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Exponential Stability Of The Nonlinear Korteweg-De Vries Equation With Internal Time-Delayed Feedback*

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Abstract

In this paper, we consider the Korteweg-de Vries equation with internal time-varying delay feedback. We prove exponential stability results using an appropriate Lyapunov functional, without imposing any assumptions on the length of the spatial domain. Finally, we present numerical simulations to illustrate the stability results obtained.

1 Introduction

The aim of this work is to investigate the stabilization of the Korteweg-de Vries equation with time-varying delay. This quasilinear equation is given by $v_t+v_x+v_{xxx}+vv_x=0$ and models the propagation of long water waves in a channel. This solitary wave phenomenon was observed for the first time in a canal by the engineer John Scott Russell in 1834. This model was obtained from Euler's equations by Boussinesq around 1877 and rediscovered in 1890 by Diederik Korteweg and Gustav de Vries, who provided a mathematical interpretation of the hydrodynamic soliton in [4]. This equation has aroused the curiosity of many mathematicians, who have invested themselves in the study of the controllability and stabilization properties of this nonlinear equation (see [7, 8, 2, 9]).

The problem of stabilization of the Korteweg-de Vries equation with boundary time-delay feedback was studied in [1, 5] and the asymptotic stability of the nonlinear KdV equation in the presence of a constant time-delay in the internal feedback was studied in [10]. Recently, the effect of a time-varying delay on the boundary and internal stabilization of the nonlinear Korteweg-de Vries equation was investigated in [6] under a technical assumption on the length L of the spatial domain that is $L<\pi\sqrt{3}$. This condition on L is also imposed in [1, 10]. The aim of this work is to remove this assumption and to prove the exponential stability of the system without any smallness conditions on the length L in the case of internal time-dependent delay.

In this work, we consider the following system

$$\begin{cases} v_t(x,t)+v_x(x,t)+v_{xxx}(x,t)+v(x,t)v_x(x,t) & \\ \qquad +a(x)v(x,t)+b(x)v(x,t-\sigma(t))=0, & t>0,\ x\in(0,L),\\ v(0,t)=v(L,t)=v_x(L,t)=0, & t>0,\\ v(x,0)=v_0(x), & x\in(0,L),\\ v(x,t-\sigma(0))=z_0(x,t-\sigma(0)), & 0<t<\sigma(0),\ x\in(0,L), \end{cases} \tag{1}$$

where $L>0$ is the length of the spatial domain and $v(x,t)$ is the amplitude of the water wave at position x at time t. We assume that the delay σ is a function of time t satisfying the following conditions

$$0<\sigma_0\le\sigma(t)\le M,\qquad \forall t\ge 0, \tag{2}$$

$$\dot\sigma(t)\le d<1,\qquad \forall t\ge 0, \tag{3}$$

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where $0 \leq d < 1$ and

$$\sigma \in W^{2,\infty}([0,T]), \qquad \forall T > 0. \tag{4}$$

In (1), a and b are nonnegative functions belonging to $L^\infty(0,L)$. We will also assume that supp $b = \omega$ and

$$b(x) \geq b_0 > 0 \quad \text{in} \quad \omega, \tag{5}$$

where ω is an open nonempty subset of $(0,L)$. The aim of the present work is to extend the results established in [6] to the case of internal time-varying delay feedback, for any length L, under a restrictive assumption on the weights of a and b. We assume that the functions a and b satisfy the following assumption:

$$\exists a_0 > 0, \quad \frac{2-d}{2-2d}\, b(x) + a_0 \leqslant a(x) \quad \text{in} \quad (0,L). \tag{6}$$

Then $\omega = supp\ b \subset supp\ a$ and $a(x) \geq b_0 + a_0 > 0$ in ω.

The outline of this paper is as follows. In Section 2, we recall the well-posedness results and show the decay of the energy associated with the nonlinear system. The exponential stability result is proved in Section 3. Finally, we illustrate our results with some numerical simulations in Section 4.

