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Oscillation of Nonlinear First Order Neutral Difference Equations *

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Abstract

In this note, oscillation criteria are obtained for a class of nonlinear neutral difference equations.

Oscillatory behaviors of neutral difference equations of the form

$$\Delta\left(x_n - p_n x_{n-\tau}\right) + q_n x_{n-\sigma} = 0,\ n = 0, 1, \dots \tag{1}$$

have been explored to some extent in a number of studies [1-3]. However, relatively few results are known for their nonlinear extensions of the form

$$\Delta\left(x_n - p_n x_{n-\tau}\right) + q_n \max_{s \in [n-\sigma, n]} x_s = 0,\ n = 0, 1, \dots . \tag{2}$$

Nonlinear functional equations involving the maximum function are important since they appear naturally in automatic control theory, see e.g. Popov [4]. Some of the qualitative theory of these equations has been developed recently, see e.g. [5-8]. In this note, we will consider their oscillatory behaviors.

We will assume that τ and σ are positive integers, that $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, and $\{q_n\}$ has a positive subsequence. Let $\mu = \max\{\tau, \sigma\}$. By a solution of (1) or (2), we mean a real sequence $\{x_n\}_{n=-\mu}^{\infty}$ which satisfies (1) or (2) respectively for $n \geq 0$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative.

It is easily seen that $\{x_n\}$ is an eventually positive solution of equation (1) if, and only if, $\{-x_n\}$ is its eventually negative solution. However, such a property is not valid for equation (2). Instead, $\{x_n\}$ is an eventually positive solution of (2) if, and only if, $\{-x_n\}$ is an eventually negative solution of the equation

$$\Delta(x_n - p_n x_{n-\tau}) + q_n \min_{s \in [n-\sigma, n]} x_s = 0,\ n = 0, 1, \dots . \tag{3}$$

Thus, all solutions of (2) are oscillatory if, and only if, both (2) and (3) do not have any eventually positive solutions.

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LEMMA 1. Assume that there exists a nonnegative integer $N \geq 0$ such that $p_{N+j\tau} \leq 1$ for $j = 0, 1, 2, \ldots$. Let $\{x_n\}$ be an eventually positive solution of (2) or (3). Set

$$y_n = x_n - p_n x_{n-\tau} \tag{4}$$

for all large n. Then $y_n > 0$ eventually.

The proof is similar to the proof of Lemma 1 in [2].

THEOREM 1. Assume that there exists a nonnegative integer $N \geq 0$ such that $p_{N+j\tau} \leq 1$ for $j = 0, 1, 2, \ldots$. Suppose further that either $p_n > 0$ or q_n does not vanish identically over any set of consecutive integers of the form $\{a, a+1, .., a+\sigma\}$. Then equation (2) has an eventually positive solution if, and only if, the following functional inequality

$$\Delta\left(x_n - p_n x_{n-\tau}\right) + q_n \max_{s\in[n-\sigma,n]} x_s \leq 0,\ n = 0, 1, \ldots \tag{5}$$

has an eventually positive solution; and equation (3) has an eventually positive solution if, and only if, the functional inequality

$$\Delta\left(x_n - p_n x_{n-\tau}\right) + q_n \min_{s\in[n-\sigma,n]} x_s \leq 0,\ n = 0, 1, \ldots \tag{6}$$

has an eventually positive solution.

The proof of Theorem 1 is similar to that of Theorem 1 in [2], and is thus omitted.

THEOREM 2. Assume that there exists a nonnegative integer $N \geq 0$ such that $p_{N+j\tau} \leq 1$ for $j = 0, 1, 2, \ldots$. Suppose further that there exists some positive integer T such that the functional inequality

$$\Delta y_n + q_n \min_{s\in[n-\sigma,n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau} \leq 0$$

does not have any eventually positive solutions. Then all solutions of (2) oscillate.

PROOF. If $\{x_n\}$ is an eventually positive solution of (2), then $\Delta y_n \leq 0$ and $y_n = x_n - p_n x_{n-\tau} > 0$ eventually. Thus,

$$\begin{aligned}
x_n &= y_n + p_n x_{n-\tau} = y_n + p_n y_{n-\tau} + p_n p_{n-\tau} x_{n-2\tau} \\
&= \ldots \geq y_n + p_n y_{n-\tau} + \ldots + \prod_{i=0}^{T-1} p_{n-i\tau} y_{n-(i+1)\tau} \\
&\geq \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{n-i\tau} y_{n-\tau}.
\end{aligned}$$

Hence,

$$\max_{s\in[n-\sigma,n]} x_s \geq \max_{s\in[n-\sigma,n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{s-\tau} \geq \max_{s\in[n-\sigma,n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau}.$$

Substituting the last inequality into (2), we have

$$\Delta y_n + q_n \max_{s\in[n-\sigma,n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau} \leq 0,$$

which is a contradiction. If $\{z_n\}$ is an eventually negative solution of (2), then $x_n = -z_n$ is an eventually positive solution of equation (3). Similarly, we have

$$\Delta y_n + q_n \min_{s\in[n-\sigma,n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau} \leq \Delta y_n + q_n \max_{s\in[n-\sigma,n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau} \leq 0,$$

This is also a contradiction. The proof is complete.

