Applied Mathematics E-Notes, 26(2026), 152-158 © ISSN 1607-2510
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A Refinement Of Cauchy's Theorem On The Zeros Of Quaternionic Polynomials*

Nisar Ahmad Rather[†], Danish Rashid Bhat[‡], Tanveer Bhat[§]

Received 6 October 2024

Abstract

In this paper, we shall present an interesting and significant refinement of a classical result of Cauchy about the moduli of the zeros of a quaternionic polynomial. As an application of this result, we shall obtain zero-free regions for polynomials with quaternionic coefficients.

1 Introduction

Concerning the location of zeros of quaternionic polynomials, the first study was done by Eilenberg and Niven [3, 7]. After these fundamental works, the question of locating the zeros of quaternionic polynomials has been vastly investigated. Let us first introduce the background information about the quaternions and quaternionic polynomials. The quaternions were first introduced by the Irish mathematician Sir William Rowan Hamilton in 1843. The quaternions are mathematical entities used to represent rotations in 3-D space. They extend the concept of complex numbers by adding two additional imaginary units, providing a concise way to perform spatial rotations. The quaternion number system is represented by the letter $\mathbb{H}$ and is generally represented as $q = \alpha + i\beta + j\gamma + k\delta \in \mathbb{H}$, where $\alpha, \beta, \gamma, \delta \in R$ and i, j, k are the fundamental quaternion units such that $i^2 = j^2 = k^2 = ijk = -1$. Depending upon the position of the coefficients, the quaternionic polynomial of degree n in indeterminate q is defined as $f(q) = q^n + q^{n-1}a_1 + \cdots + qa_{n-1} + a_n$ or $g(q) = q^n + a_1q^{n-1} + \cdots + a_{n-1}q + a_n$.

The Quaternion Companion Matrix: The $n \times n$ companion matrix of a monic quaternionic polynomial of the form $f(q) = q^n + q^{n-1}a_1 + ... + qa_{n-1} + a_n$ is given by

$$C_f = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 1 & \cdots & 0 & -a_{n-3} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix},$$

whereas the $n \times n$ companion matrix for a monic quaternionic polynomial of the form $g(q) = q^n + a_1q^{n-1} + \cdots + a_{n-1}q + a_n$, is given by

$$C_g = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}.$$

*Mathematics Subject Classifications: 20F05, 20F10, 20F55, 68Q42.
†Department of Mathematics, University of Kashmir, Srinagar-190006, India
‡Department of Mathematics, University of Kashmir, Srinagar-190006, India
§Department of Mathematics, University of Kashmir, Srinagar-190006, India. Corresponding Author.

Right Eigenvalue: Given an $n \times n$ matrix $A = [a_{\mu\nu}]$ of quaternions, $\lambda \in \mathbb{H}$ is called a right eigenvalue of A, if $Ax = x\lambda$ for some non-zero eigenvector $x = [x_1, x_2, \cdots x_n]^T$ of quaternions.

Left Eigenvalue: Given an $n \times n$ matrix $A = [a_{\mu\nu}]$ of quaternions, $\lambda \in \mathbb{H}$ is called the left eigenvalue of A, if $Ax = \lambda x$ for some non-zero eigenvector $x = [x_1, x_2, \cdots, x_n]^T$ of quaternions. To estimate the zeros of a polynomial is a long-standing classical problem. It is an interesting area of research for engineers as well as mathematicians and many results on the topic are available in the literature. One of the famous results regarding the distribution of zeros of polynomials known as the Eneström-Kakeya theorem, is as follows:

Theorem 1 *Let $f(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $0 < a_0 \leq a_1 \leq \cdots \leq a_n$. Then all the zeros of $f(z)$ lie in $|z| \leq 1$.*

Over the last two years various results were proved by several authors regarding the location of zeros of quaternionic polynomials. Recently, Carney et al. [2] extended the Eneström-Kakeya theorem to quaternionic settings by proving the following result.

Theorem 2 *If $f(q) = q^n a_n + q^{n-1} a_{n-1} + q^{n-2} a_{n-2} + ... + q a_1 + a_0$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of f lie in $|q| \leq 1$.*

For the complex case, concerning the location of the zeros, the famous Cauchy's theorem [6] can be stated as:

Theorem 3 *If $f(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $a_n \neq 0$, then all the zeros of $f(z)$ lie in $|z| \leq 1 + M$, where*

$$M = \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|.$$

Recently, Dar et al. [5] proved the following quaternionic version of Cauchy's theorem.

