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Majorization Involving The Cyclic Moving Average And Its Applications*

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Abstract

In this paper, using the Schur convexity of the elementary symmetry function and the majorization involving the cyclic moving average proposed by I. Olkin, we establish some new cyclic inequalities which yield Nesbitt's inequality. At the end of the paper, we propose an idea to prove the majorizing relation given by I. Olkin using an equivalent relationship between majorization and doubly random matrices.

1 Introduction

For any $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

$$x_{[1]} \geq \cdots \geq x_{[n]}$$

denote the components of $\boldsymbol{x}$ in decreasing order.

Definition 1 ([3, 9]) *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.*

(a) *$\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$ or $\boldsymbol{y} \succ \boldsymbol{x}$) if*

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \text{ for } k = 1, 2, \ldots, n-1 \tag{1}$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{2}$$

And if $\boldsymbol{x}$ is not a rearrangement of $\boldsymbol{y}$, then $\boldsymbol{x}$ is said to be strictly majorized by $\boldsymbol{y}$, denoted as $\boldsymbol{x} \prec\prec \boldsymbol{y}$.

(b) *Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. If $\boldsymbol{x} \prec\prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) < \varphi(\boldsymbol{y})$, then φ is said to be strictly Schur-convex. If $-\varphi$ is (strictly) Schur-convex, φ is said to be (strictly) Schur-concave.*

Definition 2 *Let $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, $a_1 \geq \cdots \geq a_n$. The vector of k-th order circular moving averages of $\boldsymbol{a}$ is*

$$\boldsymbol{a}^{(k)} = \left(a_1^{(k)}, a_2^{(k)}, \cdots, a_n^{(k)}\right),$$

where

$$a_i^{(k)} = \frac{1}{k}\left(a_i + a_{i+1} + \cdots + a_{i+k-1}\right),\ a_{n+i} = a_i,\ i = 1, 2, \cdots, n,\ 1 \leq k \leq n.$$

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In 2006, I. Olkin, one of the authors of the book [4], wrote a letter to K. Z. Guan, referring to the following interesting question.

Problem 1

$$\boldsymbol{a}^{(k+1)} \prec \boldsymbol{a}^{(k)},\ 1 \leq k \leq n-1. \tag{3}$$

However, a proof that $\boldsymbol{a}^{(k+1)} \prec \boldsymbol{a}^{(k)}$ remains elusive ([4, p. 63]). In 2010, Shi [7] proved that (3) holds when $n = 4,\ k = 2$ and $n = 5,\ k = 3$. In 2018, Zhang et al. [10] proved that the problem (3) is true by the majorization definition.

In this paper, as a supplement to paper [10], we will provide several applications of the majorizing relation (3).

2 Definitions and Lemmas

Definition 3 *Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.*

(*a*) *$\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \leq \alpha \leq 1$ implies*

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \cdots, \alpha x_n + (1-\alpha)y_n) \in \Omega.$$

(*b*) *Let $\Omega \subset \mathbb{R}^n$ be a convex set. A function $\varphi : \Omega \to \mathbb{R}$ is said to be a convex function on Ω if*

$$\varphi\left(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y}) \tag{4}$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0,1]$. If strict inequality holds in (4) whenever $\boldsymbol{x} \neq \boldsymbol{y}$ and $\alpha \in [0,1]$, then φ is said to be strictly convex. If $-\varphi$ is (strictly) convex, φ is said to be (strictly) concave.

Lemma 1 ([4, p. 97]) *Let φ be a symmetric convex (or concave, respectively) function on the symmetric convex set Ω. Then φ is a Schur-convex (or Schur-concave, respectively) function on Ω.*

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the r-th elementary symmetric function $s_r(\boldsymbol{x})$ for $1 \leq r \leq n$ is defined [3, 9] by

$$s_r(\boldsymbol{x}) = s_r(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} \prod_{j=1}^{r} x_{i_j}.$$

The duality of the r-th elementary symmetric function $s_r(\boldsymbol{x})$ for $1 \leq r \leq n$ is defined [8] by

$$s_r^*(\boldsymbol{x}) = s_r^*(x_1, x_2, \ldots, x_n) = \prod_{1 \leq i_1 < \cdots < i_r \leq n} \sum_{j=1}^{r} x_{i_j}.$$

We also assume that $s_0(\boldsymbol{x}) = s_0^*(\boldsymbol{x}) = 1$ and $s_r(\boldsymbol{x}) = s_r^*(\boldsymbol{x}) = 0$ for $r < 0$ or $r > n$.

