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New Fixed Point Results On Orthogonal $\mathcal{G}$-Metric Spaces Using Matkowski Type Contraction With Application To Nonlinear Integral Equation*

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Abstract

In the present work, using the Matkowski type contraction, we discuss the existence of fixed point results within the framework of orthogonal $\mathcal{G}$-metric spaces. Our results extend and generalize several well-known findings that are already present in the literature. Furthermore, as an application, we investigate solution to a nonlinear integral equation.

1 Introduction

The study of metric fixed point theory has been researched extensively in the past two decades or so because fixed point theory plays a key role in mathematics and applied sciences. For example, in the areas such as optimization, mathematical models, and economic theories.

The literature of the last decades is rich of papers that focus on all matters related to the generalized metric spaces. In 2005, Mustafa and Sims introduced a new class of generalized metric spaces called $\mathcal{G}$-metric spaces (see [6]) as a generalization of metric spaces $(\mathcal{X}, d)$. This was done to introduce and develop a new fixed point theory for a variety of mappings in this new setting. This helped to extend some known metric space results to this more general setting. Thereafter, the concept of $\mathcal{G}$-metric space has been studied and used to obtain various fixed point theorems by several mathematicians (see [2, 5, 8, 9, 11]). Here, we present the necessary definitions and results in $\mathcal{G}$-metric spaces. However, for more details, we refer to [6].

Definition 1 *Let $\mathcal{X}$ be a non-empty set, $\mathcal{G} : \mathcal{X}^3 \to \mathbb{R}_+$ be a function satisfying the following properties:*

(G1) $\mathcal{G}(x, y, z) = 0$ *if* $x = y = z$,

(G2) $0 < \mathcal{G}(x, x, y)$ *for all* $x, y \in \mathcal{X}$ *with* $x \neq y$,

(G3) $\mathcal{G}(x, x, y) \leq \mathcal{G}(x, y, z)$ *for all* $x, y, z \in \mathcal{X}$ *with* $y \neq z$,

(G4) $\mathcal{G}(x, y, z) = \mathcal{G}(x, z, y) = \mathcal{G}(y, z, x) = \cdots$,

(G5) $\mathcal{G}(x, y, z) \leq \mathcal{G}(x, a, a) + \mathcal{G}(a, y, z)$ *for all* $x, y, z, a \in \mathcal{X}$.

Then the function $\mathcal{G}$ is called a generalized metric, or, more specially, a $\mathcal{G}$-metric on $\mathcal{X}$, and the pair $(\mathcal{X}, \mathcal{G})$ is called a $\mathcal{G}$-metric space.

Definition 2 *Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space, and let (x_n) be a sequence of points of $\mathcal{X}$. We say that (x_n) is $\mathcal{G}$-convergent to $x \in X$ if $\lim_{n,m\to\infty} \mathcal{G}(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathcal{G}(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.*

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Proposition 1 *Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space. The following are equivalent:*

(1) (x_n) is $\mathcal{G}$-convergent to x.

(2) $\mathcal{G}(x_n, x_n, x) \to 0$ as $n \to +\infty$.

(3) $\mathcal{G}(x_n, x, x) \to 0$ as $n \to +\infty$.

(4) $\mathcal{G}(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

Definition 3 *Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A sequence (x_n) is called a $\mathcal{G}$-Cauchy sequence if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathcal{G}(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \geq N$, that is, $\mathcal{G}(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.*

Proposition 2 *Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space. Then the following are equivalent:*

(1) The sequence (x_n) is $\mathcal{G}$-Cauchy.

(2) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathcal{G}(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq N$.

