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Exact Solutions of a Linear Functional Differential Equation *

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Abstract

In this short review, we present a concise summary of some of the results in the book [1] "On the Accurate Distribution of Characteristic Roots and Stability of Linear Delay Differential Systems" by the author. It is hoped that this note will be useful for readers who are not familiar with the Chinese language used in [1].

A basic functional differential equation is

$$x'(t) + ax(t) + bx(t-\tau) = 0, \tag{1}$$

where a, b, τ are constants, and $\tau > 0$. To find all the solutions of (1), we need its characteristic equation

$$\lambda + a + be^{-\tau\lambda} = 0. \tag{2}$$

To this end, let us first consider the function

$$f(s) = se^s, \ s \in R. \tag{3}$$

The properties of the smooth function f can be derived by elementary means. Roughly, it is negative and decreasing on $(-\infty, -1)$. It is negative and increasing on $(-1, 0)$, and it is positive and increasing on $(0, \infty)$. Let us denote the restrictions of the function f on $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$ by $f_{|(-\infty,-1)}, f_{|(-1,0)}$ and $f_{|(0,\infty)}$ respectively. Let us further denote their inverses by lm_{-2}, lm_{-1} and lm_{+1} respectively. The function lm_{+1} is increasing and continuously differentiable on its domain $(0, \infty)$, and its range is $(0, \infty)$. The function lm_{-1} is increasing and continuously differentiable on $(-e^{-1}, 0)$, its range is $(-1, 0)$. The function lm_{-2} is decreasing and continuously differentiable on $(-e^{-1}, 0)$, and its range is $(-\infty, -1)$. Furthermore, $lm_{-2} < lm_{-1}$ for $x \in (-e^{-1}, 0)$.

Consider the functions

$$F(x) = -x - \tan x \cdot lm_{+1}(B\cos x), \ x \in (0, \pi/2) \cup (\pi/2, \pi) \tag{4}$$

$$F_1(x) = -x - \tan x \cdot lm_{-1}(B\cos x), \ x \in (0, \pi/2) \cup (\pi/2, \pi) \tag{5}$$

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$$F_2(x) = -x - \tan x \cdot lm_{-2}(B\cos x),\ x \in (0, \pi/2) \cup (\pi/2, \pi) \tag{6}$$

where B is a constant. For the sake of convenience, for a given constant B, we set

$$B^*_{\pm} = -\arccos\left(-\frac{1}{Be}\right) \pm \sqrt{(Be)^2 - 1},$$

whenever it is defined. If $B < -e^{-1}$, then $0 < -1/Be < 1$; and if $B > e^{-1}$, then $-1 < -1/Be < 0$.

The function F, defined by (4), has a unique inverse. Let it be denoted by $x = \theta_+(y, B)$. When $B > 0$, $\theta_+(y, B)$ is decreasing in $(-\pi/2 - B, 0)$ and its range is $(0, \pi/2)$. When $B < 0$, $\theta_+(y, B)$ is increasing in $(-\pi, -\pi/2 - B)$ and its range is $(\pi/2, \pi)$.

The function F_1, defined by (5), has a unique inverse. Let it be denoted by $x = \theta_{-1}(y, B)$. When $B < -e^{-1}$, $\theta_{-1}(y, B)$ is decreasing in $(-\pi/2 - B, B^*_+)$ and its range is $(\arccos(-1/Be), \pi/2)$. When $-e^{-1} \leq B < 0$, $\theta_{-1}(y, B)$ is decreasing in $(-\pi/2 - B, 0)$ and its range is $(0, \pi/2)$. When $0 < B \leq e^{-1}$, $\theta_{-1}(y, B)$ is decreasing in $(-\pi, -\pi/2 - B)$ and its range is $(\pi/2, \pi)$. When $B > e^{-1}$, $\theta_{-1}(y, B)$ is decreasing in $(B^*_-, -\pi/2 - B)$ and its range is $(\pi/2, \arccos(-1/Be))$.

