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Super and Strongly Faintly Continuous Multifunctions *

İdris Zorlutuna†, Yalçın Küçük‡

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Abstract

In this paper, we introduce and discuss some strong forms of faintly continuity for multifunctions. Basic properties and characterizations of such multifunctions are established.

1 Introduction

In a series of papers, Long and Herrington [6] introduced and discussed θ-topology and faintly continuous functions. In 1990, Yalvaç [17] introduced the concepts of super and strongly faintly continuous functions. The main purpose of this paper is to define super and strongly faintly continuous multifunctions and to obtain several characterizations and basic properties of such multifunctions.

Let A be a subset of a topological space (X, τ). $intA$ and clA denote the interior and closure of A respectively. A subset A of X is called regular open (regular closed) iff $A = int(cl(A))$ $(A = cl(int(A)))$. The family of all regular open subsets of (X, τ) form a base for a smaller topology τ_s on X, called semi regularizations of τ (see [6]). A point $x \in X$ said to be a δ-cluster point of the subset A of (X, τ) if $U \cap A \neq \emptyset$ for every τ-regular open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A [16] and is denoted by δ-clA. If $A = \delta$-clA then A is called δ-closed and the complement of a δ-closed set is called δ-open. A point $x \in X$ is said to be a θ-cluster point of A if $clU \cap A \neq \emptyset$ for each open neighborhood U of x. The set of all θ-cluster points of A is called the θ-closure of A [16] and is denoted by θ-clA. If $A = \theta$-clA then A is called θ-closed and the complement of a θ-closed set is called θ-open. In a similar manner, the θ-interior of a set A is defined to be the set of all $x \in A$ for which there exists a closed neighborhood of x contained in A. In a topological space (X, τ), θ-open sets form a topology τ_θ on X and (X, τ) is regular iff $\tau = \tau_\theta$ [6].

A space (X, τ) is said to be almost regular [14] if for every regular closed set F and each point x not belonging to F, there exist disjoint open sets U and V containing F and x respectively. A subset A of a topological space (X, τ) is called N-closed [1] (H-set [16]) if every open cover of A by open sets in X has a finite subfamily whose

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†Department of Mathematics, Cumhuriyet University, 58140, Sivas-Turkey
‡Department of Mathematics, Anadolu University, Eskişehir-Turkey

interior of closures (resp. closures) cover A. A space (X, τ) is called nearly compact [15] (quasi H-closed [12]) if it is N-closed subset (resp. H-set) of it.

The net $(x_\alpha)_{\alpha \in I}$ is θ-convergent (δ-convergent [9]) to x if for each θ-open (resp. regular open) set U containing x, there exists a $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$. The net $(x_\alpha)_{\alpha \in I}$ is r-convergent [4] to x if for each open set U containing x, there exists a $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in clU$.

For a given topological space (X, τ), the collection of all sets of the form $U^+ = \{T \subseteq X : T \subseteq U\}$ ($U^- = \{T \subseteq X : T \cap U \neq \emptyset\}$) with U in τ form a basis (subbasis) for a topology on 2^X, where 2^X is the set of all nonempty subsets of X (see [7]). This topology is called upper (lower) Vietoris topology and denoted by τ_V^+ (τ_V^-). A multifunction F of a set X into Y is a correspondence such that $F(x)$ is a nonempty subset of Y, for each $x \in X$. We will denote such a multifunction by $F : X \hookrightarrow Y$. For a multifunction F, the upper and lower inverse set of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$ respectively, that is $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. Given a multifunction $F : X \hookrightarrow Y$ where $F(x)$ is nonempty, we define the induced function f on X into 2^Y by setting $f(x) = F(x)$ for each $x \in X$. Note that f is single valued and f will always denote the function induced by F unless otherwise stated. A function $f : X \to Y$ is said to be super continuous [8] (strongly θ-continuous [5]) if for each $x \in X$ and each open set V containing $f(x)$, there is an open set U containing x such that $f(int(clU)) \subseteq V$ (resp. $f(clU) \subseteq V$). A multifunction $F : X \hookrightarrow Y$ is called upper semi continuous or u.s.c. [11] (lower semi continuous or l.s.c. [11]) at a point $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there is an open set U containing x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \emptyset$ for each $z \in U$). A multifunction $F : X \hookrightarrow Y$ is called upper strongly θ-continuous or u.s.θ-c. [2] (lower strongly θ-continuous or l.s.θ-c. [2]) at a point $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there is an open set U containing x such that $F(clU) \subseteq V$ ($F(z) \cap V \neq \emptyset$ for each $z \in clU$). A multifunction $F : X \hookrightarrow Y$ is called upper δ-continuous or u.δ-c. [3] (lower δ-continuous or l.δ-c. [3]) at a point $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \subseteq int(clV)$ (resp. $F(x) \cap int(clV) \neq \emptyset$), there is an open set U containing x such that $F(int(clU)) \subseteq int(clV)$ ($F(z) \cap int(clV) \neq \emptyset$ for each $z \in int(clU)$).