Lemma 2 ([3, 9]) *The elementary symmetry function $s_r(\boldsymbol{x})$ is an increasing Schur-concave function on $\mathbb{R}_+^n = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i > 0,\ i = 1, \ldots, n\}^n$. When $r > 1$, $s_r(\boldsymbol{x})$ is also a strictly Schur-concave function on $(0, +\infty)$. In particular, $s_2(\boldsymbol{x})$ is strictly Schur-concave function on $\mathbb{R}^n$.*

Corollary 1 ([8]) *If k is even integer (or oddinteger, respectively), then $s_r(\boldsymbol{x})$ is decreasing and Schur-concave function (orincreasing and Schur-convexfunction, respectively) on $\mathbb{R}_-^n = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i < 0,$ $i = 1, \ldots, n\}^n$.*

Lemma 3 ([8]) *The dual forms of the elementary symmetric functions $s_r^*(\boldsymbol{x})$ are increasing Schur-concave function on $\mathbb{R}_+^n$.*

Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$. For the Lehmer mean with n variables

$$L_p(\boldsymbol{x}) = L_p(x_1, \ldots, x_n) = \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n x_i^{p-1}},$$

Fu et al. [2] get the following results.

Lemma 4 *Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $n \geq 2$ and $p \in \mathbb{R}$.*

(*a*) *If $p \geq 2$, then for any $a > 0$, $L_p(\boldsymbol{x})$ is Schur-convex with $\boldsymbol{x} \in \left[\frac{(p-2)a}{p}, a\right]^n$;*

(*b*) *if $p < 0$, then for any $a > 0$, $L_p(\boldsymbol{x})$ is Schur-concave with $\boldsymbol{x} \in \left[a, \frac{(p-2)a}{p}\right]^n$.*

Lemma 5 *Let $I \subset \mathbb{R}$ be an open convex set and let $g : I \to \mathbb{R}$ be twice differentiable. Then the function g is convex on I if and only if $g^{''}(t) \geq 0$ for all $t \in I$.*

Lemma 6 ([3, 9]) *Let the interval $I \subset \mathbb{R}$, $\boldsymbol{x}, \boldsymbol{y} \in I^n \subset \mathbb{R}^n$. Then $\boldsymbol{x} \prec \boldsymbol{y} \Leftrightarrow \sum_{i=1}^n g(x_i) \leq$ (or $\geq$, respectively) $\sum_{i=1}^n g(y_i)$ for all convex (or concave, respectively) function $g : I \to \mathbb{R}$.*

Remark 1 *Lemma 6 is the famous Karamata inequality.*

Corollary 2 *Let $g : I \to (0, +\infty)$ be continuous, $\varphi(\boldsymbol{x}) = \prod_{i=1}^n g(x_i)$. Then*

(*a*) *φ is (strictly) Schur-convex on $I^n \Leftrightarrow \log g$ is (strictly) convex on I;*

(*b*) *φ is (strictly) Schur-concave on $I^n \Leftrightarrow \log g$ is (strictly) concave on I.*

Lemma 7 *Let $g(x) = \log \frac{t^x - 1}{x}, t > 1$, then $g(x)$ is a strictly convex function on $(0, +\infty)$.*

Proof. The result can be found in ([8], page 61, Lemma 2.3.1). However, the book [8] is written in Chinese. For the reader's convenience, we present the proof below.

