Applied Mathematics E-Notes, 26(2026), 159-162 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/ ISSN 1607-2510

Infinitely Many Smooth Solutions To Steady MHD Equations*

Jinghan Ma†

Received 25 March 2025

Abstract

Infinitely many smooth and compactly supported solutions to the steady ideal MHD equations have been constructed from some solutions of the steady incompressible Euler equations.

1 Introduction

This short note is devoted to constructing smooth and compactly supported solutions to the steady ideal magneto-hydrodynamic (MHD) equations including incompressible and compressible cases. The steady incompressible MHD equations read

$$\begin{cases} u\cdot\nabla u - B\cdot\nabla B + \nabla p = 0, \\ u\cdot\nabla B - B\cdot\nabla u = 0, \\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \qquad \text{where the space variable } x\in\mathbb{R}^d,\ d=2,3; \tag{1}$$

and the equations we consider in the compressible case are as follows

$$\begin{cases} \rho u\cdot\nabla u - (\nabla\times B)\times B + \nabla p(\rho) = 0, \\ \nabla\times(u\times B) = 0, \\ \nabla\cdot(\rho u) = \nabla\cdot B = 0, \end{cases} \qquad \text{where the space variable } x\in\mathbb{R}^d, d=2,3. \tag{2}$$

Ideal MHD equations model the dynamics of electrically conducting fluids such as liquid metals [9] or plasma [13] under the influence of magnetic fields.

For unsteady ideal MHD equations in incompressible case, infinitely many bounded weak solutions are constructed in [7] via Baire Category methods. Those solutions exhibit compact support in both time and space while do not conserve helicity and energy. Nonuniquness of H^β weak solutions with small β is also obtained in [1] through Nash's scheme or convex integration technique. Main approaches for the above two results were already successfully applied to the incompressible Euler equations decades before ([3, 4]). Infinitely many weak solutions of lower integrability than L^3 and violating the magnetic helicity conservation are constructed in [6]. Recently, nonuniqueness of Hölder continuous weak solutions that fail to conserve Elsässer energies and cross helicity is also established in [11]. For the classical MHD equations, infinitely many $W^{s,p}$ weak solutions with vanishing magnetic helicity are also built in [10, 12].

As for steady MHD equations, to our knowledge, most of the known results are about the Liouville theorem for these equations with viscosity. Without viscosity, ideal MHD equations may admit nonuniquenness. It seems that this note is the first rigorous proof of nonuniqueness of compactly supported smooth solutions to (1) and (2).

Theorem 1 *There exist infinitely many nontrivial smooth solutions to the steady ideal incompressible MHD equations (1) and isentropic compressible MHD equations (2) such that (u, B) are compactly supported.*

*Mathematics Subject Classifications: 35Q30, 35Q35, 35Q92.

†Vision Academy Unit 3006, Jian Wai Soho East B, ChaoYang District, Beijing, 100022, P.R. China

Our approach to prove Theorem 1 is constructing solutions of the form $\varphi(P)U$, borrowing the main idea of [8, 2, 5], where φ is a smooth compactly supported function and (U,P) are smooth solutions built in [8, 2] to the steady incompressible Euler equations, i.e. (U,P) satisfy

$$\begin{cases} U\cdot\nabla U+\nabla P=0,\\ \nabla\cdot U=0. \end{cases} \tag{3}$$

Meanwhile, we also utilize orthogonality property of these solutions

$$U\cdot\nabla P=0 \tag{4}$$

and also norm relation

$$|U|^2=c_0P,\ \text{where constant } c_0=\tfrac{3}{2} \text{ if } d=2;\ c_0=3,\ \text{if } d=3. \tag{5}$$

2 Proof of Theorem 1

In this section, we will prove the main result. We divide it into two subsections.