For the equation

$$\Delta (x_n - x_{n-\tau}) + q_n \max_{s\in[n-\sigma,n]} x_s = 0, \ n = 0, 1, ..., \tag{7}$$

we have the following result.

THEOREM 3. Equation (7) has nonoscillatory solutions if, and only if,

$$\Delta^2 z_{n-1} + \frac{1}{\tau} q_n z_n = 0 \tag{8}$$

also has nonoscillatory solutions.

PROOF. Assume that $\{x_n\}$ is an eventually positive solution of (7). In view of Lemma 1, we see that there is an integer N_1 such that $x_{n-\tau} > 0, y_n = x_n - x_{n-\tau} > 0$, and $\Delta y_n \leq 0$ for $n \geq N_1$. Set $m = \min\{x_{N_1-\tau}, x_{N_1-\tau+1}, ..., x_{N_1-1}\}$. For $n \geq N_1 + \tau$, there exists a positive integer k such that

$$N_1 + k\tau \leq n < N_1 + (k+1)\tau$$

and

$$x_n = x_{n-k\tau} + \sum_{j=0}^{k-1} y_{n-j\tau} \geq m + \sum_{j=0}^{k-1} y_{n-j\tau}.$$

Furthermore, since $\Delta y_n \leq 0$ for $n \geq N_1$, and since $n - k\tau + \tau \leq N_1 + 2\tau = N_2$,

$$\begin{aligned}
\tau \sum_{j=0}^{k-1} y_{n-j\tau} &= \tau y_{n-(k-1)\tau} + \tau y_{n-(k-2)\tau} + ... + \tau y_n \\
&\geq (y_{n-(k-1)\tau} + y_{n-(k-1)\tau+1} + ... + y_{n-(k-2)\tau-1}) \\
&\quad +(y_{n-(k-2)\tau} + y_{n-(k-2)\tau+1} + ... + y_{n-(k-3)\tau-1} \\
&\quad +... + (y_n + y_{n+1} + ... + y_{n+\tau-1}) \\
&\geq \sum_{i=N_2}^{n} y_i,
\end{aligned}$$

we have

$$x_n \geq m + \frac{1}{\tau} \sum_{i=N_2}^{n} y_i.$$

Let

$$z_n = m + \frac{1}{\tau} \sum_{i=N_2}^{n} y_i,$$

then $z_n > 0$ and

$$\begin{aligned} \tau\Delta^2 z_{n-1} + q_n z_n &= \Delta y_n + q_n \left(m + \frac{1}{\tau} \sum_{i=N_2}^{n} y_i \right) \\ &= \Delta y_n + q_n \max_{s\in[n-\sigma,n]} \left(m + \frac{1}{\tau} \sum_{i=N_2}^{s} y_i \right) \\ &\leq \Delta y_n + q_n \max_{s\in[n-\sigma,n]} x_s = 0. \end{aligned}$$

In view of Lemma 1, we know that equation (8) has an eventually positive solution. If $\{x_n\}$ is an eventually positive solution of the equation

$$\Delta(x_n - x_{n-\tau}) + q_n \min_{s\in[n-\sigma,n]} x_s = 0,\ n = 0, 1, ..., \tag{9}$$

we can also prove that

$$\tau\Delta^2 z_{n-1} + q_n z_{n-\sigma} \leq 0$$

has an eventually positive solution. In view of Theorem 2 in [3], we know that equation (8) has an eventually positive solution. We now show that the converse holds. Let $\{z_n\}$ be an eventually positive solution of (8), then it is positive and concave for all large n, so that $\{\Delta z_n\}$ is eventually positive and nonincreasing. Thus it is easy to see that there exists a sufficiently large integer N such that $0 < \tau\Delta z_{n-1} \leq z_{n-\tau}$ for $n \geq N$. Let

$$H_n = \begin{cases} \tau\Delta z_{n-1} & n \geq N \\ (n - N + \tau)\Delta z_{N-1} & N - \tau \leq n < N \\ 0 & n < N - \tau \end{cases},$$

and let

$$x_n = z_{N-\tau} - \tau\Delta z_{N-1} + \sum_{i=0}^{\infty} H_{n-i\tau},\ n \geq 0.$$