Proposition 3 *Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A mapping $f : \mathcal{X} \to \mathcal{X}$ is $\mathcal{G}$-continuous at $x \in \mathcal{X}$ if and only if it is $\mathcal{G}$-sequentially continuous at x, that is, whenever (x_n) is $\mathcal{G}$-convergent to x, $(f(x_n))$ is $\mathcal{G}$-convergent to $f(x)$.*

Proposition 4 *Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space. Then, the function $\mathcal{G}(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 5 *Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space, then for any $x, y, z, a \in \mathcal{X}$,*

(1) if $\mathcal{G}(x, y, z) = 0$ then $x = y = z$;

(2) $\mathcal{G}(x, y, z) \leq \mathcal{G}(x, x, y) + \mathcal{G}(x, x, z)$;

(3) $\mathcal{G}(x, y, y) \leq 2\mathcal{G}(y, x, x)$; and

(4) $\mathcal{G}(x, y, z) \leq \mathcal{G}(x, a, z) + \mathcal{G}(a, y, z)$.

Definition 4 *A $\mathcal{G}$-metric space $(\mathcal{X}, \mathcal{G})$ is called $\mathcal{G}$-complete if every $\mathcal{G}$-Cauchy sequence is $\mathcal{G}$-convergent in $(\mathcal{X}, \mathcal{G})$.*

In 2017, Eshaghi Gordji et al. [3] defined orthogonal metric spaces as a generalization of metric spaces, as follows:

Definition 5 ([3]) *Let $\perp \subseteq \mathcal{X} \times \mathcal{X}$ be a binary relation defined on a nonempty set $\mathcal{X}$. If the relation $\perp$ satisfies the following condition: there exists $x_0 \in \mathcal{X}$ such that*

$$[\forall y, y \perp x_0] \text{ or } [\forall y, x_0 \perp y],$$

then $\mathcal{X}$ is called an orthogonal set (briey, O-set) and x_0 is called an orthogonal element. We denote this O-set by $(\mathcal{X}, \perp)$.

Definition 6 ([3]) *Let $(\mathcal{X}, \perp)$ be O-set. A sequence $\{x_i\}$ is called an orthogonal sequence if*

$$[\forall i, x_i \perp x_{i+1}] \text{ or } [\forall i, x_{i+1} \perp x_i].$$

Definition 7 ([3]) *Let $(\mathcal{X}, \perp)$ be an O-set. A mapping $\mathcal{T} : X \times X \to X$ is said to be $\perp$-preserving if $x \perp y$ implies $\mathcal{T}x \perp \mathcal{T}y$.*

The following are non-trivial examples of an orthogonal set.

Example 1 *Let $\mathcal{X} = 2\mathbb{Z}$ and set a binary relation $\perp$ on $2\mathbb{Z}$ as $m \perp n$ if $m.n = 0$. Then $(2\mathbb{Z}, \perp)$ is an orthogonal set with 0 as an orthogonal element.*

Example 2 *Let $\mathcal{X}$ be a non-empty set and we consider the power set $\mathcal{P}(\mathcal{X})$. We define $\perp$ on $\mathcal{P}(\mathcal{X})$ as $\mathcal{A} \perp \mathcal{B}$ if $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then $(\mathcal{P}(\mathcal{X}), \perp)$ is an orthogonal set, as for all $\mathcal{A} \in \mathcal{P}(\mathcal{X})$, $\emptyset \cap \mathcal{A} = \emptyset$. Similarly, one can define $\perp$ on $P(\mathcal{X})$ as $\mathcal{A} \perp \mathcal{B}$ if $\mathcal{A} \cup \mathcal{B} = \mathcal{X}$. Then $(\mathcal{P}(\mathcal{X}), \perp)$ is also an orthogonal set.*

Example 3 *Let $\mathcal{X}$ be set of all matrices of order n over $\mathbb{R}$, i.e. $\mathcal{X} = \mathcal{M}_n(\mathbb{R})$. We define $\perp$ on $\mathcal{X}$ as $\mathcal{A} \perp \mathcal{B}$ if $\mathcal{AB} = \mathcal{BA}$. Then $(\mathcal{M}_n(\mathbb{R}), \perp)$ is an orthogonal set since $\mathcal{CA} = \mathcal{AC}$ for a scalar matrix $\mathcal{C} \in \mathcal{M}_n(\mathbb{R})$.*

For more details about fixed point results on orthogonal metric spaces, the readers are referred to [1, 7, 10]. Combining these two ideas leads to the concept of an orthogonal $\mathcal{G}$-metric space, which is the primary setting for the results presented here. On the other hand, the contraction condition is also often generalized. Instead of requiring a single constant less than one to bound the contraction, Matkowski-type contractions employ a more general function that is contractive but not necessarily linear. This approach can lead to fixed point theorems that are applicable to broader classes of mappings (see [4]). The aim of this paper is to combine orthogonal $\mathcal{G}$-metric spaces with Matkowski-type contractions to obtain new fixed point results. We then show how these results can be applied to establish the existence of solutions to nonlinear integral equations.

