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Analytic Solutions of Linear Neutral Functional Differential-Difference Equations *

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Abstract

This paper is concerned with the existence of analytic solutions of higher order linear neutral functional differential-difference equations.

Existence, stability, asymptotic, and oscillatory behaviors of solutions of functional differential-difference equations have been studied quite extensively by many authors [1-5]. However, it seems little has been done on the existence of analytic solutions. In this note, we are concerned with the existence of analytic solutions of the following equation

$$\frac{d^k f(z)}{dz^k} + \sum_{i=1}^{k} \varphi_i(z) \frac{d^{k-i} f(z)}{dz^{k-i}} + \sum_{l=1}^{n} \sum_{j=0}^{k} \alpha_{lj}(z) \frac{d^{k-j} f(z-\tau_j)}{dz^{k-j}} = g(z). \tag{1}$$

For this purpose, we will employ approximating equations which are ordinary differential equations without the delay terms.

LEMMA 1. Suppose there exist a negative number b and a positive number D such that the functions $\varphi_1(z), ..., \varphi_k(z)$, $\alpha_{10}(z), ..., \alpha_{nk}(z)$ and $g(z)$ are analytic in the region $|z-b| < |b| - D$, and $\sum_{l=1}^{n} |\alpha_{l0}(b)| < 1$. For $|z-b| < |b| - D$, let

$$\varphi_i(z) = \sum_{m=0}^{\infty} \varphi_{im}(z-b)^m, \ i = 1, 2, ..., k,$$

$$\alpha_{lj}(z) = \sum_{m=0}^{\infty} \alpha_{ljm}(z-b)^m, \ i = 1, 2, \ldots, k; l = 1, 2, \ldots, n; j = 0, 1, \ldots, k,$$

and

$$g(z) = \sum_{m=0}^{\infty} g_m(z-b)^m.$$

Then the ordinary differential equation

$$\frac{d^k \varphi(z)}{dz^k} = \sum_{i=1}^{k} \tilde{\varphi}_i \, (z) \frac{d^{k-i}\varphi(z)}{dz^{k-i}} + \sum_{l=1}^{n} \sum_{j=0}^{k} \tilde{\alpha}_{lj} \, (z) \frac{d^{k-j}\varphi(z)}{dz^{k-j}} + \tilde{g} \, (z) \tag{2}$$

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has an analytic solution in the region $|z-b|<|b|-D$ that satisfies

$$0 \leq \varphi^{(i)}(b) < +\infty, \ i=0,1,\ldots,k-1, \tag{3}$$

where

$$\widetilde{\varphi_i}\ (z) = \sum_{m=0}^{\infty} |\varphi_{im}|(z-b)^m,$$

$$\widetilde{\alpha_{lj}}\ (z) = \sum_{m=0}^{\infty} |\alpha_{ljm}|(z-b)^m,$$

$$\widetilde{g}\ (z) = \sum_{m=0}^{\infty} |g_m|(z-b)^m.$$

PROOF. We seek a power series solution of equation (2) in the form

$$\varphi(z) = \sum_{m=0}^{\infty} \varphi_m (z-b)^m. \tag{4}$$

Substituting (4) into (2) we obtain

$$\begin{aligned} &\left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right)\frac{(m+k)!}{m!}\varphi_{m+k} \\ = \ &\sum_{i=1}^{k}\sum_{s=0}^{m}|\varphi_{is}|\frac{(k-i+m-s)!}{(m-s)!}\varphi_{k-i+m-s} + \sum_{l=1}^{n}\sum_{j=1}^{k}|\alpha_{lj0}|\frac{(k-j+m)!}{m!}\varphi_{k-j+m} \\ &+\sum_{l=1}^{n}\sum_{j=0}^{k}\sum_{s=1}^{m}|\alpha_{ljs}|\frac{(k-j+m-s)!}{(m-s)!}\varphi_{k-j+m-s} + |g_m|. \end{aligned}$$