In view of the definition of H_n, it is clear that $0 < x_n < \infty$ for all $n \geq 0$, that

$$\max\{x_{N-\tau}, x_{N-\tau+1}, ..., x_{N-1}\} = z_{N-\tau} - \tau\Delta z_{N-1} + (\tau - 1)\Delta z_{N-1} \leq z_{N-\tau},$$

and that

$$x_n - x_{n-\tau} = H_n = \tau\Delta z_{n-1}$$

for $n \geq N$. For any n which satisfies $N \leq n \leq N + \tau - 1$, we see that

$$x_n = \tau\Delta z_{n-1} + x_{n-\tau} \leq \sum_{i=n-\tau}^{n-1} \Delta z_i + x_{n-\tau} \leq \sum_{i=N-\tau}^{n-1} \Delta z_i + z_{N-\tau} = z_n.$$

By induction, it is easy to prove that for any n which satisfies $N+k\tau \leq n < N+(k+1)\tau$ where $k = 0, 1, 2, ...$, $x_n \leq z_n$ is still valid. Thus

$$\Delta(x_n - x_{n-\tau}) + q_n \max_{s\in[n-\sigma,n]} x_s \leq \tau\Delta^2 z_{n-1} + q_n z_n = 0$$

as desired. Similarly, we have also

$$x_n \leq \sum_{i=N-\tau}^{n-1} \Delta z_i + z_{N-\tau} \leq \sum_{i=N-\tau}^{n-1+\sigma} \Delta z_i + z_{N-\tau} = z_{n+\sigma}.$$

Thus, we have

$$\begin{aligned} & \Delta(x_n - x_{n-\tau}) + q_n \min_{s\in[n-\sigma,n]} x_s \leq \tau\Delta^2 z_{n-1} + q_n \min_{s\in[n-\sigma,n]} z_{s+\sigma} \\ = \quad & \tau\Delta^2 z_{n-1} + q_n z_n = 0. \end{aligned}$$

The proof is complete.

As an important corollary, the equation

$$\Delta(x_n - x_{n-\tau}) + \frac{\alpha}{(n+1)^2} \min_{s\in[n-\sigma,n]} x_s = 0,\ n = 0, 1, 2, ...$$

in view of Theorem 3, is oscillatory if, and only if, $\alpha/\tau > 1/4$.

We make several additional remarks. Let us say that a solution of (1) or (2) is strongly oscillatory if for any given nonnegative integer N, there is a corresponding integer $m \geq N$ such that $x_m x_{m+1} < 0$. Assume that $q_n > 0$ for $n \geq 0$. In this case, let $\{x_n\}$ be a nonnegative solution of (1) or (2), and let y_n be defined by (4). We have already seen that $y_n > 0$ and $\Delta y_n \leq 0$ eventually. Note that $x_n \geq y_n$ for all large n.

THEOREM 4. Assume that there exists a nonnegative integer $N \geq 0$ such that $p_{N+j\tau} \leq 1$ for $j = 0, 1, 2, ...$. Suppose further that $q_n > 0$ for $n \geq 0$. Then every nontrivial solution of equation (1) is strongly oscillatory if, and only if, the inequality

$$\Delta\left(x_n - p_n x_{n-\tau}\right) + q_n x_{n-\sigma} \leq 0, : n = 0, 1, ...$$

does not have any eventually nonnegative solutions, and equation (2) has a nonnegative solution if, and only if, (5) has a nonnegative solution.

THEOREM 5. Assume that there exists a nonnegative integer $N \geq 0$ such that $p_{N+j\tau} \leq 1$ for $j = 0, 1, 2, ...$. Assume further that $q_n > 0$ for $n \geq 0$, If there exists some integer T such that the functional inequality

$$\Delta y_n + q_n \max_{s\in[n-\sigma,n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau} \leq 0$$

does not have an eventually positive solution, then equation (2) does not have an eventually nonnegative solution. If there exists some integer T such that the functional inequality

$$\Delta y_n + q_n \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{n-i\tau} y_{n-\tau} \leq 0$$

does not have an eventually positive solution, then every solution of equation (1) is strongly oscillatory.

THEOREM 6. Assume that $q_n > 0$ for $n \geq 0$. Then all solutions of the equation

$$\Delta\left(x_n - x_{n-\tau}\right) + q_n x_{n-\sigma} = 0,$$

are strongly oscillatory if, and only if, equation (8) is oscillatory; and equation (7) has an eventually nonnegative solution if, and only if, (8) has an eventually positive solution.

The proofs of Theorem 4, Theorem 5 and Theorem 6 are similar to the proofs of Theorem 1, Theorem 2 and Theorem 3 respectively. They will be omitted. Results analogous to Theorem 4, Theorem 5 and Theorem 6 are not valid for equation (3). For example, the sequence $\{\sin(n\pi/2) - 1\}$ is a nonpositive solution of the equation

$$\Delta(x_n - x_{n-4}) + q_n \max_{s \in [n-6,n]} x_s = 0,$$

where $\{q_n\}$ is any real sequence.

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