quad -4(a_{12}-a_{30})x+4)^{1/2}\Big\} \,/\, \{2(a_{12}x-1)\}\,. \end{aligned}$$

It can be checked that when $0<c<1$, the equation $1+a_{30}x+a_{40}x^2+a_{50}x^3=0$ has only a real root, which indicates that the curve defined by (4) has, besides the origin, only one point of intersection with the x-axis. Moreover, $x=1/a_{12}=1/\alpha$ (note that $\alpha<0$) is the asymptote of the quintic curve (see the following figure).



LEMMA 1. The system $E(c)$ has at most one singular point which is not on the curve defined by (4).

PROOF. According to (5), we know that a singular point $A_0=(x_0,y_0)$, which is not on $F=0$, must be on the straight line $2c+Ax+By=0$. Solving the system of equations

$$\begin{aligned} cx+y+a_1x^2-\alpha xy &= 0, \\ x+cy-x^2-b_2xy-2\alpha y^2 &= 0, \\ 2c+5a_1x+5a_2y &= 0, \end{aligned}$$

we obtain the unique solution

$$\begin{aligned} x_0 &= \frac{75+29c^2}{75+5c^2+3c^2\alpha-3c^4\alpha}, \\ y_0 &= -\frac{3c(75+29c^2)}{5(75+5c^2+3c^2\alpha-3c^4\alpha)}, \end{aligned}$$

as required.

LEMMA 2. The system $E(c)$ has two singular points $A_1=(x_1,y_1)$ and $A_2=(x_2,y_2)$ besides $(0,0)$ and $A_0=(x_0,y_0)$, where

$$x_1 = \frac{-5m+5\sqrt{k}}{n},$$

$$\begin{aligned} y_1 &= \frac{cx_1 + a_1 {x_1}^2}{\alpha x_1 - 1}, \\ x_2 &= \frac{-5m - 5\sqrt{k}}{n}, \\ y_2 &= \frac{cx_2 + a_1 {x_2}^2}{\alpha x_2 - 1}, \end{aligned}$$

and

$$\begin{aligned} m &= 115c^2 - 375\alpha - 3c^6\alpha + c^4(5 + 66\alpha), \\ n &= \alpha[550c^2(\alpha - 1) + 1875\alpha + 6c^6\alpha - c^4(10 + 143\alpha)], \\ k &= m^2 + (-75 + 46c^2 + 29c^4)n. \end{aligned}$$

Indeed, the proof follows by solving $\bar{P} = 0, \bar{Q} = 0$.

From the expressions of the singular points A_0, A_1 and A_2, it is easy to see that : (i) when $c = 0$, A_0 is on the x-axis, A_1 and A_2 "disappear to the infinity"; (ii) when $0 < c < 1$, A_0 is in the fourth quadrant, A_1 is in the third quadrant and A_2 is in the second quadrant; and (iii) both A_1 and A_2 are on the curve defined by (4).

LEMMA 3. (i) If $c = 0$, then A_0 is a center; and (ii) if $0 < c < 1$, then A_0 is a focus and A_1, A_2 are nodes.

PROOF. (i) If $c = 0$, then the system $E(c)$ becomes $dx/dt = y$, $dy/dt = x - x^2$. It is evident that the system $E(c)$ is a symmetrical integrable system such that $(y^2 - x^2 + 2x^3/3)$ is constant. Thus $A_0(1, 0)$ is a center. (ii) If $0 < c < 1$, A_0 is a focus, since

$$D = \text{div}(\bar{P}, \bar{Q})|_{A_0} = \frac{c(c^2 - 25)(-5 + 10\alpha + 3c^2\alpha)}{5(-75 - 5c^2 - 3c^2\alpha + 3c^4\alpha)}$$

$$\begin{aligned} J &= (\bar{P}_x\bar{Q}_y - \bar{P}_y\bar{Q}_x)|_{A_0} \\ &= \frac{(1875 - 600c^2 - 75c^4 - 1875\alpha - 475c^2\alpha + 17c^4\alpha + 45c^6\alpha)}{25(75 + 5c^2 + 3c^2\alpha - 3c^4\alpha)} \end{aligned}$$

and $D^2 - 4J < 0$. In addition, neither A_1 nor A_2 can be a focus or a center for they are on the curve (4). Furthermore, $(0, 0)$ is a saddle, and the four points $(0, 0), A_0, A_1, A_2$ are the corners of a concave quadrilateral. According to Berlinskii Theorem, A_1 and A_2 are nodes. The proof is complete.

LEMMA 4. When $0 < c < 1$, the system $E(c)$ has three singular points in the infinity and one is a node, the others are saddles.