The function F_2, defined by (6), has a unique inverse. Let it be denoted by $x = \theta_{-2}(y, B)$. When $B < -e^{-1}$, $\theta_{-2}(y, B)$ is increasing in $(B^*_+, +\infty)$ and its range is $(\arccos(-1/Be), \pi/2)$. When $-e^{-1} \leq B < 0$, $\theta_{-2}(y, B)$ is increasing in $(0, +\infty)$ and its range is $(0, \pi/2)$. When $0 < B \leq e^{-1}$, $\theta_{-2}(y, B)$ is increasing in $(-\infty, -\pi)$ and its range is $(\pi/2, \pi)$. When $B > e^{-1}$, $\theta_{-2}(y, B)$ is increasing in $(-\infty, B^*_-)$ and its range is $(\pi/2, \arccos(-1/Be))$.

In (3), let

$$\lambda = -a + \frac{1}{\tau} S, \tag{7}$$

then (2) becomes

$$Se^S = A, \tag{8}$$

where

$$A = -\tau b e^{a\tau}. \tag{9}$$

To simplify matters, let us set

$$\theta^{(j)}_{k,C} = \theta_k(j\pi, C),$$

$$\psi^{(j)}_{k,C} = j\pi + \theta^{(j)}_{k,C},$$

$$\psi^{(j)}_{(-1,-2),C} = j\pi + \arccos\left(\frac{1}{C(j\pi)e}\right)$$

and

$$\varphi^{(j)}_{k,C} = lm_k(C\cos\theta^{(j)}_{k,C}).$$

The following facts hold for the real roots of (2):

- (I) If $b < 0$, then (2) has a unique simple real root $\lambda_1 = -a + \frac{1}{\tau} lm_+ A$.
- (II) If $b = 0$, then (2) has a unique simple real root $\lambda_2 = -a$.

- (III) If $0 < b < \frac{1}{\tau}e^{-(a\tau+1)}$, then (2) has only two simple real roots $\lambda_3 = -a + \frac{1}{\tau}lm_{-1}A, \lambda_4 = -a + \frac{1}{\tau}lm_{-2}A$.
- (IV) If $b = \frac{1}{\tau}e^{-(a\tau+1)}$, then (2) has a double root $\lambda_5 = -(a + \frac{1}{\tau})$.
- (V) If $b > \frac{1}{\tau}e^{-(a\tau+1)}$, then (2) has no real roots.

The following facts hold for the pur complex roots of (8):

- (I) If $0 < A \le e^{-1}$, then all the pure complex roots of (8) are given by

$$S_{2m+1}^{(1\pm)} = \varphi_{-2,-A}^{(2m+1)} \pm \psi_{-2,-A}^{(2m+1)} i$$

  for $m = 0, 1, 2, \cdots$, where $\theta_{-2,-A}^{(2m+1)} \in (0, \pi/2)$.

- (II) If $e^{-1} < A < 3\pi/2$, then all the pure complex roots of (8) are given by

$$S_{2m+1}^{(2\pm)} = \begin{cases} \varphi_{-2,-A}^{(2m+1)} \pm \psi_{-2,-A}^{(2m+1)} i, & (2m+1)\pi > (-A)_+^* \\ -1 \pm \psi_{(-1,-2),-A}^{(2m+1)} i, & (2m+1)\pi = (-A)_+^* \\ \varphi_{-1,-A}^{(2m+1)} \pm \psi_{-1,-A}^{(2m+1)} i, & \pi \le (2m+1)\pi < (-A)_+^* \end{cases}$$

  for $m = 0, 1, 2, \cdots$, where $\theta_{-1,-A}^{(2m+1)}, \theta_{-2,-A}^{(2m+1)} \in (\arccos\left(\frac{1}{Ae}\right), \frac{\pi}{2})$.

- (III) If $A = 3\pi/2$, then all the pure complex roots of (8) are given by

$$S_{2m+1}^{(3\pm)} = \begin{cases} \varphi_{-2,-A}^{(2m+1)} \pm \psi_{-2,-A}^{(2m+1)} i, & (2m+1)\pi > (-A)_+^* \\ -1 \pm \psi_{(-1,-2),-A}^{(2m+1)} i, & (2m+1)\pi = (-A)_+^* \\ \varphi_{-1,-A}^{(2m+1)} \pm \psi_{-1,-A}^{(2m+1)} i, & 3\pi \le (2m+1)\pi < (-A)_+^* \\ \pm\frac{3\pi}{2} i, & (2m+1)\pi = \pi \end{cases}$$

  for $m = 0, 1, 2, \cdots$, where $\theta_{-1,-A}^{(2m+1)}, \theta_{-2,-A}^{(2m+1)} \in (\arccos\left(\frac{1}{Ae}\right), \frac{\pi}{2})$.

