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Some Results for Generalized Lie Ideals in Prime Rings with Derivation II *

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Abstract

Let R be a prime ring of characteristic different from two, $d : R \to R$ a non-zero derivation, and M a non-zero left ideal of R. We prove the following results: (1) if $a \in R$ and $[d(R), a]_{\sigma,\tau} = 0$, then $\sigma(a) + \tau(a) \in Z$, the center of R, (2) if $d([R, a]_{\sigma,\tau}) = 0$, then $\sigma(a) + \tau(a) \in Z$, (3) if $([R, M]_{\sigma,\tau}, a)_{\sigma,\tau} = 0$, then $a \in Z$, (4) $(d(R), a) = 0$ if, and only if, $d((R, a)) = 0$.

Let R be a ring and σ, τ be two mappings from R into itself. We write $[x, y]$, (x, y), $[x, y]_{\sigma,\tau}$, $(x, y)_{\sigma,\tau}$ for $xy - yx$, $xy + yx$, $x\sigma(y) - \tau(y)x$ and $x\sigma(y) + \tau(y)x$ respectively, and will make extensive use of the following basic commutator identities: $(xy, z) = x[y, z] + (x, z)y = x(y, z) - [x, z]y$, $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$.

An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is inner if there exists an $a \in R$ such that $D(x) = [a, x]$ for all $x \in R$. For subsets $A, B \subset R$, let $[A, B]$ ($[A, B]_{\sigma,\tau}$) be the additive subgroup generated by all $[a, b]$ ($[a, b]_{\sigma,\tau}$) for all $a \in A$ and $b \in B$. We recall that a Lie ideal L is an additive subgroup of R such that $[R, L] \subset L$. We first introduce the generalized Lie ideal in [4] as following. Let U be an additive subgroup of R, and let $\sigma, \tau : R \to R$ be two mappings. Then (i) U is a (σ, τ)-right Lie ideal of R if $[U, R]_{\sigma,\tau} \subset U$, (ii) U is a (σ, τ)-left Lie ideal of R if $[R, U]_{\sigma,\tau} \subset U$, (iii) if U is both a (σ, τ)-right Lie ideal and (σ, τ)-left Lie ideal of R, then U is a (σ, τ)-Lie ideal of R. Every Lie ideal of R is a $(1, 1)$-Lie ideal of R, where $1 : R \to R$ is the identity map. As an example, let I be the set of integers,

$$R = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} : \; x, y, z, t \in I \right\},$$

$$U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : \; x, y \in I \right\} \subset R,$$

and $\tau : R \to R$ the mapping defined by $\tau(x) = bxb$, where $b = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \in R$. Then U is a $(1, \tau)$-left Lie ideal but not a Lie ideal of R.

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Some algebraic properties of (σ,τ)-Lie ideals are considered in [1], [2] and [5], where further references can be found.

Let R be a prime ring of characteristic different from two, $d : R \to R$ a nonzero derivation, Z the center of R and $a \in R$. Lee and Lee in [6] proved that if $[d(R), d(R)] \subset Z$, then R is commutative. In the present paper, we generalize this result for generalized Lie ideal. In [3], Herstein proved that in a prime ring of characteristic different from two, $[d(R), a] = 0$ implies $a \in Z$. We shall extend Herstein's theorem by proving that $[d(R), a]_{\sigma,\tau} = 0$ implies $\sigma(a) + \tau(a) \in Z$.

Throughout this note, R will be a prime ring with characteristic different from 2, Z the center of R, d a non-zero derivation of R and U is (σ,τ)-left Lie ideal of R.

LEMMA 1. If $a \in R$ and $[d(R), a]_{\sigma,\tau} = 0$, then $\sigma(a) + \tau(a) \in Z$.

PROOF. If $a \in Z$ then the proof of the theorem is obvious. So we assume that $a \notin Z$. By hypothesis, we have for all $x \in R$,

$$\begin{aligned} 0 &= [d(x\sigma(a)), a]_{\sigma,\tau} = [d(x)\sigma(a) + xd(\sigma(a)), a]_{\sigma,\tau} \\ &= d(x)[\sigma(a), \sigma(a)] + [d(x), a]_{\sigma,\tau}\sigma(a) + x[d(\sigma(a)), a]_{\sigma,\tau} + [x, \tau(a)]d(\sigma(a)). \end{aligned}$$

