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Global Attractivity for a Differential-Difference Population Model *

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Abstract

In this paper, sufficient conditions are found for the positive equilibrium of the differential-difference equation (1) to be globally attractive.

Consider the delay differential equation

$$N'(t) = r(t)N(t)\left(\frac{1-N(t-\tau)}{1+\lambda(t)N(t-\tau)}\right)^{\alpha},\ t \geq 0, \tag{1}$$

where $r(t) \in C([0,\infty),(0,\infty))$, $\lambda(t) \in C([0,\infty),(0,\infty))$, $\tau > 0$ and α is a ratio of two odd integers so that $\alpha \geq 1$. If $\alpha = 1$, $\lambda(t) \equiv \lambda$ then (1) becomes the so called "food-limited" model

$$N'(t) = r(t)N(t)\frac{1-N(t-\tau)}{1+\lambda N(t-\tau)},\ t \geq 0, \tag{2}$$

which has been studied by many authors, see [1-6]. Equation (1) was proposed in [1] but has not been studied. In this paper, we shall prove two theorems related to attractivity of the positive equilibrium 1. Before stating our results, let us first note that equation (1) under the initial condition

$$N(t) = \phi(t),\ t \in [-\tau, 0],\ \phi \in C([-\tau,0],[0,\infty)),\ \phi(0) > 0. \tag{3}$$

has a unique positive solution $N(t)$ on $[-\tau, \infty)$.

THEOREM 1. Suppose $0 < \lambda(t) \leq 1$ for $t \geq 0$,

$$\int_0^{+\infty} \frac{r(t)}{[1+\lambda(t)]^{\alpha}}dt = \infty, \tag{4}$$

and

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} \frac{r(s)}{(\lambda(s))^{\alpha}}ds \leq 3. \tag{5}$$

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Then every solution of (1) and (3) tends to 1 as t tends to $+\infty$.

THEOREM 2. Suppose $\lambda(t) \geq 1$ for $t \geq 0$, (4) holds and

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} r(s)ds \leq 3, \tag{6}$$

then every solution of (1) and (3) tends to 1 as t tends to $+\infty$.

To prove our results, we need the following lemmas.

LEMMA 1. For any $v \in [0,1)$, $\ln(2e^{-v(1-v/2)} - 1) \geq -2v$, and for any $u \in [0,\infty)$, $\ln(2e^{u(1+u/2)} - 1) \geq 2u$.

Indeed, let $f(v) = 2e^{-v(1-v/2)} - e^{-2v}$ and $g(v) = (1-v)e^{v(1+v/2)}$. It is easy to see that $g(0) = 1$, $g'(v) = -v^2 e^{v(1+v/2)} \leq 0$, and for some $\xi \in (0,v)$,

$$f'(v) = 2e^{-2v}[1 - g(v)] = -2e^{-2v} g'(\xi) v \geq 0.$$

It follows that $f(v) \geq f(0) = 1$ for $v \in [0,1)$. The other assertion is similarly proved.

LEMMA 2. Assume that $v \in (0,1)$. Then for any $x \in [0,\infty)$,

$$\ln \frac{1 + [2e^{-v(1-v/2)} - 1]e^{-vx}}{1 + e^{-vx}} \leq -v\left(1 - \frac{v}{2}\right) + \frac{v^2}{2}x. \tag{7}$$

PROOF. Set $a = 2e^{-v(1-v/2)} - 1$ and $f(x) = \ln\left((1 + ae^{-vx}) / (1 + e^{-vx})\right)$. Observe that $f(0) = -v\,(1 - v/2)\,,\ f'(0) = v[e^{v(1-v/2)} - 1]/2$ and

$$f''(x) = \left[\frac{a}{(a + e^{vx})^2} - \frac{1}{(1 + e^{vx})^2}\right] v^2 e^{vx}.$$