Theorem 4 *If $f(q) = q^n + q^{n-1} a_1 + \cdots + q a_{n-1} + a_n$ is a quaternionic polynomial with quaternion coefficients and q is quaternionic variable, then all the zeros of $f(q)$ lie inside the ball $|q| \leq 1 + \max_{1 \leq \nu \leq n} |a_\nu|$.*

Theorem 4 was refined by Rather et al. [8] by proving:

Theorem 5 *Let $f(q) = q^n + q^{n-1} a_1 + q^{n-2} a_2 + \cdots + q a_{n-1} + a_n$ be a monic quaternionic polynomial of degree n with quaternionic coefficients and q be a quaternion variable. If $\alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_n$ are ordered positive numbers,*

$$\alpha_\nu = \frac{|a_\nu|}{r^\nu}, \ \nu = 2, 3, \cdots, n,$$

where r is a positive real number. Then all the zeros of $f(q)$ lie in the union of balls $\{q \in H : |q| \leq r(1 + \alpha_2)\}$ and $\{q \in H : |q + a_1| \leq r\}$.

2 Main Results

We begin with the following result which is a significant refinement of Theorem 4.

Theorem 6 *If $f(q) = q^n + q^p q_p + q^{p-1} q_{p-1} + \cdots + q q_1 + q_0$, $0 \le p \le n-1$ is a monic quaternionic polynomial of degree n with quaternionic coefficients and q be a quaternion variable, and $|q_\nu| \le M$, $\nu = 0, 1, \cdots, n-p$, then all the zeros of $f(q)$ lie in the ball*

$$|q| \le \left\{ (1+M)^{p+1} - 1 \right\}^{\frac{1}{n}}.$$

If in Theorem 6, we take $p = n-1$, we get following result.

Corollary 1 *Let $f(q) = q^n + q^{n-1} a_{n-1} + q^{n-2} a_{n-2} + \cdots + q a_1 + a_0$ be a monic quaternionic polynomial of degree n with quaternionic coefficients and q be a quaternion variable, and $|a_\nu| \le M$, $\nu = 0, 1, \cdots, n-1$. Then all the zeros of $f(q)$ lie in the ball*

$$|q| \le \{(1+M)^n - 1\}^{\frac{1}{n}}.$$

The following corollary is an immediate consequence of Theorem 6.

Corollary 2 *If $f(q) = q^n + q^p a_p + q^{p-1} a_{p-1} + \cdots + q a_1 + a_0$, $0 \le p \le n-1$ is a quaternionic polynomial of degree n with quaternionic coefficients, then $|q| \le (1+M)^{\frac{p+1}{n}}$.*

Remark 1 *If $p = n-1$, then Corollary 2 reduces to Theorem 4.*

Corollary 3 *If $f(q) = q^n + q^p a_p + q^{p-1} a_{p-1} + \cdots + q a_1 + a_0$, $0 \le p \le n-1$ is a quaternionic polynomial of degree n with quaternionic coefficients such that $|a_j| \le 1$, $j = 0, 1, \cdots, p$, then all the zeros of $f(q)$ lie in the ball $|q| \le 2^{\frac{p+1}{n}}$.*

From Corollary 1, we can easily deduce the following:

Corollary 4 *If $f(q) = q^n + q^p a_p + q^{p-1} a_{p-1} + \cdots + q a_1 + a_0$, $0 \le p \le n-1$ is a quaternionic polynomial of degree n with quaternionic coefficients such that $|a_j| \le 1$, $j = 0, 1, \cdots, p$, then all the zeros of $f(q)$ lie in the ball $|q| \le (2^n - 1)^{\frac{1}{n}}$.*

As an application to Corollary 4, we now present the following result regarding the location of zeros of a quaternionic polynomial.

