

# New Variants of Hankel Matrices with Fibonacci Entries: Norms and Properties <sup>\*†</sup>

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## Abstract

This study introduces novel variants of Hankel matrices, including the left Hankel matrix and its symmetric forms, as well as the newly defined CFP<sub>r</sub>L<sub>r</sub>R (Column-First-Plus-rLast-rRight) and CLP<sub>r</sub>FrL (Column-Last-Plus-rFirst-rLeft) Hankel matrices, with entries derived from Fibonacci numbers. We establish properties of these matrices, deriving exact expressions for their Euclidean norms and bounds for their spectral norms using well-known Fibonacci sum identities. Additionally, inequalities for maximum row and column sum norms are provided, along with corollaries for specific cases. The results contribute to the understanding of structured matrices in linear algebra, with potential applications in system identification, signal processing, and numerical analysis.

## 1 Introduction

Special matrices are widely studied in the research area of matrix theory. In particular, special matrices whose entries are well known number sequences and polynomials have become a very interesting research subject in recent years and some scholars have obtained some good results in this area. Many studies examined the norms of the special matrices involving famous number sequences and polynomials. They found various properties of these matrices, such as lower bounds, upper bounds, and exact values for the spectral norms, Euclidean norms, and determinants.

Hankel matrices are one of the most researched and well-known types of structured matrices, and they are widely used. Hankel matrices arise in many different theoretical and applied fields. For example, in the study by Mu and Chen [10], the comparable conditions for the system's identifiability were covered in the paper first. It was then demonstrated that the identifiability of the system was equal to the row-full-rank of the Hankel matrix made up of impulse responses. Finally, the necessary and sufficient conditions were given for the row-full-rank of the Hankel matrix made out of correlation functions. The instantaneous autocorrelation function reconstruction problem was formulated [15] as a block Hankel matrix rank minimization problem, which robustly handles amplitude changes between signal components and efficiently reduces the impact of burst missing data. This work [13] proposed two modified minimal realization techniques that use modified Hankel matrices to obtain lower-order models in the time domain. The techniques were effective, easy to compute, and also applicable to multiple-input, multiple-output systems. A self-contained survey of quick techniques for solving linear equation systems using Toeplitz or Hankel coefficient matrices was provided in [5]. They talked about Schur-type and Levinson-type algorithms. It was shown how they relate to Lanczos methods, Padé recursions, and triangular factorizations. Split algorithms were devised and their linkages with butterfly factorizations were studied in the case where the matrices had additional symmetry features. It was demonstrated in [2] that the singular values of this Hankel matrix were the square roots of the proper orthogonal decomposition energies and that the left singular vectors represented a discrete approximation of space-time proper orthogonal decomposition modes. The work of Toeplitz, Hankel and circulant matrices has led to significant recent success and can be credited with several important mathematical advances [1, 9, 11, 12, 14].

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In this study, we introduce the definitions of the left Hankel matrix and its varieties. Secondly, we produce the new concepts called CFP $r$ L $r$ R and CLP $r$ FrL Hankel matrices and the properties of the related Fibonacci numbers. Then, we present the main results and the detailed presentation.

## 2 Preliminaries

A matrix  $H = [h_{ij}] \in M_{n \times n}(\mathbb{C})$  where  $h_{ij} = h_{i+j-2}$ , is called a Hankel matrix [4]. A Hankel matrix is of the form

$$H = \begin{pmatrix} h_0 & h_1 & \dots & h_{n-2} & h_{n-1} \\ h_1 & h_2 & \dots & h_{n-1} & h_n \\ h_2 & h_3 & \dots & h_n & h_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{n-2} & h_{n-1} & \dots & h_{2n-4} & h_{2n-3} \\ h_{n-1} & h_n & \dots & h_{2n-3} & h_{2n-2} \end{pmatrix}_{n \times n}.$$

A matrix  $H_r = [h_{ij}] \in M_{n \times n}(\mathbb{C})$  such that

$$h_{ij} = \begin{cases} rh_{i+j-2}, & i + j \leq n, \\ h_{i+j-2}, & i + j > n, \end{cases}$$

is called a left  $r$ -Hankel matrix. A matrix  $SH_r = [h_{ij}] \in M_{n \times n}(\mathbb{C})$  such that

$$h_{ij} = \begin{cases} rh_{i+j-2}, & i + j > n + 1, \\ h_{i+j-2}, & i + j \leq n + 1, \end{cases}$$

is called a symmetric left  $r$ -Hankel matrix [3].