By computing,

$$g''(x) = -\frac{t^x \log^2 t}{(t^x - 1)^2} + \frac{1}{x^2}.$$

To prove $g''(x) > 0$, it is equivalent to proving that

$$x^2 t^x (\log t)^2 < (t^x - 1)^2, \tag{5}$$

extracting the square root both sides in this inequality and dividing by the same t^x, then (5) equivalent to

$$f(x) := t^{\frac{x}{2}} - t^{-\frac{x}{2}} - x \log t > 0.$$

$$f'(x) = \frac{1}{2}(t^{\frac{x}{2}} + t^{-\frac{x}{2}} - 2) \log t > 0.$$

So $f(x)$ is strictly increasing on $(0, +\infty)$, and then $f(x) > f(0) = 0$ for $x > 0$, this is $g''(x) > 0$. By Lemma 5, $g(x)$ is a strictly convex function on $(0, +\infty)$. ∎

3 Main Results and Their Proofs

Theorem 1 *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, $n \geq 3$. Then for $1 \leq k \leq n-1$, we have*

$$\sum_{1\leq i_1<\cdots<i_r\leq n} \prod_{j=1}^{r} x_{i_j}^{(k)} \leq \sum_{1\leq i_1<\cdots<i_r\leq n} \prod_{j=1}^{r} x_{i_j}^{(k+1)}. \tag{6}$$

Proof. By Lemma 2, the elementary symmetry function $s_r(\boldsymbol{x})$ is an increasing Schur-concave function on $\mathbb{R}_+^n$, it follows that the inequality (6) is obtained from the majorizing relation (3). ■

Corollary 3 *Let $a \geq b \geq c \geq d > 0$. Then*

$$\left(1+\frac{c}{a+b}\right)\left(1+\frac{d}{b+c}\right)\left(1+\frac{a}{c+d}\right)\left(1+\frac{b}{d+a}\right) \geq \left(\frac{3}{2}\right)^4 \tag{7}$$

and

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2. \tag{8}$$

Proof. In Theorem 1, taking $n = r = 4$, $k = 2$, we have

$$\frac{a+b}{2}\cdot\frac{b+c}{2}\cdot\frac{c+d}{2}\cdot\frac{d+a}{2} \leq \frac{a+b+c}{3}\cdot\frac{b+c+d}{3}\cdot\frac{c+d+a}{3}\cdot\frac{d+a+b}{3}, \tag{9}$$

making a little deformation gives inequality (7). From the inequality (9), we have

$$\frac{2(b+c+a)}{3(b+c)}\cdot\frac{2(c+d+b)}{3(c+d)}\cdot\frac{2(d+a+c)}{3(d+a)}\cdot\frac{2(a+b+d)}{3(a+b)} \geq 1,$$

by arithmetic-geometric mean inequality, it follows that

$$\begin{aligned} & \frac{2(b+c+a)}{3(b+c)}+\frac{2(c+d+b)}{3(c+d)}+\frac{2(d+a+c)}{3(d+a)}+\frac{2(a+b+d)}{3(a+b)} \\ \geq\ & 4\cdot\sqrt[4]{\frac{2(b+c+a)}{3(b+c)}\cdot\frac{2(c+d+b)}{3(c+d)}\cdot\frac{2(d+a+c)}{3(d+a)}\cdot\frac{2(a+b+d)}{3(a+b)}} \geq 4 \\ \Leftrightarrow\ & \frac{b+c+a}{b+c}+\frac{c+d+b}{c+d}+\frac{d+a+c}{d+a}+\frac{a+b+d}{a+b} \geq 6 \\ \Leftrightarrow\ & \frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2. \end{aligned}$$

■

Theorem 2 *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, $n \geq 3$. Then for $1 \leq k \leq n-1$, we have*

$$\prod_{1\leq i_1<\cdots<i_r\leq n} \sum_{j=1}^{r} x_{i_j}^{(k)} \leq \prod_{1\leq i_1<\cdots<i_r\leq n} \sum_{j=1}^{r} x_{i_j}^{(k+1)}. \tag{10}$$

Proof. By Lemma 3, the dual forms of the elementary symmetry function $s_r^*(\boldsymbol{x})$ is a Schur-concave function on $\mathbb{R}_+^n$, it follows that the inequality (10) is obtained from the majorizing relation (3). ■