2 Well-Posedness Result and the Decay of the Energy

The aim of this section is to recall the main well-posedness result for the nonlinear system (1) already established in [6] and to show the decay of the energy associated with the system.

From (6), we can find a nonnegative function ξ in $L^\infty(0,L)$ such that supp $\xi = supp\ b = \omega$ and

$$\frac{1}{1-d}\, b(x) + a_0 \leqslant \xi(x) \leqslant 2a(x) - b(x) - a_0 \quad \text{in} \quad \omega. \tag{7}$$

The Hilbert space $H = L^2(0,L) \times L^2(\omega \times (0,1))$ is provided with the time-dependent inner product

$$\left\langle \begin{pmatrix} v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^L v\tilde{v}dx + \sigma(t) \int_\omega \int_0^1 \xi(x) z\tilde{z} d\rho dx.$$

Using (2) and (5), we can show that the norm $\|\cdot\|_t$ is equivalent to the usual norm $\|\cdot\|_H$ on H:

$$\forall t \geq 0,\ \forall (v,z) \in H,\ (1+\sigma_0 b_0)\|(v,z)\|_H^2 \leq \|(v,z)\|_t^2 \leq (1+2M\|a\|_\infty)\|(v,z)\|_H^2. \tag{8}$$

We set

$$B = C([0,T], L^2(0,L)) \cap L^2((0,T), H^1(0,L)),$$

endowed with the norm

$$\|v\|_B = \|v\|_{C([0,T],L^2(0,L))} + \|v\|_{L^2((0,T),H^1(0,L))}.$$

The following theorem provides the global well-posedness result for the nonlinear system (1).

Theorem 1 *Let $L > 0$ and assume that the conditions (2)–(6) hold. Then, for any initial data $(y_0, z_0(., -\sigma(0).)) \in H$, there exists a unique solution $v \in B$ to the system (1).*

Proof. The proof can be found in [6, Theorem 2.9]. ■

For a solution v of the nonlinear system (1), we consider the following definition of the energy associated with the system (1)

$$E(t) = \int_0^L v^2(x,t)dx + \sigma(t) \int_\omega \int_0^1 \xi(x) v^2(x, t-\sigma(t)\rho) d\rho dx, \tag{9}$$

where ξ is defined by (7). This energy is composed of two terms, the first corresponds to the natural energy of the KdV equation and the second is classical when considering an internal delayed term. The energy E defined by (9), is non-increasing and satisfies

$$\begin{aligned} \frac{d}{dt}E(t) \quad &\leq \quad -v_x^2(0,t) - \int_\omega (2a(x) - b(x) - \xi(x))v^2(x,t)dx \\ &\quad - \int_\omega ((1-d)\xi(x) - b(x))v^2(x,t-\sigma(t))dx \leq 0, \end{aligned} \tag{10}$$

since, from (7), we have $2a(x) - b(x) - \xi(x) > 0$ and $(1-d)\xi(x) - b(x) > 0$.

3 Exponential Stability Result

In this section, we study the exponential stability of (1) using an appropriate Lyapunov functional that allows us to prove the exponential stability of the system without imposing any condition on the length L of the domain. We consider the following Lyapunov functional

$$V(t) = E(t) + \eta_1 V_1(t) + \eta_2 V_2(t), \tag{11}$$

where η_1 and η_2 are positive real constants chosen sufficiently small, E is the energy defined by (9), and V_1 and V_2 are defined by

$$V_1(t) = \int_0^L e^{\alpha x} v^2(x,t)dx, \tag{12}$$

$$V_2(t) = \sigma(t) \int_\omega \int_0^1 (1-\rho)v^2(x,t-\sigma(t)\rho)d\rho dx, \tag{13}$$

where $\alpha > 0$ will be taken small enough to ensure the decrease of the energy. The term V_1 is not classical for the Korteweg-de Vries equation, it is used in the context of the Kawahara equation with constant delay (see [3]), while V_2 arises from the time-dependent delay term.