2 Super Faintly Continuous Multifunctions

In this section, we define upper (lower) super faintly continuous multifunctions and we obtain many characterizations and basic properties of these multifunctions.

DEFINITION 1. A multifunction $F : X \hookrightarrow Y$ is said to be (a) upper super faintly continuous (briefly u.s.f.c.) at a point $x \in X$ if for each θ-open set V in Y with $F(x) \subseteq V$, there exists an open set U containing x such that $F(int(clU)) \subseteq V$; (b) lower super faintly continuous (briefly l.s.f.c.) at a point $x \in X$ if for each θ-open set V in Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in int(clU)$; and (c) upper (lower) super faintly continuous on X if it has the property at each point $x \in X$.

EXAMPLE 1. Let $X = \{0, 1, 2\}$ and $Y = \{a, b, c, d, e\}$. Let τ and υ be respectively topologies on X and on Y given by $\tau = \{\emptyset, X, \{0\}, \{1\}, \{0, 1\}\}$ and $\upsilon =$

$\{\emptyset, Y, \{a,b\}, \{c,d\}, \{c,d,e\}, \{a,b,c,d\}\}$. Define the multifunction $F : X \hookrightarrow Y$ by $F(0) = \{a,b,c\}$, $F(1) = \{d\}$, $F(2) = \{b,c,e\}$. Then F is u.s.f.c.

EXAMPLE 2. Let $X = \{0,1,2\}$ and $Y = \{a,b,c,d,e\}$. Let τ and υ be respectively topologies on X and on Y given by $\tau = \{\emptyset, X, \{0\}, \{1\}, \{0,1\}\}$ and $\upsilon = \{\emptyset, Y, \{a,b\}, \{c,d\}, \{c,d,e\}, \{a,b,c,d\}\}$. Define the multifunction $F : X \hookrightarrow Y$ by $F(0) = \{a,b,c\}$, $F(1) = \{d\}$, $F(2) = \{c,d\}$. Then F is l.s.f.c.

THEOREM 1. Let $F : (X,\tau) \hookrightarrow (Y,\upsilon)$ be a multifunction, then the following statements are equivalent.

(1) F is l.s.f.c.

(2) For any θ-open set $V \subseteq Y$ and for each x of X with $F(x) \cap V \neq \emptyset$, there is a regular open set U containing x such that $z \in U$ implies $F(x) \cap V \neq \emptyset$.

(3) $F : (X,\tau_s) \hookrightarrow (Y,\upsilon_\theta)$ is l.s.c.

(4) $F^-(V) \subseteq X$ is δ-open in X for every θ-open set V of Y.

(5) $F^+(K) \subseteq X$ is δ-closed in X for every θ-closed set K of Y.

(6) The induced mapping $f : (X,\tau) \to (2^Y, (\upsilon_\theta)_V^-)$ is super continuous.

(7) The induced mapping $f : (X,\tau_s) \to (2^Y, (\upsilon_\theta)_V^-)$ is continuous.

(8) For each $x \in X$ and for each net $(x_\alpha)_{\alpha\in I}$ which is δ-converging to x and any θ-open set V with $F(x) \cap V \neq \emptyset$, there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $F(x_\alpha) \cap V \neq \emptyset$.

(9) For each $y \in F(x)$ and for every net $(x_\alpha)_{\alpha\in I}$ which is δ-converging to x, there exists a subnet $(z_\beta)_{\beta\in\xi}$ of the net $(x_\alpha)_{\alpha\in I}$ and a net $(y_\beta)_{(\beta,V)\in\xi}$ in Y with $y_\beta \in F(z_\beta)$ is θ-convergent to y.