Corollary 4 *Let $a \geq b \geq c \geq d > 0$. Then*

$$\begin{aligned} & \left(1+\frac{c+d}{a+2b+c}\right)\left(1+\frac{a+c}{a+b+c+d}\right)\left(1+\frac{b+c}{2a+b+d}\right) \\ & \cdot\left(1+\frac{d+a}{b+2c+d}\right)\left(1+\frac{b+d}{b+c+d+a}\right)\left(1+\frac{a+b}{c+2d+a}\right) \geq \left(\frac{3}{2}\right)^6. \end{aligned} \tag{11}$$

Proof. In Theorem 2, taking $n = 4$, $r = k = 2$, we have

$$\begin{aligned}
&\left(\frac{a+b}{2}+\frac{b+c}{2}\right)\left(\frac{a+b}{2}+\frac{c+d}{2}\right)\left(\frac{a+b}{2}+\frac{d+a}{2}\right)\\
&\cdot\left(\frac{b+c}{2}+\frac{c+d}{2}\right)\left(\frac{b+c}{2}+\frac{d+a}{2}\right)\left(\frac{c+d}{2}+\frac{d+a}{2}\right)\\
\le\ &\left(\frac{a+b+c}{3}+\frac{b+c+d}{3}\right)\left(\frac{a+b+c}{3}+\frac{c+d+a}{3}\right)\left(\frac{a+b+c}{3}+\frac{d+a+b}{3}\right)\\
&\cdot\left(\frac{b+c+d}{3}+\frac{c+d+a}{3}\right)\left(\frac{b+c+d}{3}+\frac{d+a+b}{3}\right)\left(\frac{c+d+a}{3}+\frac{d+a+b}{3}\right)\\
\Leftrightarrow\ &\frac{a+2b+c}{2}\cdot\frac{a+b+c+d}{2}\cdot\frac{2a+b+d}{2}\frac{b+2c+d}{2}\cdot\frac{b+c+d+a}{2}\cdot\frac{c+2d+a}{2}\\
\le\ &\frac{a+2b+2c+d}{3}\cdot\frac{2a+b+2c+d}{3}\cdot\frac{2a+2b+c+d}{3}\\
&\cdot\frac{b+2c+2d+a}{3}\cdot\frac{2b+c+2d+a}{3}\cdot\frac{c+2d+2a+b}{3}\\
\Leftrightarrow\ &\left(\frac{3}{2}\right)^6\le\frac{a+2b+2c+d}{a+2b+c}\cdot\frac{2a+b+2c+d}{a+b+c+d}\cdot\frac{2a+2b+c+d}{2a+b+d}\\
&\cdot\frac{b+2c+2d+a}{b+2c+d}\cdot\frac{2b+c+2d+a}{b+c+d+a}\cdot\frac{c+2d+2a+b}{c+2d+a},
\end{aligned}$$

making a little deformation gives inequality (11). ■

Theorem 3 *Let $x_1 \ge x_2 \ge \cdots \ge x_n > 0$, $n \ge 3$. If $2b \ge a \ge 0$, then for $1 \le k \le n-1$, we have*

$$\prod_{i=1}^{n}(x_i^{(k)}+b)^{(x_i^{(k)}+a)} \ge \prod_{i=1}^{n}(x_i^{(k+1)}+b)^{(x_i^{(k+1)}+a)}. \tag{12}$$

Proof. Let $f(x) = (x+b)^{x+a}$ and $g(x) = \log f(x) = (x+a)\log(x+b)$. Then

$$g'(x) = \log(x+b) + \frac{x+a}{x+b} \quad\text{and}\quad g''(x) = \frac{1}{x+b} + \frac{b-a}{(x+b)^2} = \frac{x+2b-a}{(x+b)^2}.$$

Since $x > 0$, $2b \ge a \ge 0$ and $g''(x) > 0$, this means that $f(x)$ is a logarithmic convex function. According to the Corollary 2, the inequality (12) is obtained from majorizing relation (3). ■