By means of the Cauchy inequalities, for any fixed $r \in (0, |b|-D)$, there exists a positive number M such that

$$|\varphi_{im}| \leq \frac{M}{r^m}, \ |\alpha_{ljm}| \leq \frac{M}{r^m}, \ |g_m| \leq \frac{M}{r^m}$$

for $i=1,2,\ldots,k; j=0,1,\ldots,k; l=1,2,\ldots n$; and $m=0,1,\ldots$. We now define a sequence $\{S_m\}$ as follows: for $0 \leq m \leq k-1$, $S_m = B_m$, and for $m \geq k$,

$$\begin{aligned} S_m \ = \ &\frac{(m-k)!}{m!(1-\sum_{l=1}^{n}|\alpha_{l00}|)}\Bigg[\sum_{i=1}^{k}\sum_{s=0}^{m-k}\frac{(m-i-s)!}{(m-k-s)!}\frac{M}{r^s}S_{m-i-s} \\ &+n\sum_{j=1}^{k}\frac{(m-j)!}{(m-k)!}MS_{m-j} + n\sum_{j=0}^{k}\sum_{s=1}^{m-k}\frac{(m-j-s)!}{(m-k-s)!}\frac{M}{r^s}S_{m-j-s} + \frac{M}{r^{m-k}}\Bigg] \end{aligned}$$

where B_m are arbitrary positive numbers which satisfy $|\varphi_m| \leq B_m$ for $0 \leq m \leq k-1$. By induction, we easily see from the definition of S_m that $0 \leq \varphi_m \leq S_m$ for all $m \geq 0$.

Furthermore, for $m \geq k$, by the recursive definition of S_m,

$$
\begin{aligned}
& \left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right) S_{m+1} \\
= \quad & \frac{(m+1-k)!}{(m+1)!}\left[\sum_{i=1}^{k}\sum_{s=0}^{m+1-k}\frac{(m+1-i-s)!}{(m+1-k-s)!}\frac{M}{r^s}S_{m+1-i-s}\right. \\
& +n\sum_{j=1}^{k}\frac{(m+1-j)!}{(m+1-k)!}MS_{m+1-j} \\
& \left.+n\sum_{j=0}^{k}\sum_{s=1}^{m+1-k}\frac{(m+1-j-s)!}{(m+1-k-s)!}\frac{M}{r^s}S_{m+1-j-s}+\frac{M}{r^{m+1-k}}\right] \\
= \quad & \frac{(m+1-k)!}{(m+1)!}\left\{\left[\frac{(n+1)m!M}{(m+1-k)!}+\frac{nMm!}{r(m-k)!}\right]S_m\right. \\
& +M\sum_{i=2}^{k}\frac{(m+1-i)!}{(m+1-k)!}S_{m+1-i}+nM\sum_{j=2}^{k}\frac{(m+1-j)!}{(m+1-k)!}S_{m+1-j} \\
& +\frac{1}{r}\left[\sum_{i=1}^{k}\sum_{s=1}^{m+1-k}\frac{(m+1-i-s)!}{(m+1-k-s)!}\frac{M}{r^{s-1}}S_{m+1-i-s}+n\sum_{j=1}^{k}\frac{M(m-j)!}{(m-k)!}S_{m-j}\right. \\
& \left.\left.+n\sum_{j=0}^{k}\sum_{s=2}^{m+1-k}\frac{(m+1-j-s)!}{(m+1-k-s)!}\frac{M}{r^{s-1}}S_{m+1-j-s}+\frac{M}{r^{m-k}}\right]\right\} \\
= \quad & \left[\frac{(n+1)M}{m+1}+\frac{(m+1-k)nM}{(m+1)r}\right]S_m \\
& +M\sum_{i=2}^{k}\frac{(m+1-i)!}{(m+1)!}S_{m+1-i}+nM\sum_{j=2}^{k}\frac{(m+1-j)!}{(m+1)!}S_{m+1-j} \\
& +\frac{(m+1-k)!}{(m+1)!}\frac{1}{r}\left[\frac{m!}{(m-k)!}\left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right)S_m\right] \\
= \quad & \left[\frac{(n+1)M}{m+1}+\frac{(m+1-k)nM}{(m+1)r}+\frac{1}{r}\frac{m+1-k}{m+1}\left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right)\right]S_m \\
& +M\sum_{i=2}^{k}\frac{(m+1-i)!}{(m+1)!}S_{m+1-i}+nM\sum_{j=2}^{k}\frac{(m+1-j)!}{(m+1)!}S_{m+1-j}.
\end{aligned}
$$