PROOF. (1)$\Rightarrow$(2). Let $x \in X$ and let V be a θ-open set in Y with $F(x) \cap V \neq \emptyset$. Since f is l.s.f.c. at x, there exists an open set W of X containing x such that $z \in int(clW)$ implies $F(z) \cap V \neq \emptyset$. Put $int(clW) = U$. Then U is a regular open set in X and $z \in U$ implies $F(z) \cap V \neq \emptyset$.

(2)$\Rightarrow$(3) and (3)$\Rightarrow$(4). These are immediate.

(4)$\Rightarrow$(5). Let $K \subseteq Y$ be any θ-closed set. Then $Y - K \subseteq Y$ is a θ-open set, by (4), $F^-(Y-K) \subseteq X$ is a δ-open set. Since we can write $F^+(K) = X - F^-(Y-K)$, $F^+(K)$ is a δ-closed set in X.

(5)$\Rightarrow$(1). Let $x \in X$ and let $V \subseteq Y$ be a θ-open set with $x \in F^-(V)$. Then $Y - V \subseteq Y$ is a θ-closed set, by (5), $F^+(Y-V) \subseteq X$ is a δ-closed set. Since we can write $F^-(V) = X - F^+(Y-V)$, $F^-(V)$ is a δ-open set in X and $x \in F^-(V)$. Therefore, there is an open set U containing x such that $x \in int(clU) \subseteq F^-(V)$.

(1)$\Rightarrow$(6). For any $x \in X$, let $f(x) \in \cap_{i=1}^n V_i^-$ and $\cap_{i=1}^n V_i^-$ be a $(\upsilon_\theta)_V^-$-open set in 2^Y. Then $V_1, ..., V_n$ are θ-open sets in Y and since $f(x) \in V_i^-$ for $i = 1,2,...,n$, we have $F(x) \cap V_i \neq \emptyset$. Since F is l.s.f.c. at $x \in X$, there exists an open set U_i containing x such that $z \in int(clU_i)$ implies $F(z) \cap V_i \neq \emptyset$ for all $i = 1,2,...,n$. Put $U = \cap_{i=1}^n U_i$. Then, we obtain $f(z) \in \cap_{i=1}^n V_i^-$ for all $z \in int(clU)$ i.e. $f(int(clU)) \subseteq \cap_{i=1}^n V_i^-$.

(6)$\Rightarrow$(1). Let $x \in X$ and V be a θ-open set with $x \in F^-(V)$. Then $f(x) = F(x) \in V^-$ and V^- is a $(\upsilon_\theta)_V^-$-open set in 2^Y. Since f is super continuous at $x \in X$, there exists an open set U such that $F(int(clU)) = f(int(clU)) \subseteq V^-$. Thus we obtain an open set U containing x such that $z \in int(clU)$ implies $F(z) \cap V \neq \emptyset$.

(6)$\Leftrightarrow$(7). This is obvious from [13].

(4)⇒(8). Let $x \in X$ and $(x_\alpha)_{\alpha\in I}$ be a net δ-converging to x. By (4), for any θ-open set V in Y with $F(x)\cap V \neq \emptyset$, $F^-(V)$ is an δ-open set in X and $x \in F^-(V)$. Hence there exists an open set U such that $x \in int(clU) \subseteq F^-(V)$. For this U, since $(x_\alpha)_{\alpha\in I}$ is δ-convergent to x, there is a $\alpha_0 \in I$ such that $x_\alpha \in F^-(V)$ for all $\alpha \geq \alpha_0$ and $F(x_\alpha)\cap V \neq \emptyset$ for all $\alpha \geq \alpha_0$ and (8) follows.

(8)⇒(4). Suppose (4) is not true. Then there is a θ-open set V in Y with $x \in F^-(V)$ such that for each open set U of X containing x, $x \in int(clU) \not\subseteq F^-(V)$ i.e. there is a $x_U \in int(clU)$ such that $x_U \notin F^-(V)$. Define $D = \{(x_U, U) : U \in \tau(x), x_U \in int(clU), x_U \notin F^-(V)\}$. Now the ordering $\leq$ defined by $(x_{U_1}, U_1) \leq (x_U, U) \Leftrightarrow U \subseteq U_1$ is a direction on D and g defined by $g : D \to X$, $g((x_U, U)) = x_U$ is a net on X. The net $(x_U)_{(x_U,U)\in D}$ is δ-converging to x. But $F(x_U)\cap V \neq \emptyset$ for all $(x_U, U) \in D$. This is a contradiction.