- (IV) If $A > 3\pi/2$, then all the pure complex roots of (8) are given by

$$S_{2m+1}^{(4\pm)} = \begin{cases} \varphi_{-2,-A}^{(2m+1)} \pm \psi_{-2,-A}^{(2m+1)} i, & (2m+1)\pi > (-A)_+^* \\ -1 \pm \psi_{(-1,-2),-A}^{(2m+1)} i, & (2m+1)\pi = (-A)_+^* \\ \varphi_{-1,-A}^{(2m+1)} \pm \psi_{-1,-A}^{(2m+1)} i, & -\frac{\pi}{2} + A < (2m+1)\pi < (-A)_+^* \\ \pm\left((2m+1)\pi + \frac{3\pi}{2}\right) i, & (2m+1)\pi = -\frac{\pi}{2} + A \\ \varphi_{+1,-A}^{(2m+1)} \pm \psi_{+1,-A}^{(2m+1)} i, & \pi \le (2m+1)\pi < -\frac{\pi}{2} + A \end{cases}$$

  for $m = 0, 1, 2, \cdots$, where $\theta_{+1,-A}^{(2m+1)} \in (\frac{\pi}{2}, \pi)$ and $\theta_{-1,-A}^{(2m+1)}, \theta_{-2,-A}^{(2m+1)} \in (\arccos\left(\frac{1}{Ae}\right), \frac{\pi}{2})$.

- (V) If $-e^{-1} \le A < 0$, then all the pure complex roots of (8) are given by

$$S_{2m}^{(1\pm)} = \varphi_{-2,A}^{(2m)} \pm \psi_{-2,A}^{(2m)} i, m = 1, 2, \cdots$$

  where $\theta_{-2,A}^{(2m)} \in (0, \frac{\pi}{2})$.

- (VI) If $-\pi/2 < A < -e^{-1}$, then all the pure complex roots of (8) are given by

$$S_{2m}^{(2\pm)} = \begin{cases} \varphi_{-2,A}^{(2m)} \pm \psi_{-2,A}^{(2m)} i, & 2m\pi > A_+^* \\ -1 \pm \psi_{(-1,-2),A}^{(2m)} i, & (2m+1)\pi = A_+^* \\ \varphi_{-1,A}^{(2m)} \pm \psi_{-1,A}^{(2m)} i, & 0 \le 2m\pi < A_+^* \end{cases}$$

  for $m = 0, 1, 2, \cdots$, where $\theta_{-1,A}^{(2m)}, \theta_{-2,A}^{(2m)} \in (\arccos\left(-\frac{1}{Ae}\right), \frac{\pi}{2})$.

- (VII) If $A = -\pi/2$, then all the pure complex roots of (8) are given by

$$S_{2m}^{(3\pm)} = \begin{cases} \varphi_{-2,A}^{(2m)} \pm \psi_{-2,A}^{(2m)} i, & 2m\pi > A_+^* \\ -1 \pm \psi_{(-1,-2),A}^{(2m)} i, & (2m+1)\pi = A_+^* \\ \varphi_{-1,A}^{(2m)} \pm \psi_{-1,A}^{(2m)} i, & 0 < 2m\pi < A_+^* \\ \pm\frac{\pi}{2} i, & 2m\pi = 0 \end{cases}$$

  for $m = 0, 1, 2, \cdots$, where $\theta_{-1,A}^{(2m)}, \theta_{-2,A}^{(2m)} \in (\arccos\left(-\frac{1}{Ae}\right), \frac{\pi}{2})$.

