

Asymptotic Behavior Of Solutions Of Nonautonomous Riccati Difference Equations*

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Abstract

In this paper, we investigate the asymptotic behavior of solutions of a nonautonomous Riccati difference equation. We establish conditions on the parameters that guarantee the existence of a unique positive equilibrium point, which is locally asymptotically stable. Additionally, we demonstrate that all solutions of the equation are positive, bounded, and increasing, ultimately converging to the unique positive equilibrium point. The main result obtained here partially answers the open problem proposed by Kulenović and Ladas in 2002.

1 Introduction

Nonautonomous Riccati difference equations (NRDEs) represent a significant class of discrete dynamical systems characterized by their time-dependent coefficients. As seen in [5], NRDEs are pivotal in various fields such as mathematical biology, control theory, economics, and quantum mechanics, where they model processes that evolve over discrete time steps. The importance of studying NRDEs stems from their ability to describe complex, real-world phenomena where the parameters in the system change over time. Analyzing the asymptotic behavior of solutions to nonautonomous Riccati difference equations is crucial, as it provides insights into the long-term stability, periodicity, and convergence of the systems they model, thereby offering predictive power and a deeper understanding of the underlying dynamics.

Several interesting open problems exist in the study of rational difference equations. The dynamical behavior of the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0,$$

studied in [8] generalizes the case $k = 1$ investigated in [10], thereby solving one of the open problem there. Motivated by this, the qualitative properties of the positive solutions in terms of stability and oscillation of nonlinear difference equation

$$x_{n+1} = \frac{\alpha x_{n-m} + \delta x_n}{\beta + \gamma x_{n-k} x_{n-l} (x_{n-k} + x_{n-l})}, \quad n \in \mathbb{N}_0,$$

where $\alpha, \beta, \gamma, \delta \in (0, \infty)$ and $m, k, l \in \mathbb{N}$ such that $m < k < l$, are discussed in [1], together with some numerical illustrations. In [4], the second-order rational difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}, \quad x_0, x_{-1} \in \mathbb{R},$$

which is associated with a linear third-order difference equation in the same way that the first-order Riccati equation ($c = 0$) is associated with a linear second-order equation, is analyzed for its asymptotic stability

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based on certain conditions on the parameters a , b , and c . Further, in the complex plane, [7] treats Riccati equations and second-order linear difference equations by surveying basic properties of these equations which are analogous in the differential case. Sufficient conditions for the existence of solutions of a nonlinear difference equation which is asymptotically equivalent to the solutions of the linear difference equation are established in [13]. The conditions under which a certain type of polynomial sequence is asymptotically equivalent to some solution of the nonautonomous difference equation and every solution is asymptotically polynomial are obtained in [14]. Sufficient conditions for the existence of solutions of discrete Volterra equations with prescribed asymptotic behavior, especially asymptotically polynomial and asymptotically periodic solutions are presented in [16]. The recent papers [15, 17, 18, 19, 21, 20, 22] witnessed that the study asymptotic properties of solutions of difference equations is an active and important area of research.

In this paper, we investigate the asymptotic behavior of solutions of a nonautonomous Riccati difference equation (NRDE)

$$x_{n+1} = \frac{\alpha_n + \beta_n x_n}{A_n + B_n x_n}, \quad n \in \mathbb{N}_0, \quad (1)$$

where α_n , A_n , β_n , B_n , and x_0 are all real numbers. It is noteworthy to mention here that [6] investigates (1) when $\{A_n\}_{n \in \mathbb{N}}$, $\{B_n\}_{n \in \mathbb{N}}$, $\{\alpha_n\}_{n \in \mathbb{N}}$, and $\{\beta_n\}_{n \in \mathbb{N}}$ are periodic sequences of real numbers with period-2 and the initial point $x_0 \in \mathbb{R}$. Further, in [6] itself, the authors deduce conditions for the parameters of (1) under which it has periodic solutions. Certain well-known results, for the same difference equation, in the case of constant coefficients can be found in [2, 3, 11, 12]. Motivated by [9] which investigates (1) for the case when $\alpha_n B_n - A_n \beta_n \neq 0$, $B_n \neq 0$, $n \in \mathbb{N}_0$, we, in this paper, use a different approach to generalize the findings therein by including the case when $\alpha_n B_n - A_n \beta_n = 0$, $n \in \mathbb{N}_0$. We provide some conditions on the parameters of (1) which ensure that it has a unique positive equilibrium point that is locally asymptotically stable, and that every solution is positive, bounded, monotonically increasing, and converges to the unique equilibrium point. In this way, we answer the open problem 6.10.2 given in [10].