2.1 Solutions to (1)

As discussed in the introduction, we set

$$u(x)=\phi(P(x))U(x),\ B(x)=\psi(P(x))U(x) \tag{6}$$

with compactly supported functions $\phi(s)$ and $\psi(s)$ to be found. Then by (3), (4) and a direct calculation we get

$$\begin{aligned} &\nabla\cdot u=\phi(P)\nabla\cdot U+\phi'(P)U\cdot\nabla P=0,\\ &\nabla\cdot B=\psi(P)\nabla\cdot U+\psi'(P)U\cdot\nabla P=0,\\ &u\cdot\nabla B-B\cdot\nabla u=\phi(P)\psi(P)U\cdot\nabla U-\psi(P)\phi(P)U\cdot\nabla U=0, \end{aligned}$$

for any smooth functions $\phi(P)$ and $\psi(P)$. As for the first equation in (1), again by (3) and (4), we derive

$$\phi^2(P)U\cdot\nabla U-\psi^2(P)U\cdot\nabla U+\nabla p=-(\phi^2(P)-\psi^2(P))\nabla P+\nabla p, \tag{7}$$

hence taking

$$p=\int_{-\infty}^{P}(\phi^2(s)-\psi^2(s))ds \tag{8}$$

leads to

$$\nabla p=(\phi^2(P)-\psi^2(P))\nabla P,$$

then the first equation in (1) is valid. Therefore, we get smooth solutions (u,B,p) with velocity u and magnetic filed B defined in (6) for any smooth and compactly supported functions $\phi(s)$ and $\psi(s)$ and pressure p defined in (8).

2.2 Solutions to (2)

Note that

$$(\nabla\times B)\times B=B\cdot\nabla B-\tfrac{1}{2}\nabla|B|^2,$$

$$\begin{aligned} \nabla\times(u\times B) &= (B\cdot\nabla)u-(u\cdot\nabla)B+u\nabla\cdot B-B\nabla\cdot u\\ &= (B\cdot\nabla)u-(u\cdot\nabla)B-B\nabla\cdot u. \end{aligned}$$

The equations (2) are reduced to be the following system

$$\begin{cases} \rho u \cdot \nabla u - B \cdot \nabla B + \frac{1}{2}\nabla |B|^2 + \nabla p(\rho) = 0, \\ (B\cdot\nabla)u - (u\cdot\nabla)B - B\nabla\cdot u = 0, \\ \nabla\cdot(\rho u) = \nabla\cdot B = 0. \end{cases} \tag{9}$$

Similar to the incompressible case, define

$$u(x) = \alpha(P(x))U(x),\ B(x) = \beta(P(x))U(x),\ \rho(x) = r(P(x)) \tag{10}$$

with compactly supported functions $\alpha(s)$, $\beta(s)$ and $r(P)$ to be fixed. First of all, by (3) and (4)

$$\nabla\cdot(\rho u) = \nabla\cdot(r(P)\alpha(P)U) = r(P)\alpha(P)\nabla\cdot U + \frac{\mathrm{d}}{\mathrm{d}P}(r(P)\alpha(P))U\cdot\nabla P = 0,$$

for all smooth functions $r(P)$ and $\alpha(P)$. Similarly, we get $\nabla\cdot B = 0$ for all smooth function $\beta(P)$. As for the second line of (9), again from (3) and (4), we can derive

$$\begin{aligned} & (B\cdot\nabla)u - (u\cdot\nabla) - B\nabla\cdot u \\ = \ & \alpha(P)\beta(P)U\cdot\nabla U - \alpha(P)\beta(P)U\cdot\nabla U - \beta(P)U\nabla\cdot(\alpha(P)U) \\ = \ & 0. \end{aligned}$$

For the first line of (9), from (3), (4) and (5), we infer

$$\begin{aligned} & r(P)(\alpha(P)U)\cdot\nabla(\alpha(P)U) - \beta(P)U\cdot\nabla(\beta(P)U) + \frac{1}{2}\nabla(\beta^2(P)|U|^2) + \nabla p(r(P)) \\ = \ & r(P)\alpha^2(P)U\cdot\nabla U - \beta^2(P)U\cdot\nabla U + \frac{c_0}{2}\nabla(\beta^2(P)P) + \nabla p(r(P)) \\ = \ & \left(-r(P)\alpha^2(P) + \beta^2(P) + \frac{\mathrm{d}}{\mathrm{d}P}(\frac{c_0}{2}\beta^2(P)P) + p'(r(P))r'(P)\right)\nabla P. \end{aligned}$$