2 Preliminaries

We begin by defining the key concepts used throughout the paper.

Definition 8 (Orthogonal G-Metric Space) *Let X be a nonempty set equipped with an orthogonal relation $\perp$. A mapping $\mathcal{G} : X^3 \to [0, \infty)$ is called an orthogonal $\mathcal{G}$-metric if it satisfies the following conditions for all $x, y, z, a \in X$:*

$(OG1)$ *$\mathcal{G}(x, y, z) = 0$ if $x = y = z$;*

$(OG2)$ *$\mathcal{G}(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;*

$(OG3)$ *$\mathcal{G}(x, x, y) = \mathcal{G}(x, y, x) = \mathcal{G}(y, x, x)$;*

$(OG4)$ *$\mathcal{G}(x, y, z) = \mathcal{G}(p(x, y, z))$ where p is a permutation of x, y, z;*

$(OG5)$ *$\mathcal{G}(x, y, z) \leq \mathcal{G}(x, a, a) + \mathcal{G}(a, y, z)$ for all $x, y, z, a \in X$;*

$(OG6)$ *For $x \perp y$, $\mathcal{G}(x, y, y) = \mathcal{G}(y, x, x)$, $\mathcal{G}(x, y, y) = \mathcal{G}(y, z, z)$ and $y \perp z$ implies $\mathcal{G}(x, y, z) = \mathcal{G}(y, x, z)$.*

The triple $(X, \mathcal{G}, \perp)$ is called an orthogonal $\mathcal{G}$-metric space.

Definition 9 (Orthogonal G-Convergence) *A sequence $\{x_n\}$ in an orthogonal $\mathcal{G}$-metric space $(X, \mathcal{G}, \perp)$ is said to be orthogonally $\mathcal{G}$-convergent to a point $x \in X$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathcal{G}(x_n, x, x) < \epsilon$ for all $n \geq \mathbb{N}$.*

Definition 10 (Orthogonal G-Cauchy Sequence) *A sequence $\{x_n\}$ in an orthogonal $\mathcal{G}$-metric space $(X, \mathcal{G}, \perp)$ is said to be an orthogonal $\mathcal{G}$-Cauchy sequence if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathcal{G}(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq \mathbb{N}$.*

Definition 11 (Orthogonal G-Complete) *An orthogonal $\mathcal{G}$-metric space $(X, G, \perp)$ is said to be orthogonally $\mathcal{G}$-complete if every orthogonal $\mathcal{G}$-Cauchy sequence in X orthogonally $\mathcal{G}$-converges to some point in X.*

Definition 12 (Matkowski Functione) *A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a Matkowski function if:*

(i) φ is non-decreasing;

(ii) φ is continuous; and

(iii) $\varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$.

3 Main Results

Now, we are ready to present our fixed point theorem for orthogonal $\mathcal{G}$-metric spaces using Matkowski-type contractions.

Theorem 1 *Let $(\mathcal{X}, \mathcal{G}, \perp)$ be an orthogonally $\mathcal{G}$-complete space, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-mapping. Assume that $\mathcal{T}$ satisfies the following conditions:*

(i) T is an orthogonal preserving mapping.