Since $a \leq 1$, it follows that $f''(x) \leq 0$ for $x \geq 0$. By the mean value theorem and the fact that $e^{x(1-x/2)} \leq 1 + x$ for $x \geq 0$, we have

$$\begin{aligned} f(x) &\leq f(0) + f'(0)x = -v\left(1 - \frac{v}{2}\right) + \frac{vx}{2}[e^{v(1-v/2)} - 1] \\ &\leq -v\left(1 - \frac{v}{2}\right) + \frac{v^2}{2}x. \end{aligned}$$

LEMMA 3. The system of inequalities

$$\ln \frac{1+u}{1-u} \leq 2v, -\ln \frac{1-v}{1+v} \leq 2u, \tag{8}$$

has a unique solution $(u,v) = (0,0)$ in the region $\{(u,v) : -1 < v \leq 0 \leq u < 1\}$.

PROOF. Set $g(x) = \exp\left(2(1-x)/(1+x)\right),\ f(x) = x - g(g(x))$, and

$$h(x) = (1+x)^2[1 + g(x)]^2 - 16g(x)g(g(x)).$$

Observe that $h(1) = 0$,

$$f'(x) = 1 - g'(x)g'(g(x)) = 1 - \frac{16g(x)g(g(x))}{(1+x)^2[1+g(x)]^2},$$

and for $x > 1$

$$\begin{aligned} h'(x) &= 2[1+g(x)][(1+x)(1+g(x))-4g(x)] \\ &\quad + \frac{64}{(1+x)^2} g(x) g(g(x)) \frac{[1-g(x)]^2}{[1+g(x)]^2} > 0. \end{aligned}$$

It follows that $h(x) > h(1) = 0$ for $x > 1$, and so $f'(x) > 0$ for $x > 1$. This shows that $f(x) > f(1) = 0$ for $x > 1$. On the other hand, from (8), we have $g(\mu) \le \lambda \le 1 \le \mu \le g(\lambda)$, where $\lambda = (1-v)/(1+v)$ and $\mu = (1+u)/(1-u)$. If $u > 0$, then $\mu > 1$, and so $\mu \le g(\lambda) \le g(g(\mu)) < \mu$. This contradiction implies that $u = v = 0$. The proof is complete.

LEMMA 4. Suppose (4) holds. Then every solution of (1) and (3) that does not oscillate about 1 tends to 1 as $t \to \infty$.

The proof is similar to the arguments presented in [2] and is thus omitted.

LEMMA 5. Suppose $0 < \lambda(t) \le 1$ for $t \ge 0$ and (5) holds. Let $N(t) = N(t; 0, \phi)$ be a solution of (1) and (3) which is oscillatory about 1. Then $N(t)$ is bounded above and is strictly bounded below by 0.

PROOF. Let $t_0 > 0$ be large enough so that

$$\int_{t-\tau}^{t} \frac{r(s)}{(\lambda(t))^\alpha} ds \le 4,$$

for all $t \ge t_0$. Let t^* be a local maximum point of $N(t)$ for $t \ge t_0 + \tau$. Then $N'(t^*) = 0$ and by (1), $N(t^* - \tau) = 1$. Integrating (1) from $t^* - \tau$ to t^*,

$$\begin{aligned} N(t^*) &= \exp\left(\int_{t^*-\tau}^{t^*} r(s) N(s) \left[\frac{1 - N(s-\tau)}{\lambda(s) N(s-\tau)} \right]^\alpha ds \right) \\ &\le \exp\left(\int_{t^*-\tau}^{t^*} r(s) ds \right) \le e^4. \end{aligned}$$

