Two Special Convolution Products of \((n/2 - k - 1)\)-th Derivatives of Dirac Delta in Hypercone

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Abstract

In this paper two special convolution products \(\delta^{(n/2-k-1)}(u) \ast \delta^{(n/2-l-1)}(u)\) and \(\delta^{(n/2-k-1)}(m^2 + u) \ast \delta^{(n/2-l-1)}(m^2 + u)\) are expressed in terms of several known quantities.

Let \(x = (x_1, x_2, \ldots, x_n)\) be a point of \(\mathbb{R}^n\). We shall write

\[
x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots x_{\mu+\nu}^2 = u,
\]

where \(\mu + \nu = n\). \(\Gamma_+\) denotes the interior of the forward cone

\[
\Gamma_+ = \{ x \in \mathbb{R}^n | x_1 > 0, u > 0 \},
\]

and \(\bar{\Gamma}_+\) denotes its closure. Similarly, \(\Gamma_-\) denotes the domain

\[
\Gamma_- = \{ x \in \mathbb{R}^n | x_1 < 0, u > 0 \}
\]

We let \(R\) denote the family of functions \(\Phi\) introduced in [1, p.72]:

\[
R_\Phi(u) = 0 \text{ if } x = 2^{-k}; \quad \text{and} \quad R_\Phi(u) = 1 \quad \text{K}_n(u - n/2); \quad x \in \Gamma_+.
\]

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Here $\alpha$ is a complex parameter and $n$ the dimension of the space, the constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2 + \alpha - n)/2) \Gamma((1 - \alpha)/2) \Gamma(\alpha)}{\Gamma((2 + \alpha - \mu)/2) \Gamma((\mu - \alpha)/2)}$$

(6)

and $\mu$ is the number of positive terms of

$$u = x_1^2 + \ldots + x_\mu^2 - x_{\mu+1}^2 - \ldots - x_{\mu+\nu}^2, \ \mu + \nu = n.$$  

(7)

$R_\alpha(u)$ is a distribution of $\alpha$ and is an ordinary function if the real part of $\alpha$ is greater than or equal to $n$.

By putting $\nu = 1$ in $R_\alpha(u)$ and (6) and remembering the Legendre's duplication formula of $\Gamma(z)$ [2, p.344]

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2),$$

(8)

$R_\alpha(u)$ reduces to $M_\alpha(u)$ which is the hyperbolic kernel of Riesz [3, p.31]: $M_\alpha(u) = 0$ if $x \notin \Gamma_+$ and

$$M_\alpha(u) = \frac{1}{H_n(\alpha)} u^{(\alpha-n)/2}, \ x \in \Gamma_+.$$  

(9)

Here

$$u = x_1^2 - x_2^2 - \ldots - x_n^2,$$

(10)

and

$$H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma(\alpha/2) \Gamma((\alpha - n + 2)/2).$$

(11)

Trione in [4, p.11] proves the validity of the property

$$\langle k \rangle R_{2k}(u) = R_0(u) = \delta(x)$$

(12)

for $k = 0, 1, 2, \ldots$, where

$$\langle k \rangle = \left\{ \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_\mu^2} - \frac{\partial^2}{\partial x_{\mu+1}^2} - \ldots - \frac{\partial^2}{\partial x_{\mu+\nu}^2} \right\}^k$$

(13)

is the ultrahyperbolic operator iterated $k$-times and $\delta(x) = \delta(x_1, x_2, \ldots, x_n)$ is the Dirac delta function. From (12), $R_{2k}(u)$ is the unique elementary solution of the $n$-dimensional ultrahyperbolic operator iterated $k$-times defined by (13)

Aguirre in [5, p.149] proves the following properties:

1. $\mu$ odd and $\nu$ even ($n$ odd)

$$R_{2k}(u) = \frac{1}{(-1)^{(\mu-1)/2} \pi^{(n-1)/2} 2^{2k-1} \Gamma(k)} u^{k-n/2} \frac{\Gamma(k - n/2 + 1)}{\Gamma(k)}.$$  

(14)

where $u$ is defined by (7)

2. $\mu$ odd and $\nu$ odd ($n$ even)

$$R_{2k}(u) = \frac{1}{(-1)^{(\mu-1)/2} \pi^{(n-1)/2} 2^{2k-1} \Gamma(k)} u^{k-n/2} \frac{\Gamma(k - n/2 + 1)}{\Gamma(k)}.$$  

(15)
if $k \geq n/2$ and

$$R_{2k}(u) = \frac{1}{(-1)^{(\mu-1)/2} \pi^{(n-1)/2} 2^{k-1} \Gamma(k)} \delta^{(n/2-k-1)} \quad (16)$$

if $k < n/2$.

