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Lower Bound Estimates Of Derivative Of A Class Of Polynomials[*]

Robinson Soraisam[†], Barchand Chanam[‡]

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Abstract

For a polynomial $p(z)$ of degree n having all its zeros on $|z| = k$, $k \leq 1$, Govil [Proc. Nat. Acad. Sci., 50(1980), 183–187] proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-1} + k^n} \max_{|z|=1} |p(z)|.$$

The above inequality gives an upper bound estimate of $\max_{|z|=1} |p'(z)|$. It is natural and of interest to seek a lower bound estimate of $\max_{|z|=1} |p'(z)|$. In this attempt, we are able to obtain the lower bound estimates for this particular class of polynomials.

1 Introduction

The classical Bernstein's inequality [1] for an n^{th}-order polynomial $p(z)$ states that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

The above inequality (1) can be sharpened, if the zeros of $p(z)$ are restricted in certain ways. In this direction, Erdös conjectured and later Lax [10] in 1944 proved that if $p(z)$ has no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

It is interesting to note that in 1939, about five years earlier, Turán [15] proved his famous inequality concerning a lower bound of $\max_{|z|=1} |p'(z)|$ by imposing a restriction on the location of the zeros of the polynomial that they all lie in the closed unit disc $|z| \leq 1$, unlike Bernstein's inequality and Erdös-Lax theorem wherein they estimated an upper bound of $\max_{|z|=1} |p'(z)|$. It is also worth knowing that the restriction imposed on the zeros of the polynomial in Turán's inequality is nearly opposite to the hypotheses in the Erdös-Lax theorem. He proved that if $p(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{3}$$

In 1969, Malik [11] obtained a partial generalization of Erdös and Lax's inequality by considering the class of polynomials having no zero in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{4}$$

For $p(z) = (z + k)^n$, we can get equality for the above result.

Malik [11] was the first to establish the following result which generalizes Turán's inequality (3) by considering polynomials with all their zeros in $|z| \leq k$, $k \leq 1$ and he proved it in the same paper mentioned above [11] as an application of his inequality (4).

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{5}$$

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[†] Department of Mathematics, National Institute of Technology Manipur, Manipur, 795004, India

[‡] Department of Mathematics, National Institute of Technology Manipur, Manipur, 795004, India

In an attempt to prove the analogous inequality of (5) for the case $k \geq 1$, Govil [3] in 1973 obtained the following result

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{6}$$

There have been so many problems in the literature concerning Turán's type inequality (3) (see some of these recent papers [5, 9, 12, 13, 14] and the references therein). It is worth noting that with the establishment of inequalities (5) and (6), Turán's inequality is completely generalized on any closed disc $|z| \leq k$ for $k \leq 1$ as well as $k \geq 1$ whereas returning our attention to the analogous inequality of Malik's inequality (4), for the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, the precise estimate of maximum of $|p'(z)|$ on $|z| = 1$ is, in general, not easily obtainable. In this concern, similar to the techniques for obtaining inequality (5) as an application of Malik's inequality (4), it would then be expected that if $p(z) \neq 0$ in $|z| < k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|.$$

However, this bound does not, in general, hold as illustrated by the counter example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ for which $\max_{|z|=1} |p'(z)| = \frac{13}{6} \approx 2.166$ and $\max_{|z|=1} |p(z)| = 1.191$, but for $k = \frac{1}{3}$, we have

$$\frac{n}{1+k^n} \max_{|z|=1} |p(z)| = \frac{2 \times 1.191}{1 + \left(\frac{1}{3}\right)^2} \approx 2.144 < 2.166 = \max_{|z|=1} |p'(z)|\,.$$

In the literature (see [2, page 206], [6], [8, page 216] etc.), it is simply mentioned that this example is due to Edward B. Saff, even though its origin or source has not yet been cited as a reference. This remains a potential gap in the literature. However, imposing a strong restriction on the moduli of the derivatives of $p(z)$ and the associated inversive polynomial $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, i.e., $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, Govil [4] in 1973 proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|,$$

where here and throughout the paper $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

Again, for a particular class of polynomials of degree n not vanishing in $|z| < k$, $k \leq 1$, Govil [4] proved the following

Theorem 1 *If $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-1}+k^n} \max_{|z|=1} |p(z)|.$$

We are constantly interested to obtain a lower bound estimate for the class of polynomials having no zero in $|z| < k$, $k > 0$. Such problems do not exist so far in the literature. However, in this work, for the class of polynomials having all its zeros on $|z| = k$, $k \leq 1$, we have been able to obtain the lower bound estimates for $\max_{|z|=1} |p'(z)|$.