Hence we obtain

$$[x, \tau(a)]d(\sigma(a)) = 0,\ x \in R \tag{1}$$

Replacing x by xy, $y \in R$ in (1) and using (1), we get

$$[R, \tau(a)]Rd(\sigma(a)) = 0.$$

Since R is prime ring and $a \notin Z$, we obtain $d(\sigma(a)) = 0$. Now let us consider the following mappings on $R : D(x) = [x, \sigma(a)]$ and $H(x) = [x, a]_{\sigma,\tau}$, where D is a non-zero derivation of R such that $Hd(x) = 0$. For any $x, y \in R$, we have $H(xy) = [xy, a]_{\sigma,\tau} = x[y, \sigma(a)] + [x, a]_{\sigma,\tau}y$. Hence we get

$$H(xy) = H(x)y + xD(y) \tag{2}$$

But this can also be calculated in a different way. Indeed, $H(xy) = [xy, a]_{\sigma,\tau} = x[y, a]_{\sigma,\tau} + [x, \tau(a)]y$ and so one obtains

$$H(xy) = [x, \tau(a)]y + xH(y) \tag{3}$$

For any $r \in R$, $0 = [d(r), a]_{\sigma,\tau} = d(r)\sigma(a) - \tau(a)d(r)$, and so,

$$\begin{aligned} 0 &= d(0) = d(d(r)\sigma(a) - \tau(a)d(r)) \\ &= d^2(r)\sigma(a) + d(r)d(\sigma(a)) - d(\tau(a))d(r) - \tau(a)d^2(r) \\ &\quad [d^2(r), a]_{\sigma,\tau} - d(\tau(a))d(r). \end{aligned}$$

This implies that $d(\tau(a))d(r) = 0$ for all $r \in R$. Using [7, Lemma 1], we obtain $d(\tau(a)) = 0$. On the other hand, for any $x \in R$,

$$\begin{aligned} dH(x) &= d([x, a]_{\sigma,\tau}) = d(x\sigma(a) - \tau(a)x) \\ &= d(x)\sigma(a) + xd(\sigma(a)) - d(\tau(a))x - \tau(a)d(x) = [d(x), a]_{\sigma,\tau} = 0. \end{aligned}$$

Thus we get,

$$dH(R) = 0. \tag{4}$$

In view of (2), (3) and (4), one obtains, for any $x, y \in R$,

$$\begin{aligned} 0 &= Hd(xH(y)) = H(d(x)H(y) + xdH(y)) = H(d(x)H(y)) \\ &= Hd(x)H(y) + d(x)DH(y) = d(x)DH(y), \end{aligned}$$

and so, $d(R)DH(R) = 0$. By using [7, Lemma 1] we arrive at

$$DH(x) = 0.$$

That is,

$$[[x, a]_{\sigma,\tau}, \sigma(a)] = 0,\ x \in R. \tag{5}$$

Taking $\tau(a)x$, instead of x in (5), we get

$$\begin{aligned} 0 &= [[\tau(a)x, a]_{\sigma,\tau}, \sigma(a)] = [\tau(a)[x, a]_{\sigma,\tau} + [\tau(a), \tau(a)]x, \sigma(a)] \\ &= [\tau(a)[x, a]_{\sigma,\tau}, \sigma(a)] = \tau(a)[x, a]_{\sigma,\tau}, \sigma(a)] + [\tau(a), \sigma(a)][x, a]_{\sigma,\tau}, \end{aligned}$$

and so,

$$[\tau(a), \sigma(a)][x, a]_{\sigma,\tau} = 0,\ x \in R. \tag{6}$$

Replacing x by $xy, y \in R$ in (6) and using (6), we obtain

$$[\tau(a), \sigma(a)]R[R, \sigma(a)] = 0.$$

Since R is a prime ring and $a \notin Z$, we get

$$[\tau(a), \sigma(a)] = 0. \tag{7}$$

Now, expanding (5) and using (7) one obtains

$$\begin{aligned} 0 &= [[x, a]_{\sigma,\tau}, \sigma(a)] = [x\sigma(a) - \tau(a)x, \sigma(a)] \\ &= [x, \sigma(a)]\sigma(a) - \tau(a)[x, \sigma(a)] = [[x, \sigma(a)], a]_{\sigma,\tau}, \end{aligned}$$

that is,

$$HD(x) = 0,\ x \in R. \tag{8}$$

Linearizing (8), we get

$$\begin{aligned} 0 &= HD(xy) = H(D(x)y + xD(y)) \\ &= HD(x)y + D(x)D(y) + xHD(y) + [x, \tau(a)]D(y) \\ &= D(x)D(y) + [x, \tau(a)]D(y) = [x, \sigma(a)]D(y) + [x, \tau(a)]D(y), \end{aligned}$$

that is,

$$[x, \sigma(a) + \tau(a)]D(y) = 0,\ x, y \in R.$$

Since D is a non-zero derivation of the prime ring R, using [7, Lemma 1], we obtain $\sigma(a) + \tau(a) \in Z$.