From the definition of $V(t)$ and $E(t)$, we have, for any $t > 0$,

$$E(t) \leq V(t) \leq \left(1 + \max\{e^{\alpha L}\eta_1, \frac{\eta_2}{b_0}\}\right) E(t). \tag{14}$$

Indeed, from (5) and (7), we have

$$\begin{aligned} E(t) \quad &\leq \quad V(t) \\ &= \quad E(t) + \eta_1 \int_0^L e^{\alpha x} v^2(x,t)dx + \eta_2\sigma(t) \int_\omega \int_0^1 (1-\rho)v^2(x,t-\sigma(t)\rho)d\rho dx \\ &\leq \quad E(t) + \eta_1 e^{\alpha L} \int_0^L v^2(x,t)dx + \eta_2\sigma(t) \int_\omega \int_0^1 \frac{\xi(x)}{b_0} v^2(x,t-\sigma(t)\rho)d\rho dx \\ &\leq \quad \left(1 + \max\{e^{\alpha L}\eta_1, \frac{\eta_2}{b_0}\}\right) E(t). \end{aligned}$$

Now, we prove in the following theorem, that the energy E of the nonlinear system (1) decays exponentially.

Theorem 2 *Assume that conditions (2)–(6) are satisfied and let $L > 0$. Then, there exists a sufficiently small constant $r > 0$, such that for every $(v_0, z_0) \in H$ satisfying $\|(v_0, z_0)\|_H \leq r$, the energy of the nonlinear system (1) decays exponentially. More precisely, there exist two positive constants γ and k such that*

$$E(t) \leq kE(0)e^{-2\gamma t}, \ \forall t > 0,$$

where, for η_1, η_2 and α small enough,

$$\gamma \leq \min\left\{\frac{(9\alpha - \sqrt{2\alpha}e^{\alpha L}Lr)\pi^2\eta_1}{6L^2(1+\eta_1 e^{\alpha L})}, \frac{(1-d)\eta_2}{2M(\eta_2 + \|\xi\|_{L^\infty(0,L)})}\right\} \tag{15}$$

and

$$k \leq 1 + \max\left\{e^{\alpha L}\eta_1, \frac{\eta_2}{b_0}\right\}.$$

Proof. Firstly, we prove that V decays exponentially, so we prove that $\frac{d}{dt}V(t) + 2\gamma V(t) \leq 0$ for all $t > 0$. Assume that v is a solution of (1) with $(v_0, z_0(., -\sigma(0).)) \in D(\mathcal{A}(0))$, satisfying the condition $\|(v_0, z_0(., -\sigma(0).))\|_H \leq r$. Differentiating V_1, we get

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= 2\int_0^L e^{\alpha x} v(x,t)v_t(x,t)dx \\
&= -2\int_0^L e^{\alpha x} v(x,t)(v_x(x,t) + v_{xxx}(x,t) + v(x,t)v_x(x,t) + a(x)v(x,t))dx \\
&\quad -2\int_0^L e^{\alpha x} b(x)v(x,t)v(x,t-\sigma(t))dx.
\end{aligned}$$

Using integration by parts and the boundary conditions, we obtain

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= (\alpha + \alpha^3)\int_0^L e^{\alpha x} v^2(x,t)dx - 3\alpha\int_0^L e^{\alpha x} v_x^2(x,t)dx \\
&\quad +\frac{2}{3}\alpha\int_0^L e^{\alpha x} v^3(x,t)dx - v_x^2(0,t) - 2\int_0^L e^{\alpha x} a(x)v^2(x,t)dx \\
&\quad -2\int_0^L e^{\alpha x} b(x)v(x,t)v(x,t-\sigma(t))dx.
\end{aligned}$$