Open Problem 6.10.2 Assume that $\{\alpha_n\}_{n \in \mathbb{N}_0}$, $\{\beta_n\}_{n \in \mathbb{N}_0}$, $\{A_n\}_{n \in \mathbb{N}_0}$, $\{B_n\}_{n \in \mathbb{N}_0}$ are convergent sequences of real numbers with finite limits. Investigate the forbidden set and the asymptotic character of solutions of the nonautonomous Riccati equation (1). Here, by the forbidden set, we mean the set F of all initial conditions $x_0 \in \mathbb{R}$ through which (1) is not well-defined for $n \in \mathbb{N}_0$.

The setup of the paper is as follows. In Section 2, we formulate the main assumptions for the parameters of (1). We then establish some preliminary results in Section 3. In Section 4, we prove our main result and present an illustrative example.

2 The Main Hypotheses

We shall assume that x_0 , α_0 , β_0 , α , β , A_0 , B_0 , A , and B are positive constants such that $(A_0 - \beta_0)^2 + 4\alpha_0 B_0 \geq 0$ and $(A - \beta)^2 + 4\alpha B \geq 0$. In this section, we formulate the assumptions on the parameters of (1).

$$(H_1) \quad x_0 \leq \min \left\{ 1, \frac{\beta_0 - A_0 + \sqrt{(A_0 - \beta_0)^2 + 4\alpha_0 B_0}}{2B_0}, \frac{\beta - A + \sqrt{(A - \beta)^2 + 4\alpha B}}{2B} \right\}.$$

$$(H_2) \quad \frac{\alpha + \beta}{A_0} \leq 1 \text{ and } \frac{\alpha_0 + \beta_0}{A_0} < 1.$$

$$(H_3) \quad 0 < \frac{\beta - A + \sqrt{(A - \beta)^2 + 4\alpha B}}{2B} \leq 1.$$

In addition, we shall also suppose that

(H4) sequences $\{A_n\}_{n \in \mathbb{N}}$, $\{B_n\}_{n \in \mathbb{N}}$, $\{\alpha_n\}_{n \in \mathbb{N}}$, and $\{\beta_n\}_{n \in \mathbb{N}}$ are such that $A_n \geq A_0 > 0$, $B_n \geq B_0 > 0$, $\alpha_n \geq \alpha_0$, $\beta_n \geq \beta_0$ for all $n \in \mathbb{N}_0$, and converge to A , B , α and β , respectively. Further, $\{\alpha_n\}_{n \in \mathbb{N}}$ and $\{\beta_n\}_{n \in \mathbb{N}}$ are monotone increasing sequences such that

$$\frac{A_n}{A_{n+1}} \geq \frac{\alpha_n}{\alpha_{n+1}}, \quad \frac{A_n}{B_{n+1}} \geq \frac{\alpha_n}{\beta_{n+1}}, \quad \frac{B_n}{B_{n+1}} \geq \frac{\beta_n}{\beta_{n+1}}, \quad \frac{B_n}{A_{n+1}} \geq \frac{\beta_n}{\alpha_{n+1}}, \quad n \in \mathbb{N}. \quad (2)$$

3 Auxiliary Results

In this section, we shall prove some auxiliary results needed to prove our main result. We will start with the following.

Proposition 1 *Suppose (H₁)–(H₄) hold. Then every solution {x_n}_{n∈ℕ} of (1) is monotonically increasing.*

Proof. We have

$$x_1 - x_0 = \frac{\alpha_0 + \beta_0 x_0}{A_0 + B_0 x_0} - x_0 = \frac{\alpha_0 + (B_0 - A_0)x_0 - B_0 x_0^2}{A_0 + B_0 x_0}.$$