Consequently, $\limsup_{t\to\infty} N(t) \le e^4$. Next, let t_* be a local minimum point of $N(t)$ for $t \ge t_0 + 3\tau$. Then $N'(t_*) = 0$ and $N(t_* - \tau) = 1$. Proceeding as before and using the fact that

$$\frac{1 - N(t-\tau)}{1 + \lambda(t) N(t-\tau)} \ge \frac{1 - e^4}{1 + \lambda(t) e^4} \ge \frac{1 - e^4}{\lambda(t)(1 + e^4)}$$

for $t \ge t_0 + \tau$, we have

$$N(t_*) \ge \exp\left(\int_{t_*-\tau}^{t^*} \frac{r(s)}{\lambda^\alpha(s)} \left[\frac{1 - e^4}{1 + e^4} \right]^\alpha ds \right) \ge \exp\left(4 \left(\frac{1 - e^4}{1 + e^4} \right)^\alpha \right).$$

Hence

$$\liminf_{t\to\infty} N(t) \ge \exp\left(4 \left(\frac{1 - e^4}{1 + e^4} \right)^\alpha \right) > 0.$$

The proof is complete.

LEMMA 6. Assume that $\lambda(t) \geq 1$ for $t \geq 1$ and (6) holds. Let $N(t) = N(t, 0, \phi)$ be a solution of (1) and (3) which is oscillatory about 1. Then $N(t)$ is bounded above and is strictly bounded below by 0.

The proof is similar to the proof of Lemma 5 and is thus omitted.

We now turn to the proof of Theorem 1. Let $u = \limsup_{t\to\infty} N(t)$ and $v = \liminf_{t\to\infty} N(t)$. Then by Lemma 5, $0 < v \leq 1$ and $u \geq 1$. It suffices to show that $u = v = 1$. For any $\varepsilon \in (0, v)$, choose $t_0 = t_0(\varepsilon)$ such that

$$v_1 \equiv v - \varepsilon < N(t-\tau) < u + \varepsilon \equiv u_1,\ t \geq t_0, \tag{9}$$

and

$$\int_{t-\tau}^{t} \frac{r(s)}{\lambda^\alpha(t)} ds \leq 3 + \varepsilon,\ t \geq t_0 - \tau. \tag{10}$$

Note that $(1-x)/(1+\lambda x) \leq (1-x)/(\lambda(1+x))$ for $x \leq 1$ and $(1-x)/(1+\lambda x) \geq (1-x)/(\lambda(1+x))$ for $x \geq 1$. Thus

$$N'(t) \leq r(t)N(t)\left(\frac{1-v_1}{1+\lambda(t)v_1}\right)^\alpha \leq r(t)N(t)\left(\frac{1-v_1}{\lambda(t)(1+v_1)}\right)^\alpha,\ t \geq t_0, \tag{11}$$

and

$$N'(t) \geq r(t)N(t)\left(\frac{1-u_1}{1+\lambda(t)u_1}\right)^\alpha \geq r(t)N(t)\left(\frac{1-u_1}{\lambda(t)(1+u_1)}\right)^\alpha,\ t \geq t_0. \tag{12}$$

Consequently,

$$N'(t) \leq \frac{r(t)}{\lambda^\alpha(t)} N(t) \frac{1-v_1}{1+v_1},\ t \geq t_0, \tag{13}$$

and

$$N'(t) \geq \frac{r(t)}{\lambda^\alpha(t)} N(t) \frac{1-u_1}{1+u_1},\ t \geq t_0. \tag{14}$$

Set $R(t) = r(t)/\lambda^\alpha(t)$. Let $\{p_n\}$ be an increasing sequence such that $p_n \geq t_0 + \tau$, $\lim_{n\to\infty} p_n = +\infty$, $N'(p_n) = 0$ and $\lim_{n\to\infty} N(p_n) = u$. By (1), $N(p_n - \tau) = 1$. For $p_n - \tau \leq t \leq p_n$, by integrating (13) from $t - \tau$ to $p_n - \tau$, we get

$$N(t-\tau) \geq \exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right),\ p_n - \tau \leq t \leq p_n.$$