(1)⇒(9). Suppose F is l.s.f.c. at x_0. Let $(x_\alpha)_{\alpha\in I}$ be a net δ-converging to x_0. Let $y \in F(x_0)$ and V be any θ-open set containing y. So we have $F(x_0)\cap V \neq \emptyset$. Since F is l.s.f.c. at x_0, there exists an open set U such that $x \in int(clU) \subseteq F^-(V)$. Since the net $(x_\alpha)_{\alpha\in I}$ is δ-convergent to x_0, for this U, there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in int(clU)$. Therefore, we have the implication $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in F^-(V)$. For each θ-open set $V \subseteq Y$ containing y, define the sets $I_V = \{\alpha_0 \in I : \alpha \geq \alpha_0 \Rightarrow x_\alpha \in F^-(V)\}$ and $\xi = \{(\alpha, V) : \alpha \in I_V, y \in V$ and V is θ-open$\}$ and an ordering $\geq$ on ξ as follows: "$(\acute{\alpha}, \acute{V}) \geq (\alpha, V) \Leftrightarrow \acute{V} \subseteq V$ and $\acute{\alpha} \geq \alpha$". Define $\varphi : \xi \to I$ by $\varphi((\beta, V)) = \beta$. Then φ is increasing and cofinal in I, so φ defines a subnet of $(x_\alpha)_{\alpha\in I}$. We denote the subnet $(z_\beta)_{(\beta,V)\in\xi}$. On the other hand, for any $(\beta, V) \in \xi$, if $\beta \geq \beta_0$ then $x_\beta \in F^-(V)$ and we have $F(z_\beta)\cap V = F(x_\beta)\cap V \neq \phi$. Pick $y_\beta \in F(z_\beta)\cap V \neq \phi$. Then the net $(y_\beta)_{(\beta,V)\in\xi}$ is θ-convergent to y. To see this, let V_0 be a θ-open set containing y. Then there exists $\beta_0 \in I$ such that $\varphi((\beta_0, V_0)) = \beta_0$ and $y_{\beta_0} \in V$. If $(\beta, V) \geq (\beta_0, V_0)$ this means that $\beta \geq \beta_0$ and $V \subseteq V_0$. Therefore, $y_\beta \in F(z_\beta)\cap V = F(x_\beta)\cap V \subseteq F(x_\beta)\cap V_0$, so $y_\beta \in V_0$. Thus $(y_\beta)_{(\beta,V)\in\xi}$ is θ-convergent to y.

(9)⇒(1). Suppose (1) is not true, i.e. F is not l.s.f.c. at x_0. Then there exists a θ-open set $V \subseteq Y$ so that $x_0 \in F^-(V)$ and for each open set $U \subseteq X$ containing x_0, there is a point $x_U \in int(clU)$ for which $x_U \notin F^-(V)$. Let us consider the net $(x_U)_{U\in\tau(x_0)}$ where $\tau(x_0)$ is the system of τ-neighborhoods of x_0. Obviously $(x_U)_{U\in\tau(x_0)}$ is δ-convergent to x_0. Let $y_0 \in F(x_0)\cap V$. By (9), there is a subnet $(z_w)_{w\in W}$ of $(x_U)_{U\in\tau(x_0)}$ and $y_w \in F(z_w)$ like $(y_w)_{w\in W}$ is θ-convergent to y_0. As $y_0 \in V$ and $V \subseteq Y$ is a θ-open set, there is $\acute{w}_0 \in W$ so that $w \geq \acute{w}_0$ implies $y_w \in V$. On the other hand, $(z_w)_{w\in W}$ is a subnet of the net $(x_U)_{U\in\tau(x_0)}$ and so there is a function $h : W \to \tau(x_0)$ such that $z_w = x_{h(w)}$ and for each $U \in \tau(x_0)$, there is $\tilde{w}_0 \in W$ such that $h(\tilde{w}_0) \geq U$. If $w \geq \tilde{w}_0$, then $h(w) \geq h(\tilde{w}_0) \geq U$. Let us consider $w_0 \in W$ so that $w_0 \geq \acute{w}_0$ and $w_0 \geq \tilde{w}_0$. Therefore, $y_w \in V$ for each $w \geq w_0$. By the definition of the net $(x_U)_{U\in\tau(x_0)}$, we have $F(z_w)\cap V = F(x_{h(w)})\cap V = \emptyset$ and $y_w \notin V$. This is a contradiction and so F is l.s.f.c. at x_0.