The Euclidean norm of the matrix  $A$  is defined as

$$\|A\|_E = \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

The singular values of the matrix  $A$  are  $\sigma_i = \sqrt{\lambda_i(A^*A)}$ , where  $\lambda_i$  goes through the eigenvalues of  $A^*A$  and  $A^*$  is the conjugate transpose of matrix  $A$ . The square roots of the maximum eigenvalues of  $A^*A$  is called the spectral norm of  $A$  and is denoted by  $\|A\|_2$ .

The following inequality holds,

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E.$$

The maximum column and row sum matrix norm of the matrix  $A$  are

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \quad \text{and} \quad \|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|.$$

Define the maximum column length norm  $c_1$ , and the maximum row length norm  $r_1$  of any matrix  $A$  by

$$r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2} \quad \text{and} \quad c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2},$$

respectively. Let  $A, B$  and  $C$  be  $m \times n$  matrices. If  $A = B \circ C$ , then [6]

$$\|A\|_2 \leq r_1(B) c_1(C).$$

Let  $A$  be an  $m \times n$  matrix. We have [8]

$$\|A\|_2 \geq \frac{2|r_1 + r_2 + r_3 + \dots + r_m|}{m + n},$$

where  $r_i$  denote the row sums of  $A$ .

The following sum formulas for the Fibonacci numbers are well-known [7]:

$$\begin{aligned} \sum_{i=1}^{n-1} F_i &= F_{n+1} - 1, \\ \sum_{i=1}^{n-1} iF_i &= 2 + nF_{n+2} - F_{n+3}, \\ \sum_{i=1}^{n-1} F_i^2 &= F_n F_{n-1}, \\ \sum_{i=1}^{n-1} iF_i^2 &= \begin{cases} nF_n F_{n+1} - F_n^2 + 1, & \text{if } n \text{ is odd,} \\ nF_n F_{n+1} - F_n^2, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

### 3 The Left Hankel Matrices

**Definition 1** A left-Hankel matrix is defined by

$$LH = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_n & a_0 & a_1 & \dots & a_{n-2} \\ a_{n+1} & a_n & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-3} & a_{2n-4} & a_{2n-5} & \dots & a_1 \\ a_{2n-2} & a_{2n-3} & a_{2n-4} & \dots & a_0 \end{pmatrix}_{n \times n}.$$

**Definition 2** A matrix  $LH_r = [h_{ij}] \in M_{n \times n}(\mathbb{C})$ ,

$$LH_r = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ ra_n & a_0 & a_1 & \dots & a_{n-2} \\ ra_{n+1} & ra_n & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ra_{2n-3} & ra_{2n-4} & ra_{2n-5} & \dots & a_1 \\ ra_{2n-2} & ra_{2n-3} & ra_{2n-4} & \dots & a_0 \end{pmatrix}_{n \times n}$$

is called a left  $r$ -Hankel matrix. A matrix  $SLH_r = [h_{ij}] \in M_{n \times n}(\mathbb{C})$ ,

$$SLH_r = \begin{pmatrix} a_0 & ra_1 & ra_2 & \dots & ra_{n-1} \\ a_n & a_0 & ra_1 & \dots & ra_{n-2} \\ a_{n+1} & a_n & a_0 & \dots & ra_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-3} & a_{2n-4} & a_{2n-5} & \dots & ra_1 \\ a_{2n-2} & a_{2n-3} & a_{2n-4} & \dots & ra_0 \end{pmatrix}_{n \times n}$$