- (VIII) If $A < -\pi/2$, then all the pure complex roots of (8) are given by

$$S_{2m}^{(4\pm)} = \begin{cases} \varphi_{-2,A}^{(2m)} \pm \psi_{-2,A}^{(2m)} i, & 2m\pi > A_+^* \\ -1 \pm \psi_{(-1,-2),A}^{(2m)} i, & (2m+1)\pi = A_+^* \\ \varphi_{-1,A}^{(2m)} \pm \psi_{-1,A}^{(2m)} i, & -\frac{\pi}{2} - A < 2m\pi < A_+^* \\ \pm\left(2m\pi + \frac{\pi}{2}\right) i, & 2m\pi = -\frac{\pi}{2} - A \\ \varphi_{+1,A}^{(2m)} \pm \psi_{+1,A}^{(2m)} i, & 0 \le 2m\pi < -\frac{\pi}{2} - A. \end{cases}$$

  for $m = 0, 1, 2, \cdots$, where $\theta_{+1,A}^{(2m)} \in (\pi/2, \pi)$, $\theta_{-1,A}^{(2m)}, \theta_{-2,A}^{(2m)} \in (\arccos\left(-\frac{1}{Ae}\right), \frac{\pi}{2})$.

From the previous results, we obtain all the roots of (2) as follows:

- (I) If $b = 0$, then (2) has a unique root $\lambda_0 = -a$.
- (II) If $-\frac{1}{\tau}e^{-(a\tau+1)} \le b < 0$, then the roots of (2) are $\lambda_0^{(1)} = -a + \frac{1}{\tau} lm_+ A$, $\lambda_{2m+1}^{(1\pm)} = -a + \frac{1}{\tau} S_{2m+1}^{(1\pm)}, m = 0, 1, 2, \cdots$.
- (III) If $-\frac{3\pi}{2\tau}e^{-a\tau} < b < -\frac{1}{\tau}e^{-(a\tau+1)}$, then the roots of (2) are $\lambda_0^{(2)} = -a + \frac{1}{\tau} lm_+ A$, and $\lambda_{2m+1}^{(2\pm)} = -a + \frac{1}{\tau} S_{2m+1}^{(2\pm)}, m = 0, 1, 2, \cdots$.
- (IV) If $b = -\frac{3\pi}{2\tau}e^{-a\tau}$, then the roots of (2) are $\lambda_0^{(3)} = -a + \frac{1}{\tau} lm_+ A$, and $\lambda_{2m+1}^{(3\pm)} = -a + \frac{1}{\tau} S_{2m+1}^{(3\pm)}, m = 0, 1, 2, \cdots$.
- (V) If $b < -\frac{3\pi}{2\tau}e^{-a\tau}$, then the roots of (2) are $\lambda_0^{(4)} = -a + \frac{1}{\tau} lm_+ A$, and $\lambda_{2m+1}^{(4\pm)} = -a + \frac{1}{\tau} S_{2m+1}^{(4\pm)}, m = 0, 1, 2, \cdots$.
- (VI) If $0 < b < \frac{1}{\tau}e^{-(a\tau+1)}$, then the roots of (2) are $\lambda_{01}^{(1)} = -a + \frac{1}{\tau} lm_{-1} A, \lambda_{02}^{(1)} = -a + \frac{1}{\tau} lm_{-2} A$, and $\lambda_{2m}^{(1\pm)} = -a + \frac{1}{\tau} S_{2m}^{(1\pm)}, m = 1, 2, \cdots$.

- (VII) If $b = \frac{1}{\tau}e^{-(a\tau+1)}$, then the roots of (2) are the double root $\lambda_{00}^{(1)} = -\left(a + \frac{1}{\tau}\right)$, and $\lambda_{2m}^{(1\pm)} = -a + \frac{1}{\tau}S_{2m}^{(1\pm)}, m = 1, 2, \cdots$.
- (VIII) If $\frac{1}{\tau}e^{-(a\tau+1)} < b < \frac{\pi}{2\tau}e^{-a\tau}$, then the roots of (2) are $\lambda_{2m}^{(2\pm)} = -a + \frac{1}{\tau}S_{2m}^{(2\pm)}, m = 0, 1, 2, \cdots$.
- (IX) If $b = \frac{\pi}{2\tau}e^{-a\tau}$, then the roots of (2) are $\lambda_{2m}^{(3\pm)} = -a + \frac{1}{\tau}S_{2m}^{(3\pm)}, m = 0, 1, 2, \cdots$.
- (X) If $b > \frac{\pi}{2\tau}e^{-a\tau}$, then the roots of (2) are $\lambda_{2m}^{(4\pm)} = -a + \frac{1}{\tau}S_{2m}^{(4\pm)}, m = 0, 1, 2, \cdots$.