THEOREM 2. For a multifunction $F : (X,\tau) \hookrightarrow (Y,\upsilon)$, the following statements are equivalent.

(1) F is u.s.f.c.

(2) For any θ-open set $V \subseteq Y$ and for each x of X with $F(x) \subseteq V$, there is a regular open set U containing x such that $F(U) \subseteq V$.

(3) $F : (X, \tau_s) \hookrightarrow (Y, \upsilon_\theta)$ is u.s.c.
(4) $F^+(V) \subseteq X$ is δ-open in X for every θ-open set V of Y.
(5) $F^-(K) \subseteq X$ is δ-closed in X for every θ-closed set K of Y.
(6) The induced mapping $f : (X, \tau) \to (2^Y, (\upsilon_\theta)^+_V)$ is super continuous.
(7) The induced mapping $f : (X, \tau_s) \to (2^Y, (\upsilon_\theta)^+_V)$ is continuous.
(8) For each $x \in X$ and for each net $(x_\alpha)_{\alpha\in I}$ which is δ-converging to x and any θ-open set V with $F(x) \subseteq V$, there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $F(x_\alpha) \subseteq V$.

The proof is similar to that of Theorem 1, and is omitted.

For a given multifunction $F : X \hookrightarrow Y$, the graph multifunction $G_F : X \hookrightarrow X \times Y$ is defined as $G_F(x) = \{x\} \times F(x)$ for every $x \in X$. In [10], it was shown that for a multifunction $F : X \hookrightarrow Y$, $G^+_F(A \times B) = A \cap F^+(B)$ and $G^-_F(A \times B) = A \cap F^-(B)$ where $A \subseteq X$ and $B \subseteq Y$.

THEOREM 3. If the graph multifunction of $F : X \hookrightarrow Y$ is u(l).s.f.c., then F is u(l).s.f.c.

PROOF. We shall prove only the case where F is l.s.f.c. Let $x \in X$ and V be a θ-open set in Y such that $x \in F^-(V)$. Then $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$ and $X \times V$ is θ-open in $X \times Y$ by Theorem 5 in [6]. Since the graph multifunction G_F is l.s.f.c., there exists an open set U containing x such that $z \in int(clU)$ implies $G_F(z) \cap (X \times V) \neq \emptyset$. Therefore, we obtain $int(clU) \subseteq G^-_F(X \times V) = F^-(V)$ from above equalities. Consequently, F is l.s.f.c.

PROPOSITION 4. Let (X, τ) be a topological space, $A \subset Y$ an open set and $U \subset X$ a regular open set. Then $W = A \cap U$ is regular open set in A [3].

THEOREM 5. For a multifunction $F : X \hookrightarrow Y$, the following statements are true.
a) If F is u(l).s.f.c. and A is an open set of X, then $F|_A : A \hookrightarrow Y$ is u(l).s.f.c.
b) Let $\{A_\alpha : \alpha \in I\}$ be a regular open cover of X. Then a multifunction $F : X \hookrightarrow Y$ is u(l).s.f.c. iff the restrictions $F|_{A_\alpha} : A_\alpha \hookrightarrow Y$ are u(l).s.f.c. for every $\alpha \in I$.

The proof is obvious from the above proposition and we omit it.

THEOREM 6. If $F : X \hookrightarrow Y$ is a l.δ-c. multifunction and $G : Y \hookrightarrow Z$ is a l.s.f.c. multifunction, then $G \circ F : X \hookrightarrow Z$ is a l.s.f.c. multifunction.

PROOF. Let V be a θ-open set of Z. We know that $(G \circ F)^-(V) = F^-(G^-(V))$. Since G is l.s.f.c., $G^-(V)$ is a δ-open set in Y and since F is l.δ-c., $F^-(G^-(V))$ is an δ-open set in X by Theorem 2.2 in [3]. Thus we obtain that $(G \circ F)^-(V)$ is δ-open in X, and so $G \circ F$ is l.s.f.c.