is called a symmetric left  $r$ -Hankel matrix. Let  $A = H(a_0, a_1, a_2, \dots, a_{n-1})$  and  $B = LH(a_{n-1}, a_{n-2}, a_{n-3}, \dots, a_1, a_0)$ . We have

$$A = BE$$

where  $E$  is the exchange matrix defined by  $E = \text{antidiag}(1, 1, 1, \dots, 1)$ ,

$$\det(A) = \det(B).$$

**Theorem 1** Let  $A$  be a left  $r$ -Hankel matrix with the first row  $(F_0, F_1, F_2, \dots, F_{n-1})$ .

(i) If  $|r| \geq 1$ , then

$$\left\{ \begin{array}{l} \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2 + 1}{n}} \\ \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2}{n}} \end{array} \right\} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \begin{cases} |r| \sqrt{n(F_{2n-3}F_{2n-2} - F_nF_{n-1})}, & \text{if } n \text{ is odd,} \\ |r| \sqrt{n(F_{2n-3}F_{2n-2} - F_nF_{n-1})} & \text{if } n \text{ is even.} \end{cases}$$

(ii) If  $|r| < 1$ , then

$$\left\{ \begin{array}{l} |r| \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2 + 1}{n}} \\ |r| \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2}{n}} \end{array} \right\} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \begin{cases} \sqrt{n(F_{2n-3}F_{2n-2} - F_nF_{n-1})}, & \text{if } n \text{ is odd,} \\ \sqrt{n(F_{2n-3}F_{2n-2} - F_nF_{n-1})}, & \text{if } n \text{ is even.} \end{cases}$$

**Proof.** The matrix  $A$  is of the form

$$A = \begin{pmatrix} F_0 & F_1 & \dots & F_{n-2} & F_{n-1} \\ rF_n & F_0 & \dots & F_{n-3} & F_{n-2} \\ rF_{n+1} & rF_n & \dots & F_{n-4} & F_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ rF_{2n-3} & rF_{2n-4} & \dots & F_0 & F_1 \\ rF_{2n-2} & rF_{2n-3} & \dots & rF_n & F_0 \end{pmatrix}_{n \times n}.$$

Then we have,

$$\|A\|_E^2 = |r|^2 \sum_{i=n}^{2n-2} (2n-1-i)F_i^2 + \sum_{i=0}^{n-1} (n-i)F_i^2.$$

Hence, when  $|r| \geq 1$  we obtain

$$\|A\|_E^2 \geq \sum_{i=n}^{2n-2} (2n-1-i)F_i^2 + \sum_{i=0}^{n-1} (n-i)F_i^2,$$

that is

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2 + 1}{n}}, & \text{if } n \text{ is odd,} \\ \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2}{n}}, & \text{if } n \text{ is even.} \end{cases}$$

On the other hand, let the matrices  $B$  and  $C$  as

$$B = \begin{pmatrix} F_0 & F_1 & F_2 & \dots & F_{n-1} \\ F_n & F_0 & F_1 & \dots & F_{n-2} \\ F_{n+1} & F_n & F_0 & \dots & F_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{2n-3} & F_{2n-4} & F_{2n-5} & \dots & F_1 \\ F_{2n-2} & F_{2n-3} & F_{2n-4} & \dots & F_0 \end{pmatrix}_{n \times n} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ r & 1 & 1 & \dots & 1 \\ r & r & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & r & r & \dots & 1 \\ r & r & r & \dots & 1 \end{pmatrix}_{n \times n}$$

such that  $A = B \circ C$ . Then

$$r_1(B) = \max_i \sqrt{\sum_j |a_{ij}|^2} = \sqrt{\sum_{j=n}^{2n-2} F_j^2} = \sqrt{F_{2n-3}F_{2n-2} - F_nF_{n-1}}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |a_{ij}|^2} = \sqrt{(n-1)|r|^2 + 1}.$$