Therefore, we see that

$$
\begin{aligned}
& \frac{S_{m+1}}{S_m} \\
= \quad & \frac{1}{1-\sum_{l=1}^{n}|\alpha_{l00}|}\left[\frac{(n+1)M}{m+1}+\frac{(m+1-k)nM}{(m+1)r}+\frac{1}{r}\frac{m+1-k}{m+1}\left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right)\right.
\end{aligned}
$$

$$+M\sum_{i=2}^{k}\frac{(m+1-i)!}{(m+1)!}\frac{S_{m+1-i}}{S_m}+nM\sum_{j=2}^{k}\frac{(m+1-j)!}{(m+1)!}\frac{S_{m+1-j}}{S_m}\Bigg].$$

Letting $C_m = S_{m+1}/S_m$, we see that

$$\begin{aligned}
C_m &= \frac{1}{1-\sum_{l=1}^{n}|\alpha_{l00}|}\Bigg[\frac{(n+1)M}{m+1}+\frac{(m+1-k)nM}{(m+1)r}\\
&\quad+\frac{1}{r}\frac{m+1-k}{m+1}\left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right)+M\sum_{i=2}^{k}\frac{(m+1-i)!}{(m+1)!}\frac{1}{C_{m-1}C_{m-2}\ldots C_{m+1-i}}\\
&\quad+nM\sum_{j=2}^{k}\frac{(m+1-j)!}{(m+1)!}\frac{1}{C_{m-1}C_{m-2}\ldots C_{m+1-j}}\Bigg],
\end{aligned}$$

so that,

$$C_m \geq \frac{1}{1-\sum_{l=1}^{n}|\alpha_{l00}|}\left[\frac{1}{r}\frac{m+1-k}{m+1}\left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right)\right].$$

Therefore,

$$\liminf_{m\to\infty} C_m \geq \frac{1}{r} > 0$$

and

$$\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{C_{m-1}C_{m-2}\ldots C_{m+1-i}}\\
&= \frac{1}{\liminf_{m\to\infty} C_{m-1}C_{m-2}\ldots C_{m+1-i}}\\
&\leq \frac{1}{\liminf_{m\to\infty} C_{m-1}\cdot\liminf_{m\to\infty} C_{m-2}\cdots\liminf_{m\to\infty} C_{m+1-i}}\\
&< +\infty,
\end{aligned}$$

for $i = 2, 3, \ldots, k$. By taking superior limits on both sides of the defining equation for C_m, we now have

$$\begin{aligned}
&\limsup_{m\to\infty} C_m \leq \limsup_{m\to\infty}\frac{1}{1-\sum_{l=1}^{n}|\alpha_{l00}|}\Bigg\{\limsup_{m\to\infty}\frac{(n+1)M}{m+1}\\
&+\limsup_{m\to\infty}\left[\frac{(m+1-k)nM}{(m+1)r}\right]+\limsup_{m\to\infty}\left[\frac{1}{r}\frac{m+1-k}{m+1}\left(1-\sum_{l=1}^{n}|\alpha_{l00}|\right)\right]\\
&+M\sum_{i=2}^{k}\left[\limsup_{m\to\infty}\frac{(m+1-i)!}{(m+1)!}\limsup_{m\to\infty}\frac{1}{C_{m-1}C_{m-2}\ldots C_{m+1-i}}\right]\\
&+nM\sum_{j=2}^{k}\left[\limsup_{m\to\infty}\frac{(m+1-j)!}{(m+1)!}\limsup_{m\to\infty}\frac{1}{C_{m-1}C_{m-2}\ldots C_{m+1-j}}\right]\Bigg\} = \frac{1}{r},
\end{aligned}$$

that is,

$$\limsup_{m\to\infty} \frac{S_{m+1}}{S_m} \le \frac{1}{r}.$$

Thus, we have

$$\limsup_{m\to\infty} \sqrt[m]{S_m} \le \limsup_{m\to\infty} \frac{S_{m+1}}{S_m} \le \frac{1}{r},$$

whence

$$\rho = \frac{1}{\limsup_{m\to\infty} \sqrt[m]{S_m}} \ge r.$$

This proves that the radius of convergence ρ of the power series $\sum_{m=0}^{\infty} S_m(z-b)^m$ satisfies $\rho \ge r \in (0, |b| - D)$. Since r is arbitrary, we see that the power series $\varphi(z)$ is analytic in the region $|z-b| < |b| - D$. The proof is complete.