A multifunction $F : X \hookrightarrow Y$ is said to be point closed (resp. point compact) iff for each $x \in X$, $F(x)$ is closed (resp. compact) in Y.

THEOREM 7. Let $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ be a point compact and u.s.f.c. multifunction. If A is N-closed in X, then $F(A)$ is υ_θ-compact in Y.

PROOF. Let A be a N-closed set in X, and Σ be υ_θ-open cover of $F(A)$. If $a \in A$, then $F(a) \subseteq \cup\Sigma$. Since $\upsilon_\theta \subseteq \upsilon$ and $F(a)$ is compact, there exists a finite subfamily $\Sigma_{n(a)}$ of Σ such that $F(a) \subseteq \cup\Sigma_{n(a)}$. Let $\cup\Sigma_{n(a)}$ be V_a. V_a is a θ-open set in Y. Since F is u.s.f.c. at a, there exists an open set U_a of X such that $a \in int(clU_a) \subseteq F^+(V_a)$. Therefore, $\Psi = \{U_a : a \in A\}$ is an open cover of A. Since A is N-closed set in X, there exist $a_1, a_2, ..., a_k \in A$ such that $A \subseteq \cup\{int(clU_{a_i}) : a_i \in A, i = 1, 2, ..., k\}$. So we obtain

$$\begin{aligned} F(A) &\subseteq F(\cup\{int(clU_{a_i}) : a_i \in A, i = 1, 2, ..., k\}) \\ &\subseteq \cup\{V_{a_i} : a_i \in A, i = 1, 2, ..., k\} \\ &\subseteq \cup\{\cup\Sigma_{n(a_i)} : a_i \in A, i = 1, 2, ..., k\}. \end{aligned}$$

Thus $F(A)$ is υ_θ-compact in Y.

We know that in almost regular space (Y, υ), quasi H-closedness and υ_θ-compactness are the same [18]. Therefore, we have the following corollary.

COROLLARY 8. Let $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ be a point compact and u.s.f.c. multifunction. If X is nearly compact and F is surjective, then Y is υ_θ-compact. In addition, if (Y, υ) is almost regular, then space (Y, υ) is quasi H-closed.

3 Strongly Faintly Continuous Multifunctions

In this section, we define upper (lower) strongly faintly continuous multifunctions and we obtain many characterizations and basic properties of these multifunctions.

DEFINITION 2. A multifunction $F : X \hookrightarrow Y$ is said to be (a) upper strongly faintly continuous (briefly u.str.f.c.) at a point $x \in X$ if for each θ-open set V in Y with $F(x) \subseteq V$, there exists an open set U containing x such that $F(clU) \subseteq V$; (b) lower strongly faintly continuous (briefly l.str.f.c.) at a point $x \in X$ if for each $\theta-$open set V in Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in clU$; and (c) upper (lower) strongly faintly continuous on X if it has the property at each point $x \in X$.

As an example, let $X = \{0, 1\}$ with topology $\tau = \{\emptyset, X, \{0\}\}$, let $Y = \{a, b, c\}$ with topology $\upsilon = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$, and let $F : X \hookrightarrow Y$ be defined as $F(0) = \{a\}$ and $F(1) = \{a, b\}$. Then F is u(l).str.f.c. since the only $\theta-$open set in Y is Y itself.

COROLLARY 9. If a multifunction F is u(l).str.f.c., then F is u(l).s.f.c.

Note that the converse of the above corollary is false in general. Indeed, in Example 1, F is u.s.f.c., but F is not u.str.f.c at $1 \in X$. Also, in Example 2, F is l.s.f.c., but F is not l.str.f.c at $0 \in X$.

THEOREM 10. For a multifunction $F : (X, \tau) \hookrightarrow (Y, \upsilon)$, the following statements are equivalent.