Corollary 5 *Let $x_1, x_2, x_3 > 0$. If $2b \ge a \ge 0$, then*

$$\begin{aligned}
&(x_1+b)^{(x_1+a)}(x_2+b)^{(x_2+a)}(x_3+b)^{(x_3+a)}\\
\ge\ &\left(\frac{x_1+x_2}{2}+b\right)^{\left(\frac{x_1+x_2}{2}+a\right)}\left(\frac{x_2+x_3}{2}+b\right)^{\left(\frac{x_2+x_3}{2}+a\right)}\left(\frac{x_3+x_1}{2}+b\right)^{\left(\frac{x_3+x_1}{2}+a\right)}.
\end{aligned}\tag{13}$$

Proof. From the inequality (13) about the symmetry of x_1, x_2, x_3, we can assume that $x_1 \ge x_2 \ge x_3$. Taking $k = 1$ and $n = 3$, (12) yields the inequality (13) ■

Theorem 4 *Let $x_1 \ge x_2 \ge \cdots \ge x_n > 0$, $a_1, a_2, \ldots, a_n > 0$ and $n \ge 3$. Then for $1 \le k \le n-1$, we have*

$$\prod_{j=1}^{n}\sum_{i=1}^{n} a_i^{x_j^{(k)}} \ge \prod_{j=1}^{n}\sum_{i=1}^{n} a_i^{x_j^{(k+1)}}. \tag{14}$$

Proof. Let $f(x) = \sum_{i=1}^n a_i^x$. Then

$$(\log f(x))'' = \frac{\sum_{i=1}^n a_i^x \sum_{i=1}^n a_i^x (\log a_i)^2 - (\sum_{i=1}^n a_i^x \log a_i)^2}{(\sum_{i=1}^n a_i^x)^2}.$$

It is not difficult to verify $(\log f(x))'' \geq 0$ using the Cauchy-Schwarz inequality, so $f(x)$ is log-convex on $(0, +\infty)$ by Lemma 5. Then according to Corollary 2, the inequality (14) is obtained from majorizing relation (3). ■

Taking $k = 1$, $n = 3$ and $a_1 = a$, $a_2 = b$, $a_3 = c$, from (14) yields the following inequality (15).

Corollary 6 *Let $x_1, x_2, x_3 > 0$ and $a, b, c > 0$. Then*

$$\begin{aligned} &(a^{x_1} + b^{x_1} + c^{x_1})(a^{x_2} + b^{x_2} + c^{x_2})(a^{x_3} + b^{x_3} + c^{x_3}) \\ \geq\ & \left(a^{\frac{x_1+x_2}{2}} + b^{\frac{x_1+x_2}{2}} + c^{\frac{x_1+x_2}{2}}\right)\left(a^{\frac{x_2+x_3}{2}} + b^{\frac{x_2+x_3}{2}} + c^{\frac{x_2+x_3}{2}}\right)\left(a^{\frac{x_3+x_1}{2}} + b^{\frac{x_3+x_1}{2}} + c^{\frac{x_3+x_1}{2}}\right). \end{aligned} \tag{15}$$

Theorem 5 *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, $n \geq 3$. For $1 \leq k \leq n-1$, if $t > 1$, then*

$$\prod_{i=1}^n \frac{t^{x_i^{(k)}} - 1}{x_i^{(k)}} \geq \prod_{i=1}^n \frac{t^{x_i^{(k+1)}} - 1}{x_i^{(k+1)}}. \tag{16}$$

Proof. Theorem 5 can be obtained by combining Lemma 7 and Corollary 2 with the majorizing relation (3). ■

Remark 2 *For $t > 1$, it is easy to see that inequality (16) is equivalent to the following inequality.*

$$\prod_{i=1}^n x_i^{(k)} \leq \prod_{i=1}^n \frac{t^{x_i^{(k)}} - 1}{t^{x_i^{(k+1)}} - 1} \prod_{i=1}^n x_i^{(k+1)}. \tag{17}$$