In the same way, we differentiate V_2. Using integration by parts and the relation

$$\sigma(t)\partial_t v(x, t-\sigma(t)\rho) = (\dot\sigma(t)\rho - 1)\partial_\rho v(x, t-\sigma(t)\rho),$$

we get

$$\begin{aligned}
\frac{d}{dt}V_2(t) &= \dot\sigma(t)\int_\omega\int_0^1 (1-\rho)v^2(x,t-\sigma(t)\rho)d\rho dx \\
&\quad +2\sigma(t)\int_\omega\int_0^1 (1-\rho)v(x,t-\sigma(t)\rho)\partial_t(v(x,t-\sigma(t)\rho))d\rho dx \\
&= \dot\sigma(t)\int_\omega\int_0^1 (1-\rho)v^2(x,t-\sigma(t)\rho)d\rho dx \\
&\quad +2\int_\omega\int_0^1 (\dot\sigma(t)\rho - 1)(1-\rho)v(x,t-\sigma(t)\rho)\partial_\rho v(x,t-\sigma(t)\rho)d\rho dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt}V_2(t) &= \dot\sigma(t)\int_\omega\int_0^1 (1-\rho)v^2(x,t-\sigma(t)\rho)d\rho dx + \int_\omega [(\dot\sigma(t)\rho - 1)(1-\rho)v^2(x,t-\sigma(t)\rho)]_0^1 dx \\
&\quad -\int_\omega\int_0^1 (1+\dot\sigma(t) - 2\dot\sigma(t)\rho)v^2(x,t-\sigma(t)\rho)d\rho dx
\end{aligned}$$

$$= \int_\omega v^2(x,t)dx - \int_\omega \int_0^1 (1-\dot{\sigma}(t)\rho)v^2(x,t-\sigma(t)\rho)d\rho dx.$$

Now, we compute $\dfrac{d}{dt}V(t) + 2\gamma V(t)$. We obtain

$$\begin{aligned}\frac{d}{dt}V(t) + 2\gamma V(t) \quad \leq \quad & \int_\omega (-2a(x) + b(x) + \xi(x) + \eta_1 e^{\alpha L} b(x) + \eta_2) v^2(x,t)dx \\ & + \int_\omega (b(x) + (d-1)\xi(x) + \eta_1 e^{\alpha L} b(x)) v^2(x,t-\sigma(t))dx \\ & + 2\gamma(1+\eta_1 e^{\alpha L}) \int_0^L v^2(x,t)dx + \frac{2}{3}\eta_1 \alpha \int_0^L e^{\alpha x} v^3(x,t)dx \\ & - 3\eta_1 \alpha \int_0^L e^{\alpha x} v_x^2(x,t)dx - (1+\eta_1) v_x^2(0,t) \\ & + \eta_1 \int_0^L (\alpha + \alpha^3 - 2a(x)) e^{\alpha x} v^2(x,t)dx \\ & + \int_\omega \int_0^1 (2\gamma \xi(x)\sigma(t) + 2\gamma\eta_2\sigma(t) - \eta_2(1-d)) v^2(x,t-\sigma(t)\rho)d\rho dx.\end{aligned}$$

From Cauchy-Schwarz's inequality, we have

$$\int_0^L e^{\alpha x} v^3 dx \leq \|v\|^2_{L^\infty(0,L)} \int_0^L e^{\alpha x}|v|dx \leq \|v\|^2_{L^\infty(0,L)} \sqrt{\frac{e^{2\alpha L}}{2\alpha}}\ \|v\|_{L^2(0,L)}.$$

By the injection of $H_0^1(0,L)$ into $L^\infty(0,L)$, we have $\|v\|^2_{L^\infty(0,L)} \leq L\|v_x\|^2_{L^2(0,L)}$. We obtain

$$\int_0^L e^{\alpha x} v^3 dx \leq L\|v_x\|^2_{L^2(0,L)} \frac{e^{\alpha L}}{\sqrt{2\alpha}} \|v\|_{L^2(0,L)}.$$