(1) F is l.str.f.c.
(2) $F : (X, \tau_\theta) \hookrightarrow (Y, \upsilon_\theta)$ is l.s.c.
(3) $F : (X, \tau) \hookrightarrow (Y, \upsilon_\theta)$ is l.s.$\theta-$c.
(4) $F^-(V) \subseteq X$ is θ-open in X for every θ-open set V of Y.
(5) $F^+(K) \subseteq X$ is θ-closed in X for every θ-closed set K of Y.
(6) The induced mapping $f : (X, \tau) \to (2^Y, (\upsilon_\theta)^-_V)$ is strongly θ-continuous.
(7) The induced mapping $f : (X, \tau_\theta) \to (2^Y, (\upsilon_\theta)^-_V)$ is continuous.
(8) For each $x \in X$ and for each net $(x_\alpha)_{\alpha \in I}$ which is r-converging to x and any θ-open set V with $F(x) \cap V \neq \emptyset$, there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $F(x_\alpha) \cap V \neq \emptyset$.

(9) For each $y \in F(x)$ and for every net $(x_\alpha)_{\alpha \in I}$ which is r-converging to x, there exists a subnet $(z_\beta)_{\beta \in \xi}$ of the net $(x_\alpha)_{\alpha \in I}$ and a net $(y_\beta)_{(\beta, V) \in \xi}$ in Y with $y_\beta \in F(z_\beta)$ is θ-convergent to y.

THEOREM 11. For a multifunction $F : (X, \tau) \hookrightarrow (Y, \upsilon)$, the following statements are equivalent.

(1) F is u.str.f.c.
(2) $F : (X, \tau_\theta) \hookrightarrow (Y, \upsilon_\theta)$ is u.s.c.
(3) $F : (X, \tau) \hookrightarrow (Y, \upsilon_\theta)$ is u.s.θ-c.
(4) $F^+(V) \subseteq X$ is θ-open in X for every θ-open set V of Y.
(5) $F^-(K) \subseteq X$ is θ-closed in X for every θ-closed set K of Y.
(6) Induced mapping $f : (X, \tau) \to (2^Y, (\upsilon_\theta)^+_V)$ is strongly θ-continuous.
(7) Induced mapping $f : (X, \tau_\theta) \to (2^Y, (\upsilon_\theta)^+_V)$ is continuous.
(8) For each $x \in X$ and for each net $(x_\alpha)_{\alpha \in I}$ which is r-converging to x and any θ-open set V with $F(x) \subseteq V$, there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $F(x_\alpha) \subseteq V$.

THEOREM 12. If the graph multifunction of $F : X \hookrightarrow Y$ is u(l).str.f.c., then F is u(l).str.f.c.

PROOF. We shall only prove the case where F is u.str.f.c. Let $x \in X$ and V be a θ-open set in Y such that $x \in F^+(V)$. Then $G_F(x) \subseteq X \times V$ and $X \times V$ is θ-open in $X \times Y$ by Theorem 5 in [6]. Since the graph multifunction G_F is u.str.f.c., there exists an open set U containing x such that $G_F(clU) \subseteq X \times V$. Therefore, we obtain $clU \subseteq G_F^+(X \times V) = F^+(V)$. Consequently, F is u.str.f.c.

THEOREM 13. If $F : X \hookrightarrow Y$ and $G : Y \hookrightarrow Z$ are u(l).str.f.c. multifunctions, then $G \circ F : X \hookrightarrow Z$ u(l).str.f.c. multifunction.

The proof is similar to that of Theorem 5 by Theorem 7 and Theorem 8.

The graph $G(F)$ of the multifunction $F : X \hookrightarrow Y$ is θ-closed with respect to X if for each $(x, y) \notin G(F)$, there exist an open set U containing x and an open set V containing y such that $(clU \times V) \cap G(F) = \emptyset$.

THEOREM 14. Let $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ be a point closed multifunction. If F is u.str.f.c. and assume that Y is regular, then $G(F)$ is θ-closed with respect to X.

PROOF. Suppose $(x, y) \notin G(F)$. Then we have $y \notin F(x)$. Since Y is regular, there exist disjoint open sets V_1, V_2 of Y such that $y \in V_1$ and $F(x) \subseteq V_2$. By regularity of Y, V_2 is also θ-open in Y. Since F is u.str.f.c. at x, there exists an open set U in X containing x such that $F(clU) \subseteq V_2$. Therefore, we obtain $x \in U$, $y \in V_1$ and $(x, y) \in clU \times V_1 \subseteq (X \times Y) - G(F)$. So $G(F)$ is θ-closed with respect to X.

THEOREM 15. Let $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ be a point compact and u.str.f.c. multifunction. If A is H-set, then $F(A)$ is υ_θ-compact in Y.