PROOF. Applying the Poincare transformation: $x = v/z$, $y = 1/z$, $dt = zd\tau$, $a_1 + b_2 = -\beta$, and $a_2 + b_1 = \alpha$, the system $E(c)$ changes to

$$\begin{cases} dz/d\tau = -z[-2\alpha + cz + (a_1 + \beta)v + vz - v^2] = P^*(x, y) \\ dv/d\tau = (1 - v^2)z + v(\alpha - \beta v + v^2) = Q^*(x, y) \end{cases} . \tag{7}$$

When $0 < c < 1$, there are three singular points $B_0(0, v_0)$, $B_1(0, v_1)$, $B_2(0, v_2)$ on the straight line $z = 0$, where

$$v_0 = 0, v_1 = \frac{\beta + \sqrt{(\beta)^2 - 4\alpha}}{2}, v_2 = \frac{\beta - \sqrt{(\beta)^2 - 4\alpha}}{2}.$$

Now we analyze the properties of the points:

$$\begin{aligned}
(P_z^*)|_{z=0} &= 2\alpha - (a_1 + \beta)v + v^2, \\
(P_v^*)|_{z=0} &= 0, \\
(Q_z^*)|_{z=0} &= 1 - v^2, \\
(Q_v^*)|_{z=0} &= \alpha - 2\beta v + 3v^2 = f'(v).
\end{aligned}$$

The roots of the characteristic equations of the singular points are $(\lambda_1)^{(i)} = \alpha - a_1 v_i$, $(\lambda_2)^{(i)} = f'(v_i)$ for $i = 1, 2$, $(\lambda_1)^{(0)} = 2\alpha$ and $(\lambda_2)^{(0)} = \alpha$. Since $\alpha < 0$, $\alpha - a_1 v_2 < 0$ and $f'(v_2) > 0$, thus $B_0(0, 0)$ is a stable node and $B_2(0, v_2)$ is a saddle. Finally, according to the Euler's index-sum theorem, we know that $B_1(0, v_1)$ is a saddle. The proof is complete.

By means of Lemmas 1-4, we know that, under the condition $0 < c < 1$, there is no singular point other than the origin which lies on the non-isolated closed component of the quintic curve defined by (4). The following result now holds.

THEOREM 2. When $0 < c < 1$, the non-isolated closed component of the quintic curve defined by (4) constitutes the homoclinic cycles of the system $E(c)$ and its inner singular point is a focus.

To determine the global phase graphs of the system $E(c)$, we need to discuss whether the system has limit cycles. By means of Dulac's theorem, we obtain the following result.

THEOREM 3. The system $E(c)$ has no limit cycles.

Indeed, the y-axis is non-tangent (besides the origin), therefore the y-axis and the invariant curve defined by (4) divide the plane into five regions. Taking Dulac function F^{-1}, then

$$\begin{aligned}
\operatorname{div}(F^{-1}\bar{P}, F^{-1}\bar{Q}) &= F^{-1}[\bar{P}_x + \bar{Q}_y - (2c + Ax + By)] \\
&= F^{-1}(-3a_1 - b_2)x \\
&= F^{-1}(-2a_1 + \beta)x.
\end{aligned}$$

The right hand side of the above equality is a continuous function and it keeps the same sign in each region. Thus by Dulac's Theorem, there is no limit cycle in each region.

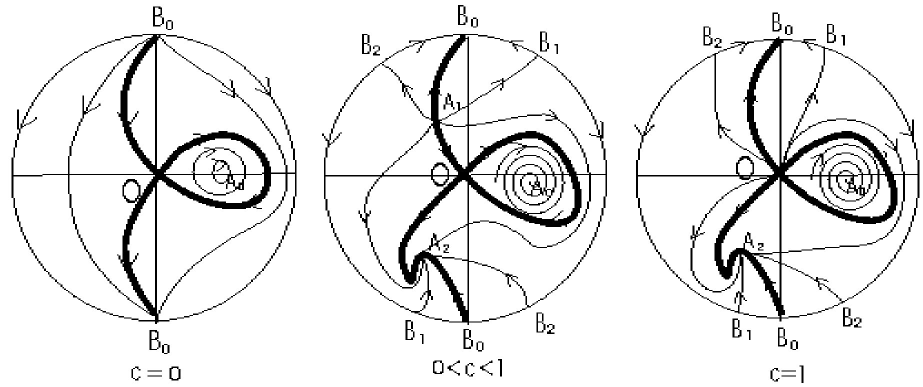
We remark that by letting $y = xu$, the system $E(c)$ is changed into a cubic system

$$\begin{cases} dx/dt = x[c + u + (a_1 + a_2 u)x] \\ du/dt = 1 - u^2 - (1 - \beta u + \alpha u^2)x \end{cases},$$

while the homoclinic cycle defined by (4) changes into the heteroclinic cycle:

$$x^2[1 - u^2 + (a_{30} + a_{21}u + a_{12}u^2)x + (a_{40} + a_{31}u)x^2 + a_{50}x^3] = 0.$$

By means of the informations obtained above, the global phase graphs of the system $E(c)$ can easily be obtained (see the following figures).



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