In view of (H₁), we obtain x₁ - x₀ ≥ 0. Assume that x_j ≥ x_{j-1} for any j ∈ {1, …, n + 1} for some n ∈ ℕ. We will prove that x_{n+2} ≥ x_{n+1}. We have

$$\begin{aligned} x_{n+2} - x_{n+1} &= \frac{\alpha_{n+1} + \beta_{n+1}x_{n+1}}{A_{n+1} + B_{n+1}x_{n+1}} - \frac{\alpha_n + \beta_n x_n}{A_n + B_n x_n} \\ &= \frac{1}{(A_{n+1} + B_{n+1}x_{n+1})(A_n + B_n x_n)} \left(A_n \alpha_{n+1} + A_n \beta_{n+1} x_{n+1} \right. \\ &\quad \left. + B_n \alpha_{n+1} x_n + B_n \beta_{n+1} x_n x_{n+1} - \alpha_n A_{n+1} - \alpha_n B_{n+1} x_{n+1} \right. \\ &\quad \left. - \beta_n A_{n+1} x_n - \beta_n B_{n+1} x_n x_{n+1} \right) \\ &= \frac{1}{(A_{n+1} + B_{n+1}x_{n+1})(A_n + B_n x_n)} \left((A_n \alpha_{n+1} - \alpha_n A_{n+1}) \right. \\ &\quad \left. + (A_n \beta_{n+1} - \alpha_n B_{n+1}) x_{n+1} + (B_n \beta_{n+1} - \beta_n B_{n+1}) x_n x_{n+1} \right. \\ &\quad \left. + (B_n \alpha_{n+1} - \beta_n A_{n+1}) x_n \right) \\ &\stackrel{(H_4)}{\geq} 0. \end{aligned}$$

Thus, in view of the principle of mathematical induction, we conclude that x_{n+1} ≥ x_n for all n ∈ ℕ. This completes the proof. ■

Proposition 2 *Suppose (H₁)–(H₄) hold. Then every solution {x_n}_{n∈ℕ} of (1) satisfies 0 ≤ x_n ≤ 1, n ∈ ℕ.*

Proof. We have

$$0 \leq x_1 = \frac{\alpha_0 + \beta_0 x_0}{A_0 + B_0 x_0} \leq \frac{\alpha_0 + \beta_0}{A_0} \stackrel{(H_2)}{\leq} 1,$$

and

$$0 \leq x_2 = \frac{\alpha_1 + \beta_1 x_1}{A_1 + B_1 x_1} \stackrel{(H_4)}{\leq} \frac{\alpha + \beta}{A_0} \stackrel{(H_2)}{\leq} 1.$$

Assume that 0 ≤ x_n ≤ 1 for some n ∈ ℕ. We prove that 0 ≤ x_{n+1} ≤ 1. We have

$$0 \leq x_{n+1} = \frac{\alpha_n + \beta_n x_n}{A_n + B_n x_n} \stackrel{(H_4)}{\leq} \frac{\alpha + \beta}{A_0} \stackrel{(H_2)}{\leq} 1.$$

Thus, in view of the principle of mathematical induction, we conclude that 0 ≤ x_n ≤ 1, n ∈ ℕ. This completes the proof. ■

Proposition 3 *Suppose (H₁)–(H₄) hold. Then every solution {x_n}_{n∈ℕ} of (1) is convergent and*

$$\lim_{n \rightarrow \infty} x_n = \frac{\beta - A + \sqrt{(A - \beta)^2 + 4\alpha B}}{2B}.$$

Proof. By virtue of Proposition 1, $\{x_n\}_{n \in \mathbb{N}}$ is monotone increasing. Also, in view of Proposition 2, we find that $\{x_n\}_{n \in \mathbb{N}}$ is bounded. Hence $\{x_n\}_{n \in \mathbb{N}}$ is convergent to some $\bar{x} \in \mathbb{R}$, and

$$\bar{x} = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \frac{\alpha_n + \beta_n x_n}{A_n + B_n x_n} \stackrel{(H_4)}{=} \frac{\alpha + \beta \bar{x}}{A + B \bar{x}},$$

whereupon

$$B\bar{x}^2 + (A - \beta)\bar{x} - \alpha = 0$$

and

$$\bar{x} = \frac{\beta - A + \sqrt{(A - \beta)^2 + 4\alpha B}}{2B}. \tag{3}$$

This completes the proof. ■

Remark 1 Note that \bar{x} , defined by (3), is the unique positive equilibrium point of (1) and $x_n \leq \bar{x}$ for $n \in \mathbb{N}$.

4 Main Result

In this section, we formulate our main result. This theorem answers the Open Problem 6.10.2 [10].