Substituting this into (1), if $N(t-\tau) \leq 1$, we have

$$\begin{aligned} N'(t) &\leq R(t)N(t)\left(\frac{1-N(t-\tau)}{1+N(t-\tau)}\right)^\alpha \leq R(t)N(t)\frac{1-N(t-\tau)}{1+N(t-\tau)} \\ &\leq R(t)N(t)\frac{1-\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1+\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}. \end{aligned}$$

If $N(t-\tau) > 1$, by (1), $N'(t) < 0$, thus

$$N'(t) \le R(t)N(t)\frac{1-\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1+\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}.$$

If $t \in (p_n - \tau, p_n)$, we have

$$N'(t) \le \min\left\{R(t)N(t)\frac{1-v_1}{1+v_1}, R(t)N(t)\frac{1-\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1+\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}\right\}. \tag{15}$$

Since $0 < x = (1-v_1)/(1+v_1) < 1$, it follows from Lemma 1 that $\ln 2e^{-x(1-x/2)-1} \ge -2x$, and so $0 < -\frac{1}{x}\ln\left(2e^{-x(1-x/2)}-1\right) \le 2$. There are two possibilities. First of all, consider the case

$$\int_{p_n-\tau}^{p_n} R(s)ds \le -\frac{1}{v_n}\ln\left(2e^{-v_0(1-v_0/2)}-1\right) \equiv A \le 3+\varepsilon,$$

where $v_0 = (1-v_1)/(1+v_1)$. Then

$$\begin{aligned}
&\ln N(p_n)\\
\le\ & \int_{p_n-\tau}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0\int_{t-\tau}^{p_n-\tau} R(s)ds\right)\right]}{1+\exp\left(-v_0\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}dt\\
=\ & \int_{p_n-\tau}^{p_n} \frac{R(t)\left\{1-\exp\left[-v_0\left(\int_{t-\tau}^{t} r(s)ds-\int_{p_n-\tau}^{t} R(s)ds\right)\right]\right\}}{1+\exp\left[-v_0\left(\int_{t-\tau}^{t} R(s)ds-\int_{p_n-\tau}^{t} R(s)ds\right)\right]}dt\\
\le\ & \int_{p_n-\tau}^{p_n} \frac{R(t)\left\{1-\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right]\right\}}{1+\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right]}dt\\
=\ & \int_{p_n-\tau}^{p_n} R(s)ds - \frac{2}{v_0}\ln\frac{1+\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right]}{1+e^{-(3+\varepsilon)v_0}}.
\end{aligned}$$

The function $f(x) = x - \left(2\ln\left[1+e^{-v_1(3+\varepsilon-x)}\right]\right)/v_1$ is increasing in $[0, 3+\varepsilon]$. Thus, by Lemmas 1 and 2, we have

$$\begin{aligned}
\ln N(p_n) &\le A - \frac{2}{v_0}\ln\frac{1+e^{-v_0(3+\varepsilon-A)}}{1+e^{-(3+\varepsilon)v_0}}\\
&= A + \frac{2}{v_0}\ln\frac{1+\left[2e^{-v_0(1-v_0/2)}-1\right]e^{-v_0(3+\varepsilon-A)}}{1+e^{-v_0(3+\varepsilon-A)}}\\
&\le A + \frac{2}{v_0}\left[-v_0\left(1-\frac{v_0}{2}\right)+\frac{v_0^2}{2}(3+\varepsilon-A)\right]\\
&= -2+(4+\varepsilon)v_0 - \frac{1-v_0}{v_0}\ln\left(2e^{-v_0(1-v_0/2)}-1\right)\\
&\le (2+\varepsilon)v_1.
\end{aligned}$$

