Applied Mathematics E-Notes, 1(2001), 1-4 ©
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Closed Form Solutions of Iterative Functional Differential Equations *

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Received 20 July 2000

Abstract

Solutions of the form $x(z) = \lambda z^{\mu}$ are found for the iterative functional differential equation $x^{(n)}(z) = (x(x(...x(z))))^k$.

Let $h = h(z)$ be a function of a complex variable. The iterates of h are defined by $h^{[0]} = I$ (the identity map), $h^{[1]} = h$, $h^{[2]} = h \circ h, ...$, etc. A number of recent studies are concerned with iterative functional differential equations. In particular, in [1], the equation

$$x'(z) = x^{[m]}(z),\ m \geq 2,$$

has been considered. Such an equation arises in problems related to motions of charged particles with retarded interactions [2]. Since it is quite different from the usual differential equations, the standard existence and uniqueness theorems cannot be applied. It is therefore of interest to find some or all of its solutions. It is shown in [1] that analytic solutions exist and series form can be given. However, we will show below that solutions of the form

$$x(z) = \lambda z^{\mu} \tag{1}$$

also exist. The idea of finding such solutions is new even though it reminds us of the usual idea of finding exponential solutions to linear ordinary differential equations with constant coefficients. The procedure for obtaining such solutions works equally well for more general equations. In this note, we will illustrate our procedure by considering a slightly more general equation of the form

$$x^{(n)}(z) = az^j \left(x^{[m]}(z)\right)^k, \tag{2}$$

where k, m, n are positive integers, j is a nonnegative integer, a is a complex number, and $x^{(n)}(z)$ is the n-th derivative of $x(z)$. We assume $m \geq 2$ and $a \neq 0$ to avoid trivial cases. It is also clear that the zero function is a trivial solution of (2), which is of no interest.

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*Mathematics Subject Classifications: 35C20, 35D10.

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Substituting (1) into (2), we obtain

$$\lambda\mu(\mu-1)\cdots(\mu-n+1)z^{\mu-n} = a\lambda^{k(1+\mu+\ldots+\mu^{m-1})}z^{k\mu^m+j}.$$

This prompts us to consider the equations

$$\lambda\mu(\mu-1)\cdots(\mu-n+1) = a\lambda^{k(1+\mu+\ldots+\mu^{m-1})}, \tag{3}$$

and

$$k\mu^m + j = \mu - n. \tag{4}$$

First of all, we assert that the polynomial

$$f(z) = kz^m - z + n + j$$

does not have any real roots if m is even, and has exactly one real root if m is odd. Indeed, for even m and real z, by solving

$$f'(z) = kmz^{m-1} - 1 = 0,$$

we see that the minimum of f occurs at the root $\rho = 1/\sqrt[m-1]{km} \in (0,1)$. Hence

$$f(z) \geq f(\rho) = \rho\left(\frac{1}{m} - 1\right) + n + j > \left(\frac{1}{m} - 1\right) + n + j > 0$$

for all real z. If m is odd, then f' has two zeros $\pm\rho$. Since

$$\min_{z\geq-\rho} f(z) = \min\{f(\rho), f(-\rho)\} = f(\rho) > 0,$$

f does not have any real roots greater than or equal to $-\rho$. Furthermore, since $f(-\rho)$ and $f(-\infty)$ have opposite signs, f has at least one real root in $(-\infty, -\rho)$. Finally, since $f'(z) > 0$ for all $z < -\rho$, f is increasing in $(-\infty, -\rho)$. So f has exactly one real root which is negative. As a consequence, the roots of f in either case cannot be $0, 1, \ldots,$ nor $n-1$.

Next, we assert that $f(z)$ has simple roots only. Suppose not, let r be a double root of f, then it is a root of f' and

$$f(z) - \frac{z}{m}f'(z) = \frac{1-m}{m}z + n + j. \tag{5}$$

Hence (5) implies that $r = m(n+j)/(m-1)$ is real and positive, which is impossible by our previous assertion.