Theorem 7 *Let $f(q) = q^n + q^{n-1} a_{n-1} + q^{n-2} a_{n-2} + \cdots + q a_1 + a_0$ be a monic quaternionic polynomial with quaternionic coefficients and $|f(q)|$ attains maximum on $|q| = t$ at the point $q = te^{I\alpha}$ where $t \in R$. Then $f(q)$ does not vanish in the ball*

$$\left| q - te^{I\alpha} \right| < \frac{t}{n\,(2^n - 1)^{\frac{1}{n}}}.$$

3 Lemmas

For the proofs of these theorems we need the following lemmas. Lemma 1 is due to Dar et al. [5].

Lemma 1 *All the left eigenvalues of a $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions lie in the union of the n Geršgorin balls defined by $B_\mu = \{q \in \mathbb{H} : |q - a_{\mu\mu}| \le \rho_\mu(A)\}$ where $\rho_\mu(A) = \sum_{\substack{\nu=1 \\ \nu \ne \mu}}^{n} |a_{\mu\nu}|$.*

Lemma 2 is due to Rather et al. [8].

Lemma 2 *Let $P(q)$ be a quaternionic polynomial with quaternionic coefficients and C_p be the companion matrix of $P(q)$. Then for any diagonal matrix $D = diag(d_1, d_2, ..., d_{n-1}, d_n)$, where $d_1, d_2, ..., d_n$ are positive real numbers, the left eigenvalues of $D^{-1} C_P D$ and the zeros of $P(q)$ are same.*

Lemma 3 *If $f(q) = q^n + q^p a_p + \cdots + qa_1 + a_0$, $0 \le p \le n-1$ is a quaternionic polynomial of degree n and if $\delta_1, \delta_2, \cdots, \delta_{p+1}$ are $p+1$ non-zero quaternions such that $\sum_{k=1}^{p+1} |\delta_k| \le 1$, then all the zeros of $f(q)$ lie in the ball $|q| \le R$, where*

$$R = \left\{ \max_{1 \le k \le p+1} \frac{|a_{p-k+1}|}{|\delta_k|} \right\}^{\frac{1}{n-p+k-1}}.$$

Proof of Lemma 3. The companion matrix for the polynomial $f(q) = q^n + q^p a_p + \cdots + qa_1 + a_0$, $0 \le p \le n-1$ is given by

$$C_f = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & -a_p \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

We take matrix $P = diag\left(\frac{1}{r^{n-1}}, \frac{1}{r^{n-2}}, \cdots, \frac{1}{r}, 1\right)$, where r is a positive real number and form the matrix

$$P^{-1}C_f P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{a_0}{r^{n-1}} \\ r & 0 & 0 & \cdots & 0 & -\frac{a_1}{r^{n-2}} \\ 0 & r & 0 & \cdots & 0 & -\frac{a_2}{r^{n-3}} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & -\frac{a_p}{r^{n-p-1}} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & r & 0 \end{bmatrix}.$$

Applying Lemma 1 to the matrix $P^{-1}C_f P$, it follows that all the left eigenvalues of $P^{-1}C_f P$ lie in the union of balls $|q| \le t$ and

$$|q + a_{n-1}| \le \frac{|a_0|}{r^{n-1}} + \frac{|a_1|}{r^{n-2}} + \cdots + \frac{|a_{n-2}|}{r} + |a_{n-1}|.$$

Since

$$\begin{aligned} |q| = |q + a_{n-1} - a_{n-1}| &\le |q + a_{n-1}| + |a_{n-1}| \\ &\le \frac{a_0}{r^{n-1}} + \frac{a_1}{r^{n-2}} + \cdots + \frac{a_{n-2}}{r} + |a_{n-1}| \\ &= \sum_{k=1}^{n} \frac{|a_{n-k}|}{r^{k-1}}. \end{aligned}$$

That is, all the left eigenvalues of the matrix $T^{-1}C_f T$ lie in the ball

$$|q| \le \max\left\{ r, \sum_{k=1}^{n} \frac{|a_{n-k}|}{r^{k-1}} \right\}. \tag{1}$$

We now choose

$$r = \max\left\{ \frac{|a_{n-k}|}{|\delta_k|} \right\}^{1/k}, \ k = 1, 2, \cdots, n.$$