(ii) There exists a Matkowski function φ such that for all $x, y \in \mathcal{X}$ with $x \perp y$,

$$\mathcal{G}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}y) \leq \varphi(M);$$

where

$$\max\left\{\mathcal{G}(x,y,y), \mathcal{G}(x,\mathcal{T}x,\mathcal{T}x), \mathcal{G}(y,\mathcal{T}y,\mathcal{T}y), \frac{\mathcal{G}(x,\mathcal{T}y,\mathcal{T}y)+\mathcal{G}(y,\mathcal{T}x,\mathcal{T}x)}{2}\right\} = M.$$

(iii) There exists an $x_0 \in \mathcal{X}$ such that $x_0 \perp \mathcal{T}(x_0)$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$ be such that $x_0 \perp \mathcal{T}(x_0)$. Define a sequence $\{x_n\}$ by $x_{n+1} = \mathcal{T}(x_n)$ for $n \geq 0$. Since $\mathcal{T}$ is orthogonal preserving, we have $x_n \perp x_{n+1}$ for all $n \geq 0$. Let $d_n = \mathcal{G}(x_n, x_{n+1}, x_{n+1})$ for $n \geq 0$. If $d_n = 0$ for some n, then $x_n = x_{n+1}$, which implies x_n is a fixed point of $\mathcal{T}$. Assume $d_n > 0$ for all n. Then, applying Condition (ii) for $x_n \perp x_{n-1}$, we have

$$\begin{aligned}
d_n &= \mathcal{G}(x_n, x_{n+1}, x_{n+1}) \\
&= \mathcal{G}\Big(\mathcal{T}(x_{n-1}),\ \mathcal{T}(x_n),\ \mathcal{T}(x_n)\Big) \\
&\leq \varphi\Big(\max\Big\{\mathcal{G}(x_{n-1}, x_n, x_n),\ \mathcal{G}\Big(x_{n-1},\ \mathcal{T}(x_{n-1}),\ \mathcal{T}(x_{n-1})\Big),\ \mathcal{G}\Big(x_n,\ \mathcal{T}(x_n),\ \mathcal{T}(x_n)\Big), \\
&\qquad \frac{\mathcal{G}\Big(x_{n-1},\ \mathcal{T}(x_n),\ \mathcal{T}(x_n)\Big) + \mathcal{G}\Big(x_n,\ \mathcal{T}(x_{n-1}),\ \mathcal{T}(x_{n-1})\Big)}{2}\Big\}\Big) \\
&= \varphi\Big(\max\Big\{d_{n-1},\ d_{n-1},\ d_n,\ \frac{\mathcal{G}(x_{n-1}, x_{n+1}, x_{n+1}) + \mathcal{G}(x_n, x_n, x_n)}{2}\Big\}\Big) \\
&\leq \varphi\Big(\max\Big\{d_{n-1},\ d_n,\ \frac{\mathcal{G}(x_{n-1}, x_n, x_n) + \mathcal{G}(x_n, x_{n+1}, x_{n+1}) + 0}{2}\Big\}\Big) \\
&\leq \varphi\Big(\max\Big\{d_{n-1},\ d_n,\ \frac{d_{n-1} + d_n}{2}\Big\}\Big) \\
&\leq \varphi(\max\{d_{n-1},\ d_n\}).
\end{aligned}$$

Now, if $\max\{d_{n-1}, d_n\} = d_n$, then we have $d_n \leq \varphi(d_n)$ and since $\varphi(t) < t$ for $t > 0$ then $\varphi(d_n) < d_n$, leading to a contradiction. Thus $\max\{d_{n-1}, d_n\} = d_{n-1}$. This yields $d_n \leq \varphi(d_{n-1})$. Since φ is non-decreasing and $\varphi(t) < t$ for $t > 0$, the sequence $\{d_n\}$ is a decreasing sequence of non-negative real numbers. Therefore, $\{d_n\}$

converges to some limit L such that $L \leq \varphi(L)$. This shows that $L = 0$. Thus, $\lim_{n\to\infty} \mathcal{G}(x_n, x_{n+1}, x_{n+1}) = 0$. Also, we have

$$\mathcal{G}(x_n, x_m, x_m) \leq \mathcal{G}(x_n, x_{n+1}, x_{n+1}) + \mathcal{G}(x_{n+1}, x_{n+2}, x_{n+2}) + ... + \mathcal{G}(x_{m-1}, x_m, x_m).$$