Next, we consider (1) under the initial condition

$$x(\theta) = \phi(\theta), \theta \in [-\tau, 0], \tag{10}$$

when $\phi(\theta) \in C[-\tau, 0]$, the initial value problem (1) and (10) possesses the unique solution

$$x(t) = X(t)\phi(0) + b\int_{-\tau}^{0} X(t - \theta - \tau)\phi(\theta)d\theta, \tag{11}$$

where $X(t)$ is the solution of the following initial value problem

$$\begin{cases} x'(t) + ax(t) + bx(t - \tau) = 0, \\ x(\theta) = H(\theta),\ \theta \in (-\infty, +\infty), \end{cases}$$

which is called the fundamental solution of (1), where $H(\theta)$ is the Heaviside function, that is,

$$H(\theta) = \begin{cases} 1, & \theta > 0, \\ 0, & \theta < 0. \end{cases}$$

In view of (11), we can find series expansions of the solutions of the problem (1) and (10). To simplify matters, let us set

$$\phi_{k,C}^{(j)} = \arctan\frac{1 + lm_k\left(C\cos\theta_k^{(j)}\right)}{j\pi + \theta_k^{(j)}},$$

and

$$\Gamma_{k,C}^{(j)}(t) = \frac{\exp\left\{\left(-a + \frac{1}{\tau}\varphi_{k,C}^{(j)}\right)t\right\}\sin\left[\frac{1}{\tau}\psi_{k,C}^{(j)} + \phi_{k,C}^{(j)}\right]}{\sqrt{\left[1 + \varphi_{k,C}^{(j)}\right]^2 + \left[\psi_{k,C}^{(j)}\right]^2}}.$$

The following facts hold for the solutions of (1):

- (I) If $-\frac{1}{\tau}e^{-(a\tau+1)} \le b < 0$, then the fundamental solution of (1) is

$$X(t) = \frac{\exp\left\{\left(-a + \frac{1}{\tau}lm_+A\right)t\right\}}{1 + lm_+A} + 2\sum_{m=0}^{+\infty}\Gamma_{-2,-A}^{(2m+1)}(t).$$

- (II) If $-\frac{3\pi}{2\tau}e^{-a\tau} < b < -\frac{1}{\tau}e^{-(a\tau+1)}$, then the fundamental solution of (1) is

$$X(t) = \frac{\exp\left\{\left(-a+\frac{1}{\tau}lm_+A\right)t\right\}}{1+lm_+A} + 2\sum_{0\le m\le k(A)}\Gamma^{(2m+1)}_{-1,-A}(t) + 2\sum_{m>k(A)}\Gamma^{(2m+1)}_{-2,-A}(t),$$

where $k(A) = -\frac{1}{2} + \frac{1}{2\pi}(-A)^*_+$.

- (III) If $b = -\frac{3\pi}{2\tau}e^{-a\tau}$, then the fundamental solution of (1) is

$$\begin{aligned} X(t) \quad = \quad & \frac{\exp\left\{\left(-a+\frac{1}{\tau}lm_+\frac{3\pi}{2}\right)t\right\}}{1+lm_+\frac{3\pi}{2}} + 2e^{-at}\frac{\cos\frac{3\pi}{2\tau}t + \frac{3\pi}{2}\sin\frac{3\pi}{2\tau}t}{1+\left(\frac{3\pi}{2}\right)^2} \\ & 2\sum_{1\le m\le k(\frac{3\pi}{2})}\Gamma^{(2m+1)}_{-1,-\frac{3\pi}{2}}(t) + 2\sum_{m>k(\frac{3\pi}{2})}\Gamma^{(2m+1)}_{-2,-\frac{3\pi}{2}}(t), \end{aligned}$$

where $k(\frac{3\pi}{2}) = -\frac{1}{2} + \frac{1}{2\pi}(-\frac{3\pi}{2})^*_+$.