Then

$$\frac{|a_{n-k}|}{|\delta_k|} \le r^k \delta_k, \ k = 1, 2, \cdots, n,$$

which gives

$$\frac{|a_{n-k}|}{r^{k-1}} \leq r|\delta_k|,$$

so that

$$\sum_{k=1}^{n} \frac{|a_{n-k}|}{r^{k-1}} \leq \sum_{k=1}^{n} r|\delta_k| = t\sum_{k=1}^{n} |\delta_k| \leq r.$$

Using this in (1), it follows that all the left eigenvalues of the matrix $P^{-1}C_fP$ lie in

$$|q| \leq \max_{1\leq k\leq n} \left\{\frac{|a_{n-k}|}{|\delta_k|}\right\}^{\frac{1}{k}}. \tag{2}$$

Since P is a diagonal matrix with real positive entries, by Lemma 2, it follows that the left eigenvalues of $P^{-1}C_fP$ are the zeros of $f(q)$. Therefore, all the zeros of $f(q)$ lie in the ball given by (2). This completes the proof of Lemma 3. ■

Lemma 4 is due to Zhenghua [11].

Lemma 4 *If $f(q) = q^n + q^p a_p + q^{p-1}a_{p-1} + \cdots + qa_1 + a_0$ is a quaternionic polynomial of degree n and $1 \leq p \leq \infty$, then*

$$\max_{|q|=r} |f'(q)| \leq n\max_{|q|=r} |f(q)|.$$

Applying Lemma 4 to the quaternionic polynomial $f(rq)$, where r is any positive real number, we get:

Lemma 5 *If $f(q) = q^n + q^p a_p + q^{p-1}a_{p-1} + \cdots + qa_1 + a_0$ is a quaternionic polynomial of degree n and $1 \leq p \leq \infty$, then*

$$\max_{|q|=r} |f'(q)| \leq \frac{n}{r}\max_{|q|=r} |f(q)|.$$

The next lemma is obtained by repeated application of Lemma 5.

Lemma 6 *If $f(q)$ is a quaternionic polynomial of degree $n \geq 1$, and r is any positive real number, then*

$$\max_{|q|=r} |f'(q)| \leq \frac{n(n-1)\cdots(n-k+1)}{r^k}\max_{|q|=r} |f(q)|, \ k = 1, 2, \cdots, n.$$

4 Proof of the Main Theorems

Proof of Theorem 6. By hypothesis, we have

$$|q_{p-k+1}| \leq M, \ k = 1, 2, \cdots, p+1. \tag{3}$$

We take

$$\delta_k = \left[\frac{(1+M)^n}{(1+M)^{p+1}-1}\right]\left[\frac{q_{p-k+1}}{(1+M)^{n-p+k-1}}\right]. \tag{4}$$

Then with the help of (3), we get

$$\sum_{k=1}^{p+1} |\delta_k| = \frac{(1+M)^n}{(1+M)^{p+1}-1}\sum_{k=1}^{p+1} |q_{p-k+1}| \frac{1}{(1+M)^{n-p+k-1}} \leq \frac{(1+M)^n}{(1+M)^{p+1}-1}\sum_{k=1}^{p+1} \frac{M}{(1+M)^{n-p+k-1}}. \tag{5}$$

Now

$$\sum_{k=1}^{p+1} \frac{M}{(1+M)^{n-p+k-1}} = \frac{M}{(1+M)^{n-p}}\sum_{k=1}^{p+1} \frac{M}{(1+M)^{k-1}}$$

$$
\begin{aligned}
&= \frac{M}{(1+M)^{n-p}}\left[\frac{1-\frac{1}{(1+M)^{p+1}}}{1-\frac{1}{(1+M)}}\right] \\
&= \frac{(1+M)^{p+1}-1}{(1+M)^{n}}. \qquad (6)
\end{aligned}
$$