PROOF. Let A be a H-set and Σ be υ_θ-open cover of $F(A)$. If $a \in A$, then $F(a) \subseteq \cup\Sigma$. Since $\upsilon_\theta \subseteq \upsilon$ and $F(a)$ is compact, there exists a finite subfamily $\Sigma_{n(a)}$ of Σ such that $F(a) \subseteq \cup\Sigma_{n(a)}$. Let $\cup\Sigma_{n(a)}$ be V_a. V_a is a θ-open set in Y. Since F is u.str.f.c. at a, there exists an open set U_a of X such that $a \in clU_a \subseteq F^+(V_a)$. Therefore, $\Psi = \{U_a : a \in A\}$ is an open cover of A. Since A is H-set, there exist

$a_1, a_2, ..., a_k \in A$ such that $A \subseteq \cup\{clU_{a_i} : a_i \in A, i = 1, 2, ..., k\}$. So we obtain

$$\begin{aligned} F(A) &\subseteq F(\cup\{clU_{a_i} : a_i \in A, i = 1, 2, ..., k\}) \\ &\subseteq \cup\{V_{a_i} : a_i \in A, i = 1, 2, ..., k\} \\ &\subseteq \cup\{\cup\Sigma_{n(a_i)} : a_i \in A, i = 1, 2, ..., k\}. \end{aligned}$$

Thus $F(A)$ is $\upsilon_\theta-$compact in Y.

COROLLARY 16. Let $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ be a point compact and u.str.f.c. multifunction. If X is quasi H-closed and F is surjective, then Y is υ_θ-compact. In addition, if (Y, υ) is almost regular, then space (Y, υ) is quasi H-closed.

THEOREM 17. Let $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ be a point closed and u.str.f.c. multifunction. If F satisfies $x_1 \neq x_2 \Rightarrow F(x_1) \neq F(x_2)$ and Y is a regular space, then X will be Hausdorff.

PROOF. Let distinct x_1, x_2 belong to X. Then $F(x_1) \neq F(x_2)$. Since F is point closed and Y is regular, for all $y \in F(x_1)$ with $y \notin F(x_2)$, there exist θ-open sets V_1, V_2 containing y and $F(x_2)$ respectively such that $V_1 \cap V_2 = \emptyset$. Since F is u.str.f.c. and $F(x_2) \subset V_2$, there exists an open set U containing x_2 such that $F(clU) \subseteq V_2$. Thus $x_1 \notin clU$. Therefore, U and $X - clU$ are disjoint open sets separating x_1 and x_2.

The following example shows that if upper strongly faintly continuity is replaced by upper semi continuity, Theorem 17 will be false.

EXAMPLE 3. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}\{c\}\{a, c\}\}$ and $Y = [0, 1]$ with the usual topology. Define the multifunction $F : X \hookrightarrow Y$, by $F(x) = [0, 1/4]$ for $x = a$, $F(x) = Y$ for $x = b$ and $F(x) = \{1/3\}$ for $x = c$. Then F is point closed, u.s.c. and Y is regular, but X is not Hausdorff.

We know that a space (Y, υ) is regular iff $\upsilon = \upsilon_\theta$. Therefore, for a multifunction which is defined on a regular space, strongly θ-continuousness and strongly faintly continuousness are equivalent. Hence, the proofs of the following corollaries are similar to those of [2]. First, if $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ is a one-to-one point compact, u.str.f.c. multifunction and Y is a T_3-space, then X is Urysohn. Next, if $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ is a multifunction and Y is a regular space, and if G_F is u.str.f.c., then X is a regular space. Here a multifunction $F : (X, \tau) \hookrightarrow (Y, \upsilon)$ is one-to-one in case $x_1 \neq x_2 \Rightarrow F(x_1) \cap F(x_2) = \emptyset$ for all x_1,$x_2 \in X$. However, if upper strongly faintly continuity is replaced by upper semi continuity, the last corollary will be false in general. Indeed, let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $Y = [0, 1]$ with the usual topology. Define the multifunction $F : X \hookrightarrow Y$, by $F(x) = (1/3, 2/3)$ for $x = a$, $F(x) = (1/4, 3/4)$ for $x = b$ and $F(x) = Y$ for $x = c$. Then the graph of F is u.s.c. and Y is regular, but X is not regular.

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