Next, suppose

$$A < \int_{p_n-\tau}^{p_n} R(s) \leq 3+\varepsilon.$$

Choose $\xi_n \in (p_n - \tau, p_n)$ such that

$$\int_{\xi_n}^{p_n} R(s)ds \equiv A.$$

Then by (15) and Lemma 1,

$$\begin{aligned}
& \ln N(p_n) \\
\leq\ & v_0 \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0 \int_{t-\tau}^{p_n-\tau} R(s)ds\right)\right]}{1+\exp\left(-v_0 \int_{t-\tau}^{p_n-\tau} R(s)ds\right)} dt \\
\leq\ & v_0 \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} \frac{R(t)\left\{1-\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right]\right\}}{1+\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right]} dt \\
=\ & v_0 \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} R(s)ds - \frac{2}{v_0} \ln \frac{1+\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{p_n} R(s)ds\right)\right]}{1+\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{\xi_n} R(s)ds\right)\right]} \\
=\ & v_0 \int_{p_n-\tau}^{p_n} R(s)ds + (1-v_0)A - \frac{2}{v_0} \ln \frac{1+\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{p_n} R(s)ds\right)\right]}{1+\exp\left[-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{\xi_n} R(s)ds\right)\right]} \\
\leq\ & (3+\varepsilon)v_0 + (1-v_0)A - \frac{2}{v_0} \ln \frac{2}{1+e^{-Av_0}} \\
=\ & -2 + (4+\varepsilon)v_0 - \frac{1-v_0}{v_0} \ln\left(2e^{-v_0(1-v_0/2)} - 1\right) \\
\leq\ & (2+\varepsilon)v_0,
\end{aligned}$$

where we have used the fact that the function

$$g(x) = -\frac{2}{v_1} \ln \frac{1+\exp\left[-v_1\left(3+\varepsilon-x\right)\right]}{1+\exp\left[-v_1\left(3+\varepsilon+A-x\right)\right]} + v_1 x$$

is increasing on $[0, 3+\varepsilon]$. In either cases, we have proved that $\ln N(p_n) \leq (2+\varepsilon)v_1$ for $n = 1, 2, \ldots$. Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$\ln u \leq 2\frac{1-v}{1+v}. \tag{16}$$

Next, let $\{q_n\}$ be an increasing sequence such that $q_n \geq t_0 + \tau$, $\lim_{n\to\infty} q_n = +\infty$, $N'(q_n) = 0$ and $\lim_{n\to\infty} N(q_n) = -v$. By (1), $N(q_n - \tau) = 1$. For $q_n - \tau \leq t \leq q_n$, integrating (14) form $t - \tau$ to $q_n - \tau$, we have

$$N(t-\tau) \leq \exp\left(-\frac{1-u_1}{1+u_1} \int_{t-\tau}^{p_n-\tau} R(s)ds\right), q_n - \tau \leq t \leq q_n.$$

Substituting this into (1), if $N(t-\tau) \geq 1$, we have

$$\begin{aligned} N'(t) &= r(t)N(t)\left(\frac{1-N(t-\tau)}{1+\lambda(t)N(t-\tau)}\right)^{\alpha} \\ &\geq R(t)N(t)\frac{1-N(t-\tau)}{1+\lambda(t)N(t-\tau)} \\ &\geq R(t)N(t)\frac{1-\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)}{1+\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)} \end{aligned}$$

for $q_n - \tau \leq t \leq q_n$. If $N(t-\tau) < 1$, then by (1), $N'(t) > 0$, thus

$$N'(t) \geq R(t)N(t)\frac{1-\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)}{1+\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)},$$

where $u_0 = (1-u_1)/(1+u_1)$. Thus

$$-N'(t) \leq \min\left\{-R(t)N(t)u_0, -R(t)N(t)\frac{1-\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)}{1+\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)}\right\} \tag{17}$$

for $q_n - \tau \leq t \leq q_n$. Noting that $0 < -u_0 < 1$, one can easily see that

$$0 < -\frac{1}{u_0}\ln\left(2e^{-u_0(1-u_0/2)} - 1\right) < 3.$$

There are two cases to consider. In the first case,

$$B \equiv \int_{q_n-\tau}^{q_n} R(s)ds \leq (3+\varepsilon) + \frac{1}{u_0}\ln\left(2e^{-u_0(1-u_0/2)} - 1\right).$$