Next, we let $p_{lj} = 1 + \tau_j b^{-1}$ for $l = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, k$, where b is an arbitrary negative number. Consider the equation

$$\frac{d^k f(z)}{dz^k} + \sum_{i=1}^{k} \varphi_i(z) \frac{d^{k-i} f(z)}{dz^{k-i}} + \sum_{l=1}^{n}\sum_{j=0}^{k} \alpha_{lj}(z) \frac{d^{k-j} f(p_{lj} z - \tau_j)}{dz^{k-j}} = g(z). \tag{5}$$

When b tends to negative infinity, this equation tends formally to the given equation (1). For this reason, the equation (5) will be called the approximating equation for equation (1).

We now state our main result in this note.

THEOREM 1. Suppose equation (1) satisfies the following hypotheses: (i) for all negative number b which is sufficiently small, there exists a positive number D (independent of b) such that $D < |b|$, and the functions $\varphi_1(z), \ldots, \varphi_k(z)$, $\alpha_{10}(z), ..., \alpha_{nk}(z)$ and $g(z)$ are analytic in the region $|z-b| < |b| - D$, as well as $\sum_{l=1}^{n} |\alpha_{l0}(b)| < 1$, and (ii) the analytic solution $\varphi(z) = \sum_{m=0}^{\infty} \varphi_m (z-b)^m$ of equation (2) (as asserted in Lemma 1) is bounded in the region $|z-b| < |b| - D$. Then there exists a number $Q < 0$ which satisfies $|Q| > D$ and equation (1) has an analytic solution $f(z)$ in the half plane $\{z | \operatorname{Re} z < Q\}$.

PROOF. According to Lemma 1, equation (2) has an analytic solution (4) in the region $|z-b| < |b| - D$. We assert that the approximating equation (5) has an analytic solution $f(z|b)$ in the region $|z-b| < |b| - D$ which satisfies

$$|f^{(i)}(b|b)| \le \varphi^{(i)}(b),\ i = 0, 1, \ldots, k-1. \tag{6}$$

For this purpose we assume $f(z|b)$ has the form

$$f(z|b) = \sum_{m=0}^{\infty} f_m (z-b)^m. \tag{7}$$

Substituting (7) into (5), we obtain the relation

$$\left(1 + \sum_{l=1}^{n} \alpha_{l00}\right) \frac{(m+k)!}{m!} f_{m+k}$$

$$+\sum_{i=1}^{k}\sum_{s=0}^{m}\frac{(k-i+m-s)!}{(m-s)!}\varphi_{is}f_{k-i+m-s}+\sum_{l=1}^{n}\sum_{j=1}^{k}\frac{(k-j+m)!}{m!}\alpha_{lj0}f_{k-j+m}$$
$$+\sum_{l=1}^{n}\sum_{j=0}^{k}\sum_{s=1}^{m}\frac{(k-j+m-s)!}{(m-s)!}\alpha_{ljs}f_{k-j+m-s}=g_m,$$

for $m = 0, 1, \ldots$. By induction, we may then show that $|f_m| \leq \varphi_m$ for all m. Hence, equation (5) has an analytic solution $f(z|b)$ in the region $|z-b| < |b| - D$ that satisfies (6). In view of condition (ii), we see further that

$$\left|\sum_{m=0}^{\infty} f_m(z-b)^m\right| \leq \left|\sum_{m=0}^{\infty} \varphi_m(z-b)^m\right| \leq K$$

in the region $|z-b| < |b| - D$, that is, the function $f(z|b)$ is bounded in the region $|z-b| < |b| - D$. Choose $Q < 0$ and $|Q| > D$. Let B be a bounded region such that its closure is contained in the region $\{z|\ \mathrm{Re}z < Q\}$. There is a negative number b_0, such that

$$\{z||z-b| < |b| - |Q|\} \supset B$$

provided $b < b_0$.