COROLLARY 2. Let R be a prime ring of characteristic different from two, d be a nonzero derivation of R and U is (σ,τ)-left Lie ideal of R. If $[d(R),U]_{\sigma,\tau}=0$, then $\sigma(u)+\tau(u)\in Z$ for all $u\in U$.

THEOREM 3. Let R be a prime ring of characteristic different from two, d a nonzero derivation of R and $a\in R$. If $d([R,a]_{\sigma,\tau})=0$, then $\sigma(a)+\tau(a)\in Z$.

PROOF. For all $r\in R$,

$$0=d([\tau(a)r,a]_{\sigma,\tau})=d(\tau(a)[r,a]_{\sigma,\tau})=d(\tau(a))[r,a]_{\sigma,\tau}+\tau(a)d([r,a]_{\sigma,\tau})$$

and so,

$$d(\tau(a))[r,a]_{\sigma,\tau}=0,\ r\in R. \tag{9}$$

Taking $rs, s\in R$ instead of r in (9), we have

$$0=d(\tau(a))[rs,a]_{\sigma,\tau}=d(\tau(a))r[s,\sigma(a)]+d(\tau(a))[r,a]_{\sigma,\tau}s.$$

Using (9), we get

$$d(\tau(a))R[R,\sigma(a)]=0.$$

Since R is a prime ring, we see that $d(\tau(a))=0$ or $a\in Z$. If $a\in Z$, then $\sigma(a)+\tau(a)\in Z$. The proof is then complete. Therefore, let us assume that $a\notin Z$. For $r\in R$,

$$\begin{aligned}0&=d([r\sigma(a),a]_{\sigma,\tau})=d([r,a]_{\sigma,\tau}\sigma(a))\\&=d([r,a]_{\sigma,\tau})\sigma(a)+[r,a]_{\sigma,\tau}d(\sigma(a))\end{aligned}$$

and so,

$$[r,a]_{\sigma,\tau}d(\sigma(a))=0,\ r\in R. \tag{10}$$

Replacing r by rs, $s\in R$ in (10), we get

$$\begin{aligned}0&=[rs,a]_{\sigma,\tau}d(\sigma(a))=r[s,a]_{\sigma,\tau}d(\sigma(a))+[r,\tau(a)]sd(\sigma(a))\\&=[r,\tau(a)]sd(\sigma(a)).\end{aligned}$$

That is,

$$[r,\tau(a)]Rd(\sigma(a))=0,\ r\in R.$$

Since R is a prime ring, and since we have assumed that $a\notin Z$, we obtain

$$d(\sigma(a))=0.$$

Now, from our hypothesis, we have, for any $r\in R$,

$$\begin{aligned}0&=d([r,a]_{\sigma,\tau})=d(r\sigma(a)-\tau(a)r)\\&=d(r)\sigma(a)+rd(\sigma(a))-d(\tau(a))r-\tau(a)d(r).\end{aligned}$$

Since $d(\sigma(a))=0$ and $d(\tau(a))=0$, we get

$$[d(r),a]_{\sigma,\tau}=0,\ r\in R.$$

By Lemma 1, we have $\sigma(a)+\tau(a)\in Z$.

COROLLARY 4. Let R be a prime ring of characteristic different from two, d a nonzero derivation of R and U is (σ,τ)-left Lie ideal of R. If $d([R,U]_{\sigma,\tau})=0$, then $\sigma(u)+\tau(u)\in Z$ for all $u\in U$.

EXAMPLE. Let
$$R=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in I\right\}$$
and
$$\sigma\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} d & -c\\ -b & a\end{pmatrix}, \tau\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a & -b\\ -c & d\end{pmatrix}$$
be two autumorphisms of R and $a=\begin{pmatrix} 1 & 0\\ 0 & 2\end{pmatrix}\notin Z$. If we define $d:R\to R$ by $d\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & -b\\ c & 0\end{pmatrix}$, then d is a derivation of R such that $d([R,a]_{\sigma,\tau})=0$, but $\sigma(a)+\tau(a)\in Z$.

THEOREM 6. Let R be a prime ring of characteristic different from two, M a non-zero ideal of R and $a\in R$. If $([R,M]_{\sigma,\tau},a)_{\sigma,\tau}=0$, then $a\in Z$.