2 Lemmas

We shall need the following lemmas in order to prove the above theorems. The first lemma is due to Govil et al. [7].

Lemma 2 *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1} |p'(z)| \geq n \max_{|z|=1} |p(z)| - \max_{|z|=1} |q'(z)|.$$

Lemma 3 *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k}\max_{|z|=1}|p(z)|.$$

Lemma 3 is due to Malik [11].

Lemma 4 *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p'(z)|\leq n\frac{n|a_0|+k^2|a_1|}{(1+k^2)\,n|a_0|+2k^2|a_1|}\max_{|z|=1}|p(z)|.$$

Lemma 4 is due to Govil et al. [7].

3 Main results

For the class of polynomials having all its zeros on $|z|=k$, $k\leq 1$, we obtain the lower bound estimates for $\max_{|z|=1}|p'(z)|$ as follows:

Theorem 5 *If $p(z)$ is a polynomial of degree n having all its zeros on $|z|=k$, $k\leq 1$, then*

$$\max_{|z|=1}|p'(z)|\geq \frac{n}{1+k}\max_{|z|=1}|p(z)|\,. \tag{7}$$

Proof. The proof of this theorem follows the same lines as that of Theorem 6, by applying Lemma 3 instead of Lemma 4. So, we omit the proof. ■

Further, we obtain an improvement of Theorem 5 by involving some of the coefficients of the polynomial. More precisely, we prove the following theorem.

Theorem 6 *If $p(z)=\sum_{\nu=0}^{\nu=n}a_\nu z^n$ is a polynomial of degree n having all its zeros on $|z|=k$, $k\leq 1$, then*

$$\max_{|z|=1}|p'(z)|\geq n\frac{n|a_n|+|a_{n-1}|}{(1+k^2)\,n|a_n|+2|a_{n-1}|}\max_{|z|=1}|p(z)|. \tag{8}$$

Proof. By Lemma 2, we have

$$\max_{|z|=1}|p'(z)|\geq n\max_{|z|=1}|p(z)|-\max_{|z|=1}|q'(z)|. \tag{9}$$

Since $p(z)$ has all its zeros on $|z|=k$, $k\leq 1$, $q(z)$ has all its zeros on $|z|=\frac{1}{k}$, $\frac{1}{k}\geq 1$. That is, $q(z)$ has no zero in $|z|<\frac{1}{k}$, $\frac{1}{k}\geq 1$ and applying Lemma 4 to $q(z)$, we have

$$\begin{aligned}\max_{|z|=1}|q'(z)| &\leq n\frac{n|a_n|+\frac{1}{k^2}\,|a_{n-1}|}{\left(1+\frac{1}{k^2}\right)n\,|a_n|+2\frac{1}{k^2}\,|a_{n-1}|}\max_{|z|=1}|q(z)|\\ &= n\frac{nk^2\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}\max_{|z|=1}|q(z)|.\end{aligned}$$

Using the simple fact that on $|z|=1$, $|p(z)|=|q(z)|$, in the right hand side of the above inequality, we have

$$\max_{|z|=1}|q'(z)|\leq n\frac{nk^2\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}\max_{|z|=1}|p(z)|. \tag{10}$$

Combining (9) and (10), we have

$$\begin{aligned}\max_{|z|=1}|p'(z)| &\geq n\left\{1-\frac{nk^2\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}\right\}\max_{|z|=1}|p(z)|\\ &= n\frac{n\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}\max_{|z|=1}|p(z)|.\end{aligned}$$

This completes the proof of Theorem 6. ■

Remark 1 It is really of interest to show that the bound given by (7) is a lower bound estimate of $\max_{|z|=1}|p'(z)|$, for this, it is worth verifying that this bound is smaller than that of bound (1) due to Bernstein. For this, it is sufficient to show that

$$\frac{n}{1+k} \leq n,$$

i.e.

$$1+k \geq 1$$

which is true as $k > 0$.

Remark 2 To show that the bound of Theorem 6 improves upon Theorem 5, it is sufficient to show that

$$\frac{n|a_n| + |a_{n-1}|}{(1+k^2)\,n|a_n| + 2|a_{n-1}|} \geq \frac{1}{1+k},$$

which is equivalent to

$$\frac{n|a_n|k^2 + |a_{n-1}|}{|a_{n-1}| + n|a_n|} \geq k,$$

i.e.

$$nk \geq \left|\frac{a_{n-1}}{a_n}\right|,$$

which clearly holds by Vieta's formula that gives the relation between the product of the roots and coefficients of the polynomial equation.

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