Since we have $a_0 \leq a(x)$ and from Poincaré's inequality, we get

$$\begin{aligned}\frac{d}{dt}V(t) + 2\gamma V(t) \quad \leq \quad & \int_\omega (-2a(x) + b(x) + \xi(x) + \eta_1 e^{\alpha L} b(x) + \eta_2) v^2(x,t)dx \\ & + \int_\omega (b(x) + (d-1)\xi(x) + \eta_1 e^{\alpha L} b(x)) v^2(x,t-\sigma(t))dx \\ & + \left(2\gamma \frac{L^2}{\pi^2}(1+\eta_1 e^{\alpha L}) + \frac{L\eta_1\sqrt{2\alpha}e^{\alpha L} r}{3} - 3\eta_1\alpha\right) \int_0^L v_x^2(x,t)dx \\ & + \eta_1(\alpha + \alpha^3 - 2a_0) \int_0^L e^{\alpha x} v^2(x,t)dx - (1+\eta_1) v_x^2(0,t) \\ & + \int_\omega \int_0^1 (2\gamma \xi(x)\sigma(t) + 2\gamma\eta_2\sigma(t) - \eta_2(1-d)) v^2(x,t-\sigma(t)\rho)d\rho dx.\end{aligned}$$

To obtain $\dfrac{d}{dt}V(t) + 2\gamma V(t) \leq 0$, from (7), we can choose η_1, η_2, α, γ and r such that

$$\eta_1 \leq \inf_{x\in\omega} \left\{ \frac{2a(x) - b(x) - \xi(x)}{e^{\alpha L} b(x)}, \frac{(1-d)\xi(x) - b(x)}{e^{\alpha L} b(x)} \right\},$$
$$\eta_2 \leq \inf_{x\in\omega} \{2a(x) - b(x) - \xi(x) - \eta_1 e^{\alpha L} b(x)\},$$

and

$$\gamma \leq \min\left\{ \frac{(9\alpha - \sqrt{2\alpha}e^{\alpha L} L r)\pi^2 \eta_1}{6L^2(1+\eta_1 e^{\alpha L})}, \frac{(1-d)\eta_2}{2M(\eta_2 + \|\xi\|_{L^\infty(0,L)})} \right\}, \tag{16}$$

where $\alpha > 0$ can be chosen such that $\alpha + \alpha^3 - 2a_0 \leq 0$ in $(0, \alpha_0)$ and r can be chosen such that $9\alpha - \sqrt{2\alpha}Le^{\alpha L}r > 0$, which means that

$$0 < r < \frac{9\sqrt{\alpha}}{\sqrt{2}Le^{\alpha L}}.$$

We integrate $\frac{d}{dt}V(t) + 2\gamma V(t) \leq 0$ over $(0, t)$, and we get $V(t) \leq V(0)e^{-2\gamma t}$, for all $t > 0$. Since E and V are equivalent, from (14), we obtain

$$E(t) \leq \left(1 + \max\{e^{\alpha L}\eta_1, \frac{\eta_2}{b_0}\}\right) E(0)e^{-2\gamma t}, \ \forall t > 0.$$

Finally, we note that $D(\mathcal{A}(0))$ is dense in H. Therefore we can take $(v_0, z_0(., -\sigma(0).)) \in H$. ■

4 Numerical Simulations and Conclusion

In this section, we illustrate the stability result obtained in this study with some numerical simulations that adapt the schemes used in [6, 10]. We choose the following parameters, $T = 10$ and the feedback terms are constant in their support supp a = supp $b = (0, L/2)$, $a(x) = 2$, $b(x) = 1$ and $\xi(x) = 2.1$. The initial conditions are $v_0(x) = 1-\cos(2\pi x)$, $z_0(x, \rho) = (1-\cos(2\pi x))\cos(2\pi\rho)$ and the delay is $\sigma(t) = 0.3(1, 1-\sin(t))$. We can observe that the decay rate γ decreases when the length of the spatial domain L increases as shown in the estimation (16).

The following figure represents $t \mapsto \ln(E(t))$ for different values of L.