We have

$$\|A\|_2 \leq |r| \sqrt{n(F_{2n-3}F_{2n-2} - F_nF_{n-1})}.$$

When  $|r| < 1$ , we also obtain

$$\begin{cases} |r| \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2 + 1}{n}} \\ |r| \sqrt{\frac{(2n-1)F_{2n-2}F_{2n-1} - (n-1)F_nF_{n-1} - (2n-1)F_{2n-1}F_{2n} - F_{2n-1}^2}{n}} \end{cases} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \begin{cases} \sqrt{n(F_{2n-3}F_{2n-2} - F_nF_{n-1})}, & \text{if } n \text{ is odd,} \\ \sqrt{n(F_{2n-3}F_{2n-2} - F_nF_{n-1})}, & \text{if } n \text{ is even.} \end{cases}$$

■

**Corollary 1** Let  $A$  be a right  $r$ -Hankel matrix with the first row  $(F_0, F_1, F_2, \dots, F_{n-1})$ .

- (i) If  $|r| \geq 1$ , then  $\|A\|_1 = \|A\|_\infty = |r|(F_{2n} - F_{n+1})$ .
- (ii) If  $|r| < 1$ , then  $\|A\|_1 = \|A\|_\infty = F_{n+1} - 1$ .

**Corollary 2** Let  $A$  be a right  $r$ -Hankel matrix with the first row  $(F_0, F_1, F_2, \dots, F_{n-1})$ . We have

$$\|A\|_2 \geq \frac{|r_1 + r_2 + r_3 + \dots + r_n|}{n},$$

$$\|A\|_2 \geq \frac{|r \sum_{i=n}^{2n-2} (2n-1-i)F_i + \sum_{i=0}^{n-1} (n-i)F_i|}{n},$$

and

$$\|A\|_2 \geq \frac{|r[(1-2n)(F_{n+1} + F_{2n-1}) + nF_{n+2} + F_{2n+2} - F_{n+3}] - nF_n + F_{n+3} - n - 2|}{n}.$$

### 4 The CFPrLrR and CLPrFrL Hankel Matrices

**Definition 3** A column-first-plus- $r$ -last- $r$ -right Hankel matrix with the first row  $(a_0, a_1, a_2, \dots, a_{n-1})$ , denoted by CFPrLrRH( $a_0, a_1, a_2, \dots, a_{n-1}$ ) means a square matrix of the form

$$A = \begin{pmatrix} ra_0 & ra_1 + ra_0 & ra_2 + ra_1 & \dots & a_{n-1} + ra_{n-2} \\ ra_1 & ra_2 + ra_1 & ra_3 + ra_2 & \dots & a_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ra_{n-3} & ra_{n-2} + ra_{n-3} & a_{n-1} + ra_{n-2} & \dots & a_{2n-4} \\ ra_{n-2} & a_{n-1} + ra_{n-2} & a_n & \dots & a_{2n-3} \\ a_{n-1} & a_n & a_{n+1} & \dots & a_{2n-2} \end{pmatrix}_{n \times n}.$$

**Definition 4** A column-last-plus-r-first-r-left Hankel matrix with the first row  $(a_0, a_1, a_2, \dots, a_{n-1})$ , denoted by  $CLPrFrLH(a_0, a_1, a_2, \dots, a_{n-1})$  means a square matrix of the form

$$B = \begin{pmatrix} a_0 + ra_1 & \dots & ra_{n-3} + ra_{n-2} & ra_{n-2} + ra_{n-1} & ra_{n-1} \\ a_n & \dots & ra_{n-4} + ra_{n-3} & ra_{n-3} + ra_{n-2} & ra_{n-2} \\ a_{n+1} & \dots & ra_{n-5} + ra_{n-4} & ra_{n-4} + ra_{n-3} & ra_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-3} & \dots & a_n & a_0 + ra_1 & ra_1 \\ a_{2n-2} & \dots & a_{n+1} & a_n & ra_0 \end{pmatrix}_{n \times n}.$$