Consider the function $h(x) = t^x - 1$. Then $(\log h(x))' = \frac{(t^x)'}{t^x - 1}$,

$$\begin{aligned} (\log h(x))'' &= \frac{(t^x)''(t^x - 1) - ((t^x)'^2}{(t^x - 1)^2} \\ &= \frac{t^x(\log t)^2(t^x - 1) - t^{2x}(\log t)^2}{(t^x - 1)^2} \\ &= -\frac{t^x(\log t)^2}{(t^x - 1)^2} \leq 0. \end{aligned}$$

This means that $h(x)$ is log-concave on $(0, +\infty)$, according to Corollary 2, from majorizing relation (3), we obtained the inequality

$$\prod_{i=1}^n \left(t^{x_i^{(k)}} - 1\right) \leq \prod_{i=1}^n \left(t^{x_i^{(k+1)}} - 1\right),$$

this is

$$\prod_{i=1}^n \frac{t^{x_i^{(k)}} - 1}{t^{x_i^{(k+1)}} - 1} \leq 1.$$

This means that for $t > 1$, inequality (17) is a strengthen of inequality (6) for $r = n$.

It is well-known that the gamma function is a log-convex function (see [6]), therefore, the following theorem holds.

Theorem 6 *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, $n \geq 3$. Then for $1 \leq k \leq n-1$, we have*

$$\prod_{i=1}^{n} \Gamma(x_i^{(k)}) \geq \prod_{i=1}^{n} \Gamma(x_i^{(k+1)}),$$

where $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\,\mathrm{d}t$ is the gamma function.

Define

$$F(x) = \frac{\Gamma(2x)}{x\Gamma^2(x)}, \quad G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)}.$$

In 1997, M. Merkle [5] has proven that for $x > 0$, the function F is strictly log-convex and the function G is strictly log-concave. Therefore, the following theorem holds.

Theorem 7 *Let $x_1, x_2, \ldots, x_n > 0$, $x_1 \geq x_2 \geq \cdots \geq x_n$, $n \geq 3$. Then for $1 \leq k \leq n-1$, we have*

$$\prod_{i=1}^{n} F(x_i^{(k)}) \geq \prod_{i=1}^{n} F(x_i^{(k+1)}) \quad \text{and} \quad \prod_{i=1}^{n} G(x_i^{(k)}) \leq \prod_{i=1}^{n} G(x_i^{(k+1)}).$$

Let $n \geq 1$, $J_p = [0, \infty)$ if $p \in (0, \infty)$ and $J_p = (0, \infty)$ if $p \in (-\infty, 0)$. For every $p \in \mathbb{R}^* = \mathbb{R} - 0$ consider the functions:

$$f_p : J_p^n \to J_p \ \text{ defined by } f_p(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \quad \text{for } (x_1, x_2, \ldots, x_n) \in J_p^n.$$

In 2019, Andrica et al. [1] has proven the following assertions,

(a) If $p \in [1, \infty)$, then f_p is convex on J_p^n.

(b) If $p \in (-\infty, 0) \cup (0, 1]$, then f_p is concave on J_p^n.

By Lemma 1, we have

(a) If $p \in [1, \infty)$, then f_p is Schur convex on J_p^n.

(b) If $p \in (-\infty, 0) \cup (0, 1]$, then f_p is Schur concave on J_p^n.

Based on the above properties, combining the majorizing relation (3), the following theorem can be proved.

Theorem 8 *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, $n \geq 3$. Then for $1 \leq k \leq n-1$, if $p \in [1, \infty)$, we have*

$$\left(\sum_{i=1}^{n} (x_i^{(k)})^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} (x_i^{(k+1)})^p\right)^{1/p}, \tag{18}$$

if $p \in (-\infty, 0) \cup (0, 1]$, then the inequality (18) is reversed.