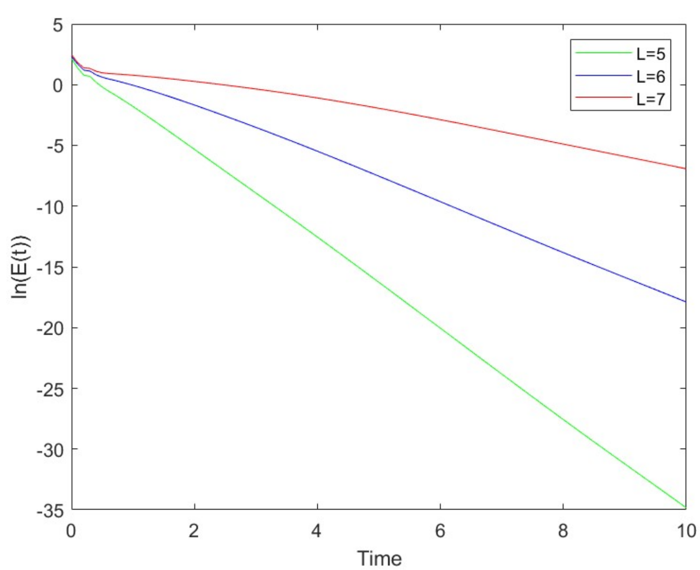

Figure 1: Representation of $t \mapsto \ln(E(t))$ for different values of L.

In this work, we present an internal stability result for the nonlinear Korteweg-de Vries equation with time-varying delay. We recall the well-posedness result of the system and prove the exponential stability using an appropriate Lyapunov functional, without any smallness assumption on the length L of the spatial domain. Finally, we present some numerical simulations to illustrate the results obtained.

We mention some possible directions for future research: the stabilization of the following nonlinear KdV equation with boundary time-varying delay, without any condition on L.

$$\begin{cases} v_t(x,t) + v_x(x,t) + v_{xxx}(x,t) + v(x,t)v_x(x,t) = 0, & t > 0,\ x \in (0,L), \\ v(0,t) = y(L,t) = 0, & t > 0, \\ v_x(L,t) = \lambda v_x(0,t) + \beta v_x(0, t-\sigma(t)), & t > 0, \\ v(x,0) = v_0(x), & x \in (0,L), \\ v_x(0, t-\sigma(0)) = z_0(t-\sigma(0)), & 0 < t < \sigma(0), \end{cases} \tag{17}$$

where λ and β are real constants satisfying $|\lambda| + |\beta| + d < 1$, with d defined in (3). The main challenge is to find a better multiplier in the expression of V_1, within the Lyapunov functional V, in order to establish the exponential stability of (17) for any length L of the domain.

References

[1] L. Baudouin, E. Crépeau and J. Valein, Two approaches for the stabilization of nonlinear KdV equation with boundary time-delay feedback, IEEE Transactions on Automatic Control, 64(2019), 1403–1414.

[2] E. Cerpa, Control of a Korteweg-de Vries equation: a tutorial, Mathematical Control & Related Fields, 4(2014), 45–99.

[3] B. Chentouf, Well-posedness and exponential stability of the Kawahara equation with a time-delayed localized damping, Math. Methods Appl. Sci., 45(2022), 10312–10330.

[4] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary wave, Phil. Mag., 39(1895), 422–443.

[5] H. Parada, E. Crépeau and C. Prieur, Delayed stabilization of the Korteweg-de Vries equation on a star-shaped network, Math. Control Signals Systems, 24(2022), 1–47.

[6] H. Parada, C. Timimoun and J. Valein, Stability results for the KdV equation with time-varying delay, Systems Control Lett., 177(2023), 105547.

[7] G. Perla Menzala, C. F. Vasconcellos and E. Zuazua, Stabilization of the Korteweg-de Vries equation with localized damping, Quart. Appl. Math., 60(2002), 111–129.

[8] L. Rosier, Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain, ESAIM, Control Optim. Calc. Var., 2(1997), 33–55.

[9] L. Rosier and B.-Y. Zhang, Control and stabilization of the Korteweg-de Vries equation: recent progresses, J. Syst. Sci. Complex., 22(2009), 647–682.

[10] J. Valein, On the asymptotic stability of the Korteweg-de Vries equation with time-delayed internal feedback, Math. Control Relat. Fields, 12(2022), 667–694.