**Remark 1** A column-first-plus-r-last-r-right Toeplitz matrix  $A = H_r + r\mathbb{T}^{ru}$  and  $\|A\|_2 \leq \|H_r\|_2 + |r| \|\mathbb{T}^{ru}\|_2$ , where  $H_r$  is the  $r$ -Hankel matrix below in the form

$$H_r = \begin{pmatrix} ra_0 & ra_1 & ra_2 & \dots & a_{n-1} \\ ra_1 & ra_2 & ra_3 & \dots & a_n \\ ra_2 & ra_3 & ra_4 & \dots & a_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ra_{n-2} & a_{n-1} & a_n & \dots & a_{2n-3} \\ a_{n-1} & a_n & a_{n+1} & \dots & a_{2n-2} \end{pmatrix}_{n \times n}$$

and  $\mathbb{T}^{ru}$  is the left-upper triangular matrix below in the form

$$\mathbb{T}^{ru} = \begin{pmatrix} 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & a_1 & a_2 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n-2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{n \times n}.$$

Similarly,  $B$  a column-last-plus-r-first-r-left Hankel matrix  $B = SLH_r + r\mathbb{T}^{rl}$ , where  $RLH_r$  is the left symmetric  $r$ -Hankel matrix and  $\mathbb{T}^{rl}$  is the right-lower triangular matrix. Then we have  $\|B\|_2 \leq \|SLH_r\|_2 + |r| \|\mathbb{T}^{rl}\|_2$ .

**Theorem 2** Let  $A$  be a column-first-plus-r-last-r-right Hankel matrix with the first row  $(F_0, F_1, F_2, \dots, F_{n-1})$ .

(i) If  $|r| \geq 1$ , then

$$\left\{ \begin{array}{l} \sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2 + 1}{n}} \\ \sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2}{n}} \end{array} \right\} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \begin{cases} |r| \sqrt{n(F_{2n-1} F_{2n-2} - F_{n-1} F_{n-2})}, & \text{if } n \text{ is odd,} \\ |r| \sqrt{n(F_{2n-1} F_{2n-2} - F_{n-1} F_{n-2})}, & \text{if } n \text{ is even.} \end{cases}$$

(ii) If  $|r| < 1$ , then

$$\left\{ \begin{array}{l} |r| \sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2 + 1}{n}} \\ |r| \sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2}{n}} \end{array} \right\} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \begin{cases} \sqrt{n(F_{2n-1} F_{2n-2} - F_{n-1} F_{n-2})}, & \text{if } n \text{ is odd,} \\ \sqrt{n(F_{2n-1} F_{2n-2} - F_{n-1} F_{n-2})}, & \text{if } n \text{ is even.} \end{cases}$$

**Proof.** The matrix  $A$  is of the form

$$A = \begin{pmatrix} rF_0 & rF_0 + rF_1 & rF_1 + rF_2 & \dots & F_{n-1} + rF_{n-2} \\ rF_1 & rF_1 + rF_2 & rF_2 + rF_3 & \dots & F_n \\ rF_2 & rF_2 + rF_3 & rF_3 + rF_4 & \dots & F_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ rF_{n-2} & F_{n-1} + rF_{n-2} & F_n & \dots & F_{2n-3} \\ F_{n-1} & F_n & F_{n+1} & \dots & F_{2n-2} \end{pmatrix}_{n \times n}$$

and

$$A = \begin{pmatrix} rF_0 & rF_2 & rF_3 & \dots & F_{n-1} + rF_{n-2} \\ rF_1 & rF_3 & rF_4 & \dots & F_n \\ rF_2 & rF_4 & rF_5 & \dots & F_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ rF_{n-2} & F_{n-1} + rF_{n-2} & F_n & \dots & F_{2n-3} \\ F_{n-1} & F_n & F_{n+1} & \dots & F_{2n-2} \end{pmatrix}_{n \times n}$$

Then we have

$$\|A\|_E^2 = |r|^2 \sum_{i=1}^{n-2} iF_i^2 + \sum_{i=n}^{2n-2} (2n-1-i)F_i^2 + (n-1)(F_{n-1} + rF_{n-2})^2 + F_{n-1}^2.$$