On the other hand Aguirre and Trione in [6, p.123] prove the following formula

$$R_\alpha(u) * R_{2k}(u) = R_{\alpha+2k}(u) \quad (17)$$

for all $\mu$ and $\nu$ where the symbol $*$ stands for convolution and $\mu+\nu = n$ is the dimension of the space.

In this paper we find two formulae for two special convolution products

$$\delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u)$$

and

$$\delta^{(n/2-k-1)}(m^2 + u) * \delta^{(n/2-l-1)}(m^2 + u)$$

using the formulae (16), (17) and the following formula [10, p.123]

$$\delta^{(k-1)}(m^2 + u) = \sum_{\nu=0}^{n/2-k-1} \frac{(m^2)^\nu}{\nu!} \delta^{(k+\nu-1)}(u) \quad (18)$$

which holds when $n$ is even and $k < n/2 - 1$.

**THEOREM 1.** Let $k$ and $l$ be non-negative integers and $n$ an even positive integer. Then

$$\delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u) = A_{k,l,n} \delta^{(n/2-k-l-1)}(u) \quad (19)$$

under conditions (i) $\mu$ and $\nu$ are both odd, and (ii) $0 \leq k + l \leq n/2 - 1$, where [5, p.148]

$$A_{k,l,n} = \frac{1}{2} (-1)^{(\mu-1)/2} \pi^{(n-2)/2} 2^{k-1} \Gamma(k) \Gamma(l) \Gamma(k+l), \quad (20)$$

and

$$\delta^{(n/2-k-1)}(u) = \frac{(n/2-k-1)!}{(-1)^{n/2-k-1}} \text{res}_{\beta = -(n/2-k)} u^\beta \quad (21)$$

and $u$ is defined by (7)

**PROOF.** From (16) and (17), we have,

$$\delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u) = (-1)^{(\mu-1)/2} \pi^{(n-2)/2} 2^{k-1} \Gamma(k) \times \Gamma(l) (-1)^{(\mu-1)/2} \pi^{(n-2)/2} 2^{l-1} (R_{2k}(u) * R_{2l}(u))$$

$$= \Gamma(k) \Gamma(l) (-1)^{\mu-1} \pi^{n-2} 2^{2(k+l)-2} R_{2(k+l)}(u) \quad (22)$$

if $\mu$ and $\nu$ are both odd. Now from (22) and (16), we have

$$\delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u) = \frac{1}{2} (-1)^{(\mu-1)/2} \pi^{(n-2)/2} 2^{k-1} \Gamma(k) \Gamma(l) \Gamma(k+l) \delta^{(n/2-k-l-1)}(u) \quad (23)$$
if \( \mu \) and \( \nu \) are both odd and \( k + l < n/2 \). From (23) we deduce (19). The proof is complete.

We remark that our new formula (19) is not a consequence of the convolution product

\[
\delta^{(s)}(P_+) \ast \delta^{(t)}(P_+) = (-1)^{\epsilon+s+t} h_{s,t,n} P_{-}^{n/2-s-t-2}
\]

which appear in [7], where

\[
h_{s,t,n} = 2^{-2}(-1)^{(\nu-1)/2} \pi^{(n+2)/2} \frac{\Gamma(n/2 - s - 1)\Gamma(n/2 - t - 1)}{\Gamma(n/2 - s - t - 1)\Gamma(n - s - t - 2)}
\]

and [11, p.278]

\[
\delta^{(s)}(P_+) = (-1)^s \text{res}_{\lambda=-s-1, s=0,1,2,...} P_+^\lambda
\]

and [11, p.254]

\[
P_+^\lambda \varphi = \int_{P>0} (P(x))^\lambda \varphi(x) dx
\]

and \( P(x) = u(x) = u \) with \( u \) defined by (7).

In fact, putting \( s = n/2 - k - 1 \) and \( t = n/2 - l - 1 \) in (24) and (25), we have

\[
\delta^{(n/2-k-1)}(P_+) \ast \delta^{(n/2-l-1)}(P_+)
\]

\[
= 2^{-2}(-1)^{(\nu-1)/2} \pi^{(n-2)/2} \frac{\Gamma(k)\Gamma(l)(-1)^{n-k-l}}{\Gamma(k+l)\Gamma(k+l-n/2+1)} P_{-}^{k+l-n/2}
\]

provided \( n/2 \leq k + l \leq n - 2 \), where [11, p.269]

\[
P_+^\lambda \varphi = \int_{P<0} (-P(x))^\lambda \varphi(x) dx.
\]

Therefore our new convolution product formula (19) is in some way complementary to the formula (28).