Theorem 1 Suppose (H_1) – (H_4) hold. Then, (1) has a unique positive equilibrium point that is locally asymptotically stable and every solution of (1) is positive, bounded, monotonically increasing, and converges to the unique positive equilibrium point.

Proof. Let $\varepsilon > 0$ be arbitrarily chosen. Take $\delta \in (0, \varepsilon]$ and suppose that $(\bar{x} - x_0) < \delta$. Keeping in mind Proposition 1 and Proposition 3, we have $\bar{x} - x_n \geq 0$, $n \in \mathbb{N}$. Also, in view of Proposition 1, we have $x_0 \leq x_1$ and $x_{n+1} \geq x_n$, $n \in \mathbb{N}$. Then $x_n \geq x_0$, $n \in \mathbb{N}$, and

$$\bar{x} - x_n \leq \bar{x} - x_0 < \delta \leq \varepsilon, \quad n \in \mathbb{N}.$$

Thus, \bar{x} is locally stable. Since $x_0 \leq x_1 \leq \bar{x}$, there exists $\gamma > 0$ such that $(\bar{x} - x_0) < \gamma$. Also, since $\lim_{n \rightarrow \infty} x_n = \bar{x}$, we can conclude that \bar{x} is locally asymptotically stable. This completes the proof. ■

To illustrate the above theorem, we provide an example given below.

Example 1 Let $A_n = B_n = \frac{n+2}{3(n+1)}$, $\alpha_n = \beta_n = \frac{n+1}{10(n+2)}$, $n \in \mathbb{N}$, $x_0 = \frac{3}{40}$, $\alpha = \beta = \frac{1}{10}$, $A_0 = B_0 = A = B = \frac{1}{3}$, and $\alpha_0 = \beta_0 = \frac{1}{20}$. Then the inequalities (2) are reduced to the inequality

$$\frac{A_n}{A_{n+1}} \geq \frac{\alpha_n}{\alpha_{n+1}},$$

i.e.,

$$\frac{(n+2)^2}{(n+1)(n+3)} \geq \frac{(n+1)(n+3)}{(n+2)^2},$$

equivalently,

$$(n+2)^4 \geq (n+1)^2(n+3)^2, \quad n \in \mathbb{N}.$$

Also, $A_n \geq A_0$, $\alpha_n \geq \alpha_0$, $n \in \mathbb{N}$, and $A_n \rightarrow A$, $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Thus, (H_4) holds. Next,

$$\frac{\alpha + \beta}{A_0} = \frac{3}{5} < 1 \quad \text{and} \quad \frac{\alpha_0 + \beta_0}{A_0} = \frac{3}{10} < 1,$$

i.e., (H_2) holds. Further,

$$\frac{\beta - A + \sqrt{(A - \beta)^2 + 4\alpha B}}{2B} = \frac{\alpha - A + \sqrt{(A - \alpha)^2 + 4\alpha A}}{2A} \leq 1,$$

equivalently,

$$\alpha + \sqrt{(A - \alpha)^2 + 4\alpha A} \leq 3A,$$

which, using the given data, yields $\frac{8}{15} < 1$ and

$$\frac{\beta - A + \sqrt{(A - \beta)^2 + 4\alpha B}}{2B} = \frac{3}{10} > x_0,$$

$$\frac{\beta_0 - A_0 + \sqrt{(A_0 - \beta_0)^2 + 4\alpha_0 B_0}}{2B_0} = \frac{3}{20} > x_0.$$

Therefore, (H_1) and (H_3) hold. Hence, in view of Theorem 1 we conclude that NRDE (1) has a unique positive equilibrium point that is locally asymptotically stable.

5 Concluding Remarks

In [9], the dynamical behavior including boundedness, invariant intervals, semicycles, global attractivity, and global asymptotic stability of all nonnegative solutions of NRDE (1) has been investigated. However, that paper assumes only $\alpha_n B_n - A_n \beta_n \neq 0$, $B_n \neq 0$, $n \in \mathbb{N}_0$. Here, we use a different approach to generalize the findings of [9] by including the case when $\alpha_n B_n - A_n \beta_n = 0$, $n \in \mathbb{N}_0$. The investigation described in this paper addresses the open problem 6.10.2 [10] regarding the asymptotic character of solutions of the nonautonomous Riccati difference equation posed in (1).

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