Using (6) in (5), we obtain $\sum_{k=1}^{p+1}|\delta_{k+1}| \leq 1$. Applying Lemma 3 with δ_k, $k=1,2,\cdots,p+1$ defined by (4), it follows that all the zeros of $f(q)$ lie in the ball

$$
\begin{aligned}
|q| &\leq \max_{1\leq k\leq p+1}\left|\frac{1}{\delta_k}q_{p-k+1}\right|^{\frac{1}{n-p+k-1}} \\
&= \max_{1\leq k\leq p+1}\left[\frac{(1+M)^{n-p+k-1}\left\{(1+M)^{p+1}-1\right\}}{(1+M)^{n}}\right]^{\frac{1}{n-p+k-1}} \\
&= (1+M)\max_{1\leq k\leq p+1}\left[\frac{(1+M)^{p+1}-1}{(1+M)^{n}}\right]^{\frac{1}{n-p+k-1}} \\
&= (1+M)\left[\frac{(1+M)^{p+1}-1}{(1+M)^{n}}\right]^{\frac{1}{n}} \\
&= \left[(1+M)^{p+1}-1\right]^{\frac{1}{n}}.
\end{aligned}
$$

This completes the proof of Theorem 6. ■

Proof of Theorem 7. Let t be any positive real number and let $w = te^{I\alpha}$, $\alpha \in R$. Then by hypothesis

$$\max_{|q|=t} f(q) = \left|q\left(te^{I\alpha}\right)\right| = |q(w)|.$$

Now consider a polynomial

$$
\begin{aligned}
R(q) &= f\left(\frac{t}{n}q + w\right) \\
&= f(w) + \left(\frac{t}{n}\right)f'(w)q + \left(\frac{t}{n}\right)^{2}f''(w)\frac{q^2}{2!} + \cdots + \left(\frac{t}{n}\right)^{n}f^{(n)}(w)\frac{q^n}{n!}.
\end{aligned}
$$

If $T(q) = q^n R\left(\frac{1}{q}\right)$, then we have

$$T(q) = f(w)q^n + \left(\frac{q}{n}\right)f'(w)q^{n-1} + \cdots + \left(\frac{t}{n}\right)^{n}\frac{f^{(n)}(w)}{n!} = \sum_{j=0}^{n}\left(\frac{t}{n}\right)^{n-j}\frac{q^{(n-j)}(w)q^j}{(n-j)!}.$$

Since $w = te^{i\alpha}$, by using Lemma 6, we obtain

$$
\begin{aligned}
\left|f^{(n-j)}(w)\right| &= \left|f^{(n-j)}(te^{I\alpha})\right| \\
&\leq \frac{n(n-1)\cdots(j+1)}{t^{n-j}}\max_{|q|=t}|f(q)| \\
&\leq \frac{n(n-1)\cdots(j+1)}{t^{n-j}}\max_{|q|=t}|f(q)| \\
&\leq \frac{n(n-1)\cdots(j+1)}{t^{n-j}}|f(w)|
\end{aligned}
$$

$$\begin{aligned} &\leq \frac{n^{n-j}}{t^{n-j}}\,|f(w)| \\ &= \left(\frac{n}{j}\right)^{(n-j)} |f(w)|\,, \quad j = 0,1,\cdots,(n-1). \end{aligned}$$

This implies

$$\left|\left(\frac{t}{n}\right)^{n-j}\frac{f^{(n-j)}(w)}{(n-j)!}\right| = \left(\frac{t}{n}\right)^{n-j}\frac{\left|f^{(n-j)}(w)\right|}{(n-j)!} \leq \left(\frac{t}{n}\right)^{(n-j)}\left|f^{(n-j)}(w)\right| \leq |f(w)|\,, \quad j = 0,1,\cdots,(n-1).$$

Which shows that the polynomial $T(q)$ satisfies the conditions of Corollary 4. Consequently, all the zeros of $T(q)$ lie in the ball $|q| \leq (2^n-1)^{\frac{1}{n}}$. Since $R(q) = q^n T\left(\frac{1}{q}\right)$, all the zeros of $T(q)$ lie in the ball $|q| \geq \frac{1}{(2^n-1)^{\frac{1}{n}}}$. Replacing q by $(q-w)\left(\frac{n}{t}\right)$ and noting that $f(q) = R\left(q-w\right)\left(\frac{n}{t}\right)$, we conclude that the quaternionic polynomial $f(q)$ does not vanish in the ball

$$|q-w| < \frac{t}{n\left(2^n-1\right)^{\frac{1}{n}}},$$

which is the desired result.
This completes the proof of Theorem 7. ■

Acknowledgement. We are grateful to the referees for their valuable suggestions, which have certainly enhanced the presentation of this paper.

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