PROOF. Let m be a noncentral element of M. Consider the following mappings on $R: d_1(r)=[r,m]_{\sigma,\tau}$ and $d_2(r)=(r,a)_{\sigma,\tau}$. Then for any $r\in R$, we have $d_2d_1(r)=d_2([r,m]_{\sigma,\tau})=([r,m]_{\sigma,\tau},a)_{\sigma,\tau}=0$. That is,
$$d_2d_1(r)=0,\ r\in R. \tag{11}$$
For $r\in R$,
$$\begin{aligned} d_1(r\sigma(m)) &= [r\sigma(m),m]_{\sigma,\tau}\\ &= r[\sigma(m),\sigma(m)]+[r,m]_{\sigma,\tau}\sigma(m)=d_1(r)\sigma(m),\end{aligned}$$
and so we get,
$$d_1(r\sigma(m))=d_1(r)\sigma(m),\ r\in R. \tag{12}$$
Let us consider (11) together with (12), we obtain
$$\begin{aligned} 0 &= d_2d_1(r\sigma(m))=d_2(d_1(r)\sigma(m))=(d_1(r)\sigma(m),a)_{\sigma,\tau}\\ &= d_1(r)[\sigma(m),\sigma(a)]+(d_1(r),a)_{\sigma,\tau}\sigma(m),\end{aligned}$$
and so we have
$$d_1(R)[\sigma(m),\sigma(a)]=0. \tag{13}$$
Putting in (13) $rs,\ s\in R$ for r we obtain
$$0=d_1(rs)[\sigma(m),\sigma(a)]=rd_1(s)[\sigma(m),\sigma(a)]+[r,\tau(m)]s[\sigma(m),\sigma(a)],$$
and using (13) we get
$$[r,\tau(m)]R[\sigma(m),\sigma(a)]=0,\ r\in R.$$
Since R is a prime ring, we have $m\in Z$ or $\sigma([m,a])=0$. If $m\in Z$, then $\sigma([m,a])=0$. That is, $[M,a]=0$ since σ is automorphism of R. For all $x\in R$ and $m\in M$, $0=$

$[xm, a] = x[m, a] + [x, a]m = [x, a]m$. That is, $[x, a]RM = (0)$, the zero ideal of R. By the primeness of R we have $a \in Z$.

THEOREM 7. Let R be a prime ring of characteristic different from two, d is a nonzero derivation of R and $a \in R$. Then $(d(R), a) = 0$ if, and only if, $d((R, a)) = 0$.

PROOF. Suppose $(d(R), a) = 0$. In this case we claim that $d(a) = 0$. If $a = 0$ then $d(a) = 0$. So we assume that $a \neq 0$. For any $x \in R$, the relation $(d(x), a) = 0$ gives

$$0 = (d(xa), a) = (d(x)a+xd(a), a) = d(x)[a, a]+(d(x), a)a+x(d(a), a)-[x, a]d(a) \quad (14)$$

and so,

$$[x, a]d(a) = 0, \ x \in R. \quad (15)$$

If we take xy, $y \in R$ instead of x in (15), we obtain

$$[R, a]Rd(a) = (0).$$

Since R is a prime ring we have $a \in Z$ or $d(a) = 0$. Now, if $a \in Z$ then $0 = (d(a), a) = d(a)a + ad(a) = 2d(a)a$. Then $d(a)a = 0$. Since we assumed that $0 \neq a$ and R is a prime ring, we get $d(a) = 0$. Thus, we conclude that $d(a) = 0$. Hence for any $r \in R$ we have $d((r, a)) = (d(r), a) + (r, d(a))$, and so, $d((R, a)) = 0$. Conversely, for $x \in R$,

$$\begin{aligned} 0 &= d((ax, a)) = d(a(x, a) + [a, a]x) \\ &= d(a(x, a)) = d(a)(x, a) + ad((x, a)). \end{aligned}$$

Hence we have

$$d(a)(x, a) = 0, \ x \in R. \quad (16)$$

In (16) replace x by xy and use (16), we get

$$0 = d(a)(xy, a) = d(a)x[y, a] + d(a)(x, a)y = d(a)x[y, a],$$

that is, $d(a)R[R, a] = 0$. Since R is a prime ring, we have $d(a) = 0$ or $a \in Z$. If $d(a) = 0$, then for any $r \in R$ we have $0 = d((r, a)) = (d(r), a) + (r, d(a)) = (d(r), a)$, and so $(d(R), a) = 0$. If $a \in Z$, then $0 = d((a, a)) = 4d(a)a$. Since the characteristic of R is different from 2, $d(a)a = 0$. Since $a \in Z$, we get $d(a)Ra = 0$, and so, $d(a) = 0$ or $a = 0$. Thus, $d(a) = 0$ is obtained. Finally, $(d(R), a) = 0$ as required.

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