To make it vanish, we require

$$-r(P)\alpha^2(P) + \beta^2(P) + \frac{\mathrm{d}}{\mathrm{d}P}(\frac{c_0}{2}\beta^2(P)P) + p'(r(P))r'(P) = 0.$$

Thus we choose $\beta(P)$ and $r(P)$ as increasing and compactly supported functions with support satisfying

$$\mathrm{supp}\ \beta(P) = \mathrm{supp}\ r(P) \supset \mathrm{supp}\ r'(P). \tag{11}$$

Then we are able to define $\alpha(P)$ such that

$$\begin{aligned} \alpha^2(P) &= \frac{1}{r(P)}[\beta^2(P) + \frac{\mathrm{d}}{\mathrm{d}P}(\frac{c_0}{2}\beta^2(P)P) + p'(r(P))r'(P)] \\ &= \frac{(1+\frac{c_0}{2})\beta^2(P) + c_0P\beta(P)\beta'(P)}{r(P)} + \frac{p'(r(P))}{r(P)}r'(P), \end{aligned} \tag{12}$$

which will satisfy our requirement. Finally, (u, B, ρ) defined in (10) are smooth solutions to (2) where $\beta(P)$ and $r(P)$ are increasing and compactly supported functions with support satisfying (11), and $\alpha(P)$ is defined in (12).

Acknowledgment. The author would like to thank Prof. Wentao Cao from Capital Normal University for his valuable discussion and suggestion. Moreover, she would also like to thank the referees for their constructive comments and suggestions towards improvement of this article.

References

[1] R. Beekie, T. Buckmaster and V. Vicol, Weak solutions of ideal MHD which do not conserve magnetic helicity, Ann. PDE, 6(2020), 40 pp.

[2] P. Constantin, J. La and V. Vicol, Remarks on a paper by Gavrilov: grad-shafranov equations, steady solutions of the three dimensional incompressible euler equations with compactly supported velocities, and applications, Geom. Funct. Anal., 29(2019), 1773–1793.

[3] C. De Lellis and L. Székelyhidi Jr., The Euler equations as a differential inclusion, Ann. of Math., 170(2009), 1417–1436.

[4] C. De Lellis and L. Székelyhidi Jr., Dissipative continuous Euler flows, Invent. Math., 193(2013), 377–407.

[5] F. Fanelli and E. Feireisl, Some remarks on steady solutions to the Euler system in $\mathbb{R}^d$, Appl. Math. Lett., 116(2021), 107031.

[6] D. Faraco, L. Sauli and L. Székelyhidi Jr., Magnetic helicity, weak solutions and relaxation of ideal MHD, Commun. Pure Appl. Math., 77(2024), 2387–2412.

[7] D. Faraco, S. Lindberg and L. L. Székelyhidi Jr., Bounded solutions of ideal MHD with compact support in space-time, Arch. Ration. Mech. Anal., 239(2021), 51–93.

[8] A. V. Gavrilov, A steady Euler flow with compact support, Geom. Funct. Anal., 29(2019), 190–197.

[9] J. -F. Gerbeau, C. Le Bris and T. Lelièvre, Mathematical Methods for the Magnetohydrodynamics of Liquid Metals, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, (2006).

[10] Y. Li, Z. Zeng and D. Zhang, Non-uniqueness of weak solutions to 3D magnetohydrodynamic equations, J. Math. Pures Appl., 165(2022), 232–285.

[11] C. Miao, Y. Nie and W. Ye, On onsager-type conjecture for the Elsässer energies of the ideal MHD equations, arXiv:2504.06071, (2025).

[12] C. Miao and W. Ye, On the weak solutions for the MHD systems with controllable total energy and cross helicity, J. Math. Pures Appl., 181(2024), 190–227.

[13] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, Comm. Pure Appl. Math., 36(1983), 635–664.