By (17) and Lemma 1,

$$\begin{aligned} -\ln N(q_n) &\leq -u_0\int_{q_n-\tau}^{q_n} R(s)ds \leq -(3+\varepsilon)u_0 - \ln\left(2e^{-u_0(1-u_0/2)} - 1\right) \\ &\leq -(1+\varepsilon)u_0. \end{aligned}$$

In the second case,

$$B < \int_{q_n-\tau}^{q_n} R(s)ds \leq 3+\varepsilon.$$

We choose $\eta_n \in (q_n - \tau, q_n)$ such that $B \equiv \int_{q_n-\tau}^{\eta_n} R(s)ds$. Then by (7) and Lemma 1,

$$\begin{aligned} & -\ln N(q_n) \\ \leq\ & -u_0\int_{q_n-\tau}^{\eta_n} R(s)ds + \int_{\eta_n}^{q_n}\frac{R(t)\left[\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)-1\right]}{1+\exp\left(-u_0\int_{t-\tau}^{q_n-\tau}R(s)ds\right)}dt \end{aligned}$$

$$
\begin{aligned}
&\leq -u_0\int_{q_n-\tau}^{\eta_n} R(s)ds + \int_{\eta_n}^{q_n} R(t)\frac{\exp\left[-u_0\left(3+\varepsilon-\int_{q_n-\tau}^{t} R(s)ds\right)\right]-1}{1+\exp\left[-u_0\left(3+\varepsilon-\int_{q_n-\tau}^{t} R(s)ds\right)\right]}dt \\
&= -u_0\int_{q_n-\tau}^{\eta_n} R(s)ds - \int_{\eta_n}^{q_n} R(s)ds - \frac{2}{u_0}\ln\frac{1+\exp\left[-u_0\left(3+\varepsilon-\int_{q_n-\tau}^{q_n} R(s)ds\right)\right]}{1+\exp\left[-u_0\left(3+\varepsilon-\int_{q_n-\tau}^{\eta_n} R(s)ds\right)\right]} \\
&= (1-u_0)B - \int_{\eta_n}^{q_n} R(s)ds + 2\left(1-\frac{u_0}{2}\right) \\
&\quad +\frac{2}{u_0}\ln\frac{1+\exp\left[-u_0\left(3+\varepsilon-\int_{q_n-\tau}^{q_n} R(s)ds\right)\right]}{2} \\
&\leq 2-(4+\varepsilon)u_0 + \frac{1-u_0}{u_0}\ln\left(2e^{-u_0(1-u_0/2)-1}\right) \\
&\leq -(2+\varepsilon)u_0,
\end{aligned}
$$

where we have used the fact that

$$h(x) = -x - \frac{2}{u_0}\ln\frac{1+\exp\left(-u_0(3+\varepsilon-x)\right)}{2}$$

is increasing on $[0, 3+\varepsilon]$. In either cases, we have proved that $-\ln N(q_n) \leq -(2+\varepsilon)u_0$ for $n = 1, 2, \dots$. Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$-\ln v \leq -2\frac{1-u}{1+u}. \tag{18}$$

Let $y = -(1-u)/(1+u)$ and $x = (1-v)/(1+v)$, then in view of (16), (18) and Lemma 3, we get $x = y = 0$. This then shows that $u = v = 1$. The proof is complete.

By means of methods similar to those in the proof of Theorem 1, and by noting that if $\lambda \geq 1$, then $(1-x)/(1+\lambda x) \leq (1-x)/(1+x)$ for $x \leq 1$, and $(1-x)/(1+\lambda x) \geq (1-x)/(1+x)$ for $x \geq 1$, we may prove Theorem 2. The details are omitted.

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