- (IV) If $b < -\frac{3\pi}{2\tau}e^{-a\tau}$, then the fundamental solution of (1) is

$$\begin{aligned} X(t) \quad = \quad & \frac{\exp\left\{\left(-a+\frac{1}{\tau}lm_+A\right)t\right\}}{1+lm_+A} + 2\sum_{0\le m\le k_1(A)}\Gamma^{(2m+1)}_{+1,-A}(t) \\ & +2\sum_{k_1(A)<m\le k(A)}\Gamma^{(2m+1)}_{-1,-A}(t) + 2\sum_{m>k(A)}\Gamma^{(2m+1)}_{-2,-A}(t), \end{aligned}$$

where $k_1(A) = -\frac{3}{4} + \frac{A}{2\pi}$, $k(A) = -\frac{1}{2} + \frac{1}{2\pi}(-A)^*_+$.

- (V) If $0 < b < \frac{1}{\tau}e^{-(a\tau+1)}$, then the fundamental solution of (1) is

$$X(t) = \frac{\exp\left\{\left(-a+\frac{1}{\tau}lm_{-1}A\right)t\right\}}{1+lm_{-1}A} + \frac{\exp\left\{\left(-a+\frac{1}{\tau}lm_{-2}A\right)t\right\}}{1+lm_{-2}A} + 2\sum_{m=1}^{+\infty}\Gamma^{(2m)}_{-2,A}(t).$$

- (VI) If $b = \frac{1}{\tau}e^{-(a\tau+1)}$, then the fundamental solution of (1) is

$$X(t) = 2\left(\frac{1}{3}+\frac{t}{\tau}\right)\exp\left\{\left(-a+\frac{1}{\tau}\right)t\right\} + 2\sum_{m=1}^{+\infty}\Gamma^{(2m)}_{-2,A}(t).$$

- (VII) If $\frac{1}{\tau}e^{-(a\tau+1)} < b < \frac{\pi}{2\tau}e^{-a\tau}$, then the fundamental solution of (1) is

$$X(t) = 2\sum_{0\le m\le k(A)}\Gamma^{(2m)}_{-1,A}(t) + 2\sum_{m>k(A)}\Gamma^{(2m)}_{-2,A}(t).$$

where $k(A) = \frac{1}{2\pi}A^*_+$.

- (VIII) If $b = \frac{\pi}{2\tau} e^{-a\tau}$, then the fundamental solution of (1) is

$$X(t) = 2e^{-at} \frac{\cos \frac{\pi}{2\tau} t + \frac{\pi}{2} \sin \frac{\pi}{2\tau} t}{1 + \left(\frac{\pi}{2}\right)^2} + 2 \sum_{1 \le m \le k(A)} \Gamma^{(2m)}_{-1,A}(t) + 2 \sum_{m > k(A)} \Gamma^{(2m)}_{-2,A}(t).$$

  where $k(A) = \frac{1}{2\pi} A^*_+$.

- (IX) If $b > \frac{\pi}{2\tau} e^{-a\tau}$, then the fundamental solution of (1) is

$$X(t) = 2 \sum_{0 \le m \le k_1(A)} \Gamma^{(2m)}_{+1,A}(t) + 2 \sum_{k_1(A) < m \le k(A)} \Gamma^{(2m)}_{-1,A}(t) + 2 \sum_{m > k(A)} \Gamma^{(2m)}_{-2,A}(t).$$

  where $k_1(A) = -\frac{1}{4} - \frac{A}{2\pi}$, $k(A) = \frac{1}{2\pi} A^*_+$.

As our final remark, it is well known that (1) is asymptotically stable if, and only if, the roots of the characteristic equation (2) are in the left half-plane. Since we have found all the roots of (2), the asymptotic stability of (1) can thus be decided. However, we will not give the corresponding criteria since they have already given in a number of places (see e.g. [2, Appendix]).

References

[1] H. S. Ren, On the Accurate Distribution of Characteristic Roots and Stability of Linear Delay Differential Systems, Northeastern Forestry University Press, Harbin, 1999 (in Chinese).

[2] J. K. Hale and Sjoerd M. Verduyn Lunel, Introduction To Functional Differential Equations, Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, 1993.