Let $\{b_k\}_{k=1}^{\infty}$ be a sequence of negative numbers decreasing to negative infinity, with $b_1 < b_0$. The sequence of analytic solutions $\{f(z|b_k)\}$ of equation (5) is a bounded family in B, because of condition (ii). Hence there exists a subsequence $\{c_k\}$ of the sequence $\{b_k\}$, such that the sequence of functions $\{f(z|c_k)\}$ converges to a limit function analytic in B, the convergence being uniform in every closed subset of B. By the standard device of expressing the region $\{z|\ \mathrm{Re}z < Q\}$ as the union of a sequence of such region B, each region including the preceding, the limit function can be continued analytically throughout the region $\{z|\ \mathrm{Re}z < Q\}$, so that a function $f(z)$ is obtained, analytic in the region $\{z|\ \mathrm{Re}z < Q\}$, and satisfying the relation

$$f(z) = \lim_{k\to\infty} f(z|d_k),$$

where $\{d_k\}$ is some subsequence of $\{b_k\}$, where it is understood that for each value of z in $\{z|\ \mathrm{Re}z > Q\}$ the function $f(z|d_k)$ is defined only for sufficiently large values of k, and where the limit is uniform in every closed bounded subset of $\{z|\ \mathrm{Re}z < Q\}$.

Now $f(z|b)$ is a solution of equation (5). Hence, since the p_{lj} appearing in (5) tends to 1 as $b \to -\infty$, it follows that $f(z)$ is a solution of (1). This completes the proof.

As an immediate corollary, suppose $\varphi_1(z), \ldots, \varphi_k(z)$, $\alpha_{10}(z), \ldots, \alpha_{nk}(z)$ and $g(z)$ are rational functions, then there exists a positive number D such that D is strictly larger than the absolute values of all the poles of these functions. Thus, the functions $\varphi_i(z), \alpha_{lj}(z)$ and $g(z)$ are analytic in the region $\{z|\ |z-b| < |b| - D\}$ where b is a negative number which is sufficiently small. If we now assume further that $\sum_{l=1}^{n} |\alpha_{l0}(b)| < 1$ and that condition (ii) in Theorem 1 hold, then the conclusion of Theorem 1 will be true.

EXAMPLE. Consider the equation

$$f'''(x) + \frac{1}{1+x}f'''(x-\tau_1) - \frac{5x}{(1+x)^3}f'(x-\tau_2) = \frac{x}{(1+x)^5}, \tag{8}$$

where $\alpha_1(x) = 1/(1+x)$, $\alpha_2(x) = -5x/(1+x)^3$, and $g(x) = x/(1+x)^5$ are analytic in $|x-b| < |b|-2$ for each negative number b which is sufficiently small. The approximating equation of (8) is

$$f'''(x) + \frac{1}{1+x} f'''(p_1 x - \tau_1) - \frac{5x}{(1+x)^3} f'(p_2 x - \tau_2) = \frac{x}{(1+x)^5}, \tag{9}$$

where $p_1 = 1 + \tau_1 b^{-1}$, $p_2 = 1 + \tau_2 b^{-1}$. Since the equation

$$\varphi'''(x) = \frac{1}{1+x} \varphi'''(x) + \frac{5x}{(1+x)^3} \varphi'(x) + \frac{x}{(1+x)^5}$$

has a bounded analytic solution $\varphi(x) = x/(1+x)$ in $|x-b| < |b| - 2$. By Theorem 1, there exists $Q < 0$ such that $|Q| > 2$ and equation (8) has an analytic solution in $(-\infty, Q)$.

References

[1] J. Hale, Theory of functional differential equations, Springer-Verlag, 1997.

[2] A. D. Myshkis, Linear Differential Equations with Delayed argument, Nauka, Moscow, 1972

[3] E. Pinney, Ordinary Difference-Differential Equations, University of California Press, Berkeley, Los Angeles, 1958.

[4] K. Schmitt, Delay and Functional Equations and Their Applications, Academic Press, New York and London, 1972, 197-217.

[5] W. Strodt, Analytic solutions of nonlinear difference equations, Annals Math., 44(3)(1943), 375-396.