Hence, when  $|r| \geq 1$ , we obtain

$$\|A\|_E^2 \geq \sum_{i=1}^{n-2} iF_i^2 + \sum_{i=n}^{2n-2} (2n-1-i)F_i^2 + (n-2)F_n^2 + F_{n-1}^2,$$

that is

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2 + 1}{n}}, & \text{if } n \text{ is odd,} \\ \sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2}{n}}, & \text{if } n \text{ is even.} \end{cases}$$

On the other hand, let the matrices  $B$  and  $C$  as

$$B = \begin{pmatrix} F_0 & F_2 & F_3 & \dots & F_{n-1} + rF_{n-2} \\ F_1 & F_3 & F_4 & \dots & F_n \\ F_2 & F_4 & F_5 & \dots & F_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{n-2} & F_{n-1} + rF_{n-2} & F_n & \dots & F_{2n-3} \\ F_{n-1} & F_n & F_{n+1} & \dots & F_{2n-2} \end{pmatrix}_{n \times n} \quad \text{and} \quad C = \begin{pmatrix} r & r & r & \dots & 1 \\ r & r & r & \dots & 1 \\ r & r & r & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{n \times n}$$

such that  $A = B \circ C$ . Then

$$r_1(B) = \max_i \sqrt{\sum_j |a_{ij}|^2} = \sqrt{\sum_{j=n}^{2n-2} F_j^2 + F_{n-1}^2} = \sqrt{F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2}}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |a_{ij}|^2} = \sqrt{(n-1)|r|^2 + 1}.$$

We have

$$\|A\|_2 \leq |r| \sqrt{n(F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2})}.$$

When  $|r| < 1$ , we also obtain

$$\begin{cases} |r|\sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2 + 1}{n}} \\ |r|\sqrt{\frac{(2n-1)(-F_{2n-1}^2 - F_n F_{n-1}) + (n-1)F_n F_{n-1} + nF_n F_{n+1} + (n-2)F_n^2 + F_{2n-1}^2}{n}} \end{cases} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \begin{cases} \sqrt{n(F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2})}, & \text{if } n \text{ is odd,} \\ \sqrt{n(F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2})}, & \text{if } n \text{ is even.} \end{cases}$$

■

**Corollary 3** Let  $A$  be a column-first-plus- $r$ -last- $r$ -right Hankel matrix with the first row  $(F_0, F_1, F_2, \dots, F_{n-1})$ .

(i) If  $|r| \geq 1$ , then

$$\|A\|_1 = |r|(F_n - 1) + F_{n+1} + |r|F_{n-2} \quad \text{and} \quad \|A\|_\infty = |r|(F_n - 2) + F_{n+1} + |r|F_{n-2}.$$

(ii) If  $|r| < 1$ , then

$$\|A\|_1 = F_{2n-1} - F_n \quad \text{and} \quad \|A\|_\infty = F_{2n-1} - F_n + |r|F_{n-2}.$$

**Corollary 4** Let  $A$  be a column-first-plus- $r$ -last- $r$ -right Hankel matrix with the first row  $(F_0, F_1, F_2, \dots, F_{n-1})$ . We have

$$\|A\|_2 \geq \frac{|r \sum_{i=1}^{n-2} iF_i + \sum_{i=n}^{2n-2} (2n-1-i)F_i + (n-1)(F_{n-1} + rF_{n-2}) + F_{n-1}^2|}{n}$$

and

$$\|A\|_2 \geq \frac{|r(2 + (n-1)F_{n+1} - F_{n+2}) + (2n-1)(F_{2n-1} - F_{n+1}) + (2-2n)F_{2n}|}{n} + \frac{|F_{2n+1} + nF_{n+2} - F_{n+3} + (n-1)(F_{n-1} + rF_{n-2}) + F_{n-1}^2|}{n}.$$

## 5 Conclusion

This study presents new variants of Hankel matrices that have not been defined in the literature. In this paper, we firstly define the left Hankel matrix and its varieties with Fibonacci numbers as entries. Then we introduce the Euclidean norm equality and the spectral norm inequalities of these matrices. We also obtain some bounds for the spectral norms of the CLP $r$ L $r$ R Hankel matrices.

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