For any $\epsilon > 0$, choose N large enough such that $\mathcal{G}(x_n, x_{n+1}, x_{n+1}) < \epsilon/2$ for all $n > N$. Therefore, for $m, n > N$, $\mathcal{G}(x_n, x_m, x_m)\Sigma_{i=1}^{nm-1}\mathcal{G}(x_i, x_{i+1}, x_{i+1}) < \epsilon$. This implies $\{xn\}$ is an orthogonal $\mathcal{G}$-Cauchy sequence, and since $\mathcal{X}$ is orthogonally $\mathcal{G}$-complete, there exists $x^* \in \mathcal{X}$ such that $x_n \to x^*$. Now let us prove that x^* is the fixed point of $\mathcal{T}$. From condition (ii), we have

$$\begin{aligned} \mathcal{G}\Big(\mathcal{T}x_n, \mathcal{T}x^*, \mathcal{T}x^*\Big) \quad \leq \quad & \varphi\Big(\max\Big\{\mathcal{G}(x_n, x^*, x^*), \mathcal{G}(x_n, \mathcal{T}x_n, \mathcal{T}x_n), \mathcal{G}(x^*, \mathcal{T}x^*, \mathcal{T}x^*), \\ & \frac{\mathcal{G}(x_n, \mathcal{T}x^*, \mathcal{T}x^*) + \mathcal{G}(x^*, \mathcal{T}x_n, \mathcal{T}x_n)}{2}\Big\}\Big). \end{aligned}$$

Since $\mathcal{G}(\mathcal{T}x_n, \mathcal{T}x^*, \mathcal{T}x^*) = \mathcal{G}(x_{n+1}, \mathcal{T}x^*, \mathcal{T}x^*)$, taking limit as $n \to \infty$, we get

$$\mathcal{G}(x^*, \mathcal{T}x^*, \mathcal{T}x^*) \leq \varphi\Big(\mathcal{G}(x^*, \mathcal{T}x^*, \mathcal{T}x^*)\Big),$$

which implies $\mathcal{G}(x^*, \mathcal{T}x^*, \mathcal{T}x^*) = 0$. Hence $x^* = \mathcal{T}x^*$. This completes the proof. ■

Theorem 2 (Uniqueness) *In addition to the hypotheses of the Theorem 1, suppose that for any two fixed points x and y of $\mathcal{T}$, it holds that $x \perp y$. Then the fixed point of $\perp$ is unique.*

Proof. Suppose x and y are two fixed points such that $x = \mathcal{T}x$ and $y = \mathcal{T}y$. By assumption, $x \perp y$. Using condition (ii) of the Theorem 1,

$$\begin{aligned} \mathcal{G}(x, y, y) &= \mathcal{G}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}y) \\ &\leq \varphi\Big(\max\{\mathcal{G}(x, y, y), \mathcal{G}(x, \mathcal{T}x, \mathcal{T}x), \mathcal{G}(y, \mathcal{T}y, \mathcal{T}y), \frac{\mathcal{G}(x, \mathcal{T}y, \mathcal{T}y) + \mathcal{G}(y, \mathcal{T}x, \mathcal{T}x)}{2}\}\Big) \\ &= \varphi\Big(\max\{\mathcal{G}(x, y, y), 0, 0, \mathcal{G}(x, y, y)\}\Big) \\ &= \varphi(\mathcal{G}(x, y, y)). \end{aligned}$$

It follows that $\mathcal{G}(x, y, y) = 0$. Hence $x = y$ which shows the uniqueness of the fixed point. ■

4 Application to Nonlinear Integral Equations

In this section we employ our main result in nonlinear integral equation.

Theorem 3 *Consider the nonlinear integral equation:*

$$x(t) = h(t) + \int_a^b k(t, s, x(s))ds, \text{ for all } t \in [a, b],$$

where, $[a, b]$ is a closed interval in $\mathbb{R}$, $h : [a, b] \to \mathbb{R}$, is continuous and $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Assume the following conditions:

(i) There exists a Matkowski function φ such that

$$|k(t, s, x) - k(t, s, y)| \leq \varphi(|x - y|)/(b - a),$$

for all $t, s \in [a, b]$ and $x, y \in \mathbb{R}$.