Let $\mu_1, \ldots, \mu_m$ be the roots of (4). In view of the above results, $\mu_1, \ldots, \mu_m$ are pairwise distinct and each one of them is different from $0, 1, \ldots,$ or $n-1$. Furthermore, in view of (3) and (4), we have

$$\lambda\mu_i(\mu_i-1)\cdots(\mu_i-n+1) = a\lambda^{k(1-\mu_i^m)/(1-\mu_i)} = a\lambda^{(k+n+j-\mu_i)/(1-\mu_i)}$$

for $i = 1, \ldots, m$, from which we obtain

$$\lambda_i = \left[\frac{\mu_i(\mu_i-1)\cdots(\mu_i-n+1)}{a}\right]^{(1-\mu_i)/(k+n+j-1)}, \; i = 1, \ldots, m. \tag{6}$$

In other words, we have found m distinct solutions of the form:

$$x_i(z) = \lambda_i z^{\mu_i},\ i = 1, 2, ..., m, \tag{7}$$

where $\mu_1, ..., \mu_m$ are roots of (4) and $\lambda_1, ..., \lambda_m$ are defined by (6).

THEOREM 1. Let D be a domain of the complex plane C which does not include the negative real axis (nor the origin). Then there exist m distinct (single valued and analytic) power functions of the form (7) which are solutions of (2) defined on D.

We remark that each solution $x_i(z) = \lambda_i z^{\mu_i}$ has a nontrivial fixed point α_i. Indeed, from $\lambda_i \alpha_i^{\mu_i} = \alpha_i$, we find

$$\alpha_i = \lambda_i^{1/(1-\mu_i)} = [\mu_i(\mu_i - 1) \cdots (\mu_i - n + 1)]^{1/(k+n+j-1)} \neq 0. \tag{8}$$

In terms of the fixed point α_i, we may therefore write $x_i(z)$ in the form

$$x_i(z) = \alpha_i^{1-\mu_i} z^{\mu_i}. \tag{9}$$

COROLLARY. Let $\mu_1, ..., \mu_m$ be the roots of (4), and $\alpha_1, ..., \alpha_m$ given by (8). Then in a neighborhood of each point α_i, $i = 1, ..., m$, equation (2) has an analytic solution of the form

$$\begin{aligned} x_i(z) &= \alpha_i + \mu_i(z - \alpha_i) + \frac{\mu_i(\mu_i - 1)}{2!\alpha_i}(z - \alpha_i)^2 + ... \\ &\quad + \frac{\mu_i(\mu_i - 1)...(\mu_i - n + 1)}{n!\alpha_i^{n-1}}(z - \alpha_i)^n + ... \end{aligned}$$

Indeed, in view of (9),

$$\begin{aligned} x_i(z) &= \alpha_i^{1-\mu_i} z^{\mu_i} = \alpha_i \left(1 + \frac{z - \alpha_i}{\alpha_i}\right)^{\mu_i} \\ &= \alpha_i \left[1 + \frac{\mu_i}{1!}\left(\frac{z - \alpha_i}{\alpha_i}\right) + \frac{\mu_i(\mu_i - 1)}{2!}\left(\frac{z - \alpha_i}{\alpha_i}\right)^2 + ...\right] \end{aligned}$$

as required.

As an example, consider the equation

$$x'(z) = x(x(z)).$$

Then (4) is reduced to

$$\mu^2 - \mu + 1 = 0,$$

which has roots $\mu_\pm = (1 \pm \sqrt{3}i)/2$. And from (3), we find $\lambda_- = \mu_-^{1/\mu_-} \approx 2.145 - 1.238i$, $\lambda_+ = \mu_+^{1/\mu_+} \approx 2.145 + 1.238i$. Since $|\mu_\pm| = 1$ and $\mu_\pm^6 = 1$, $\alpha_\pm = \mu_\pm$ are roots of unity. This shows that the requirements in the main Theorem in [1] does not hold. Therefore, we have found analytic solutions which cannot be guaranteed by the main Theorem in [1].

As our final remarks, note that the parameters j and k in (2) can be extended to real or even complex numbers. In this case, (4) might have more than one real roots, but the solutions (7) remain valid. It is possible that (4) has double roots, then the solutions in (7) corresponding to these roots are the same. Also some of its roots μ_i might equal to $0, 1, ...,$ or $n-1$. But then $\lambda_i = 0$ by (3), and $x_i(z)$ in (7) becomes the trivial solution.

References

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