THEOREM 2. Let \( k \) and \( l \) be non-negative integers and \( n \) an even positive integer. Then

\[
\delta^{(n/2-k-1)}(m^2 + u) \ast \delta^{(n/2-l-1)}(m^2 + u)
\]

\[
= \delta^{(n/2-k-1)}(u) \ast \delta^{(n/2-l-1)}(u) + a_{\mu,n} \sum_{\nu=1}^{k-1} \frac{(m^2)^\nu}{\nu!} A_{k-\nu,l} \delta^{(n/2-k-l+\nu-1)}(u)
\]

\[
+ a_{\mu,n} \sum_{j=1}^{l-1} \frac{(m^2)^j}{j!} A_{k,j} \delta^{(n/2-k-l+j-1)}(u)
\]

\[
+ a_{\mu,n} \sum_{\nu=1}^{k-1} \sum_{j=1}^{l-1} \frac{(m^2)^{\nu+j}}{\nu!} A_{k-\nu,l-j} \delta^{(n/2-k-l+\nu+j-1)}(u)
\]

under conditions a) \( \mu \) and \( \nu \) are both odd, and b) \( 0 \leq k + l \leq n/2 - 1 \), where

\[
A_{s,t} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}
\]
and

$$a_{\mu,n} = \frac{1}{2} (-1)^{\mu-1/2} \pi^{(n-1)/2}. \quad (32)$$

**Proof.** From (18), we have

$$\delta^{(n/2-k-1)}(m^2 + u) \ast \delta^{(n/2-l-1)}(m^2 + u)$$

$$= \sum_{\nu=0}^{k-1} \sum_{j=0}^{l-1} \frac{(m^2)^{\nu+j}}{\nu!} \left\{ A_{k-\nu,l-j} \delta^{(n/2-k-l+\nu+j-1)}(u) \right\}. \quad (33)$$

Now using the formula (19), we have

$$\delta^{(n/2-k-1)}(m^2 + u) \ast \delta^{(n/2-l-1)}(m^2 + u)$$

$$= \sum_{\nu=0}^{k-1} \sum_{j=0}^{l-1} \frac{(m^2)^{\nu+j}}{\nu!} \left\{ A_{k-\nu,l-j,n} \delta^{(n/2-k-l+\nu+j-1)}(u) \right\}. \quad (34)$$

where $A_{k-\nu,l-j,n}$ is defined by (32). From (34) and (19), we obtain

$$\delta^{(n/2-k-1)}(m^2 + u) \ast \delta^{(n/2-l-1)}(m^2 + u)$$

$$= \delta^{(n/2-k-1)}(u) \ast \delta^{(n/2-l-1)}(u) + a_{\mu,n} \sum_{\nu=1}^{k-1} \frac{(m^2)^{\nu}}{\nu!} A_{k-\nu,l-j} \delta^{(n/2-k-l+\nu-1)}(u)$$

$$+ a_{\mu,n} \sum_{\nu=1}^{k-1} \sum_{j=1}^{l-1} \frac{(m^2)^{\nu+j}}{\nu! j!} A_{k-\nu,l-j} \delta^{(n/2-k-l+j-1)}(u) \quad (35)$$

where $a_{\mu,n}$ is defined by the (32). The formula (35) coincides with (30).

It’s clear that putting $m^2 = 0$ in (30) we obtain (19).

We remark that (30) is not a consequence of the equality

$$\delta^{(s-1)}(m^2 + P) \ast \delta^{(t-1)}(m^2 + P)$$

$$= \delta^{(s-1)}(P_+) \ast \delta^{(t-1)}(P_+) + \sum_{r=1}^{n/2-s-t} \frac{(m^2)^{r}}{r!} C_{n/2-s-r,n/2-t-r,t} P_{-}^{n/2-k-t-r} \quad (36)$$

which appear in [8], where

$$C_{n/2-s-r,n/2-t-r,t}$$

$$= \pi^{n/2+1/2-2} (-1)^{(u-1)/2} \frac{\Gamma(n/2 - s - r) \Gamma(n/2 - t - r)}{\Gamma(n/2 - s - t + 1) \Gamma(n - s - r - t - r)}. \quad (37)$$

In fact, putting $s = n/2 - k - 1$ and $t = n/2 - l - 1$ in (36) and (37), we have

$$\delta^{(n/2-k-1)}(m^2 + P) \ast \delta^{(n/2-l-1)}(m^2 + P)$$

$$= \delta^{(n/2-k-1)}(P_+) \ast \delta^{(n/2-l-1)}(P_+) + \sum_{r=1}^{k+l-n/2} \frac{(m^2)^{r}}{r!} C_{k-r,l-r,t} P_{-}^{k+l-n/2-r}$$
under conditions $\mu$ and $\nu$ odd and $k + l \geq n/2 + r$, where

$$C_{k-r,l-r,l} = \pi^{n/2+1}2^{-2}(-1)^{(\nu-1)/2} \frac{\Gamma(k-r)\Gamma(l-r)}{\Gamma(k+l-n/2)\Gamma(k+l-2r)}.$$ 

Therefore (30) is in some way complementary to (36). 

References


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