(ii) There exists a function $x_0 \in C([a,b])$ where $C([a,b])$ is the space of all continuous functions on $[a,b]$, such that

$$x_0(t) \leq h(t) + \int_a^b k(t,s,x_0(s))ds.$$

Then, the nonlinear integral equation has a unique solution in $C([a,b])$.

Proof. Let $\mathcal{X} = C([a,b])$. Define a $\mathcal{G}$-metric on $\mathcal{X}$ by

$$\mathcal{G}(x,y,z) = \sup_{t\in[a,b]} (|x(t)-y(t)| + |y(t)-z(t)| + |z(t)-x(t)|).$$

It is known that $(\mathcal{X},\mathcal{G})$ is a complete $\mathcal{G}$-metric space. We define $x \perp y$ in $\mathcal{X}$ if $x(t)y(t)$ for all $t \in [a,b]$. Define an operator $\mathcal{T} : X \to X$ by

$$T(x)(t) = h(t) + \int_a^b k(t,s,x(s))ds.$$

Let $x, y \in \mathcal{X}$ be such that $x \perp y$, which means $x(t) \leq y(t)$ for all $t \in [a,b]$. Then,

$$\begin{aligned} |\mathcal{T}(x)(t) - \mathcal{T}(y)(t)| &= |\int_a^b k(t,s,x(s))ds - \int_a^b k(t,s,y(s))ds| \\ &\leq \int_a^b |k(t,s,x(s)) - k(t,s,y(s))|ds \\ &\leq \int_a^b \Big(\varphi(|x(s)-y(s)|)/(b-a)\Big)ds \\ &\leq \varphi\Big(\sup_{s\in[a,b]} (|x(s)-y(s)|)\Big). \end{aligned}$$

This shows,

$$|\mathcal{T}(x)(t) - \mathcal{T}(y)(t)| \leq \varphi\Big(\sup_{s\in[a,b]} (|x(s)-y(s)|)\Big).$$

Since we have

$$|x(t)-y(t)| \leq |x(t)-y(t)| + |y(t)-z(t)| + |z(t)-x(t)|$$

and similarly for other terms, we have

$$\begin{aligned} & \mathcal{G}(\mathcal{T}x,\mathcal{T}y,\mathcal{T}y) \\ \leq\ & \varphi\Big(\sup_{s\in[a,b]} (|x(s)-y(s)|)\Big) + \varphi\Big(\sup_{s\in[a,b]} (|x(s)-y(s)|)\Big) + \varphi\Big(\sup_{s\in[a,b]} (|x(s)-y(s)|)\Big) \\ =\ & 3\varphi\Big(\sup_{s\in[a,b]} (|x(s)-y(s)|)\Big) \\ \leq\ & 3\varphi\Big(\sup_{s\in[a,b]} (|x(s)-y(s)| + |y(s)-y(s)| + |y(s)-x(s)|\Big) \\ =\ & 3\varphi(\mathcal{G}(x,y,y)) \\ \leq\ & \varphi\Big(\max\Big\{\mathcal{G}(x,y,y), \mathcal{G}(x,\mathcal{T}x,\mathcal{T}x), \mathcal{G}(y,\mathcal{T}y,\mathcal{T}y), \frac{\mathcal{G}(x,\mathcal{T}(y),\mathcal{T}y) + \mathcal{G}(y,\mathcal{T}x,\mathcal{T}x)}{2}\Big\}\Big). \end{aligned}$$

From the condition (ii), $x_0 \perp \mathcal{T}(x_0)$. We also know that $\mathcal{T}$ is orthogonal preserving since $x \perp y$ implies $\mathcal{T}(x)(t)\mathcal{T}(y)(t)$. Therefore all the conditions of Theorems 1 and 2 are satisfied. Hence, $\mathcal{T}$ has a unique fixed point $x \in C([a,b])$, which is the solution of the nonlinear integral equation. ■

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