Theorem 9 *Let $x_1, x_2, \ldots, x_n > 0$, $x_1 \geq x_2 \geq \cdots \geq x_n$, $n \geq 3$. Then for $1 \leq k \leq n-1$, we have*

$$\sum_{i=1}^{n} \frac{1}{x_i^{(k)}} \geq \sum_{i=1}^{n} \frac{1}{x_i^{(k+1)}}. \tag{19}$$

Proof. Let $f(x) = \frac{1}{x}$. Then $f''(x) = \frac{2}{x^3} \geq 0$, this means that $f(x)$ is convex on $(0, +\infty)$, according to Lemma 6, combined with the majorizing relation (3), the inequality (19) can be proven. ■

Corollary 7 (Nesbitt's inequality) *For $a, b, c > 0$, we have*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof. Without loss of generality, we may assume that $a \geq b \geq c > 0$. In Theorem 9, taking $n = 3$, we have

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{3}{a+b+c} + \frac{3}{b+c+a} + \frac{3}{c+a+b} = \frac{9}{a+b+c}. \tag{20}$$

Multiplying both sides of the above inequality by $\frac{a+b+c}{2}$, and making a little deformation gives inequality (20). ∎

Theorem 10 *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and $n \geq 3$. For $1 \leq k \leq n-1$, if $p \geq 2$, then for any $a > 0$, $\boldsymbol{x} \in \left[\frac{(p-2)a}{p}, a\right]^n$, we have*

$$\frac{\sum_{i=1}^{n}(x_i^{(k)})^p}{\sum_{i=1}^{n}(x_i^{(k)})^{p-1}} \geq \frac{\sum_{i=1}^{n}(x_i^{(k+1)})^p}{\sum_{i=1}^{n}(x_i^{(k+1)})^{p-1}}, \tag{21}$$

if $p < 0$, then for any $a > 0$, $\boldsymbol{x} \in \left[a, \frac{(p-2)a}{p}\right]^n$, the inequality (21) is reversed.

Proof. From Lemma 4 and the majorizing relation (3), it follows that Theorem 10 can be proved. ∎

Corollary 8 *For any $a > 0$ and $p \geq 2$, if $a \geq x_i \geq \frac{(p-2)a}{p}$, $i = 1, 2, 3$, then*

$$\begin{aligned} &\left(\frac{x_1+x_2}{2}\right)^p + \left(\frac{x_2+x_3}{2}\right)^p + \left(\frac{x_3+x_1}{2}\right)^p \\ \geq &(x_1+x_2+x_3)\left[\left(\frac{x_1+x_2}{2}\right)^{p-1} + \left(\frac{x_2+x_3}{2}\right)^{p-1} + \left(\frac{x_3+x_1}{2}\right)^{p-1}\right]. \end{aligned} \tag{22}$$

if $p < 0$ and $\frac{(p-2)a}{p} \geq x_i \geq a$, $i = 1, 2, 3$, then the inequality (22) is reversed.

Proof. Note that the inequality (22) is symmetric, we can assume that $x_1 \geq x_2 \geq x_3$. Taking $n = 3$, $k = 2$ in Theorem 10, we can obtain the desired result. ∎

4 An Open Question

Definition 4 ([3, 9]) *An $n \times n$ matrix $Q = (q_{ij})$ is doubly stochastic if $q_{ij} \geq 0$ for $i, j = 1, \ldots, n$, and all row and column sums of Q are equal to 1, this is $\sum_i q_{ij} = 1$, $j = 1, \ldots, n$; $\sum_j q_{ij} = 1$, $i = 1, \ldots, n$.*

Theorem 11 *If $\boldsymbol{x} \prec \boldsymbol{y} \in \mathbb{R}^n$, then there exists a doubly random matrix $Q = (q_{ij})$ such that $\boldsymbol{x} = \boldsymbol{y}Q$. $\boldsymbol{x} \prec \boldsymbol{y} \Leftrightarrow$ exist a doubly stochastic matrix Q, such that $\boldsymbol{x} = \boldsymbol{y}Q$.*

To prove

$$a^{(k+1)} \prec a^{(k)},$$

we just need to prove the existence of a doubly random matrix $Q = (q_{ij})$ such that $a^{(k+1)} = a^{(k)}Q$, that is

$$a^{(k)}Q - a^{(k+1)} = O.$$

We can prove

$$a^{(k)}Q - a^{(k+1)} = O \Leftrightarrow \|a^{(k)}Q - a^{(k+1)}\| = 0,$$

where the norm of the square matrix $C(a_{i_j})$

$$\|C\| = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^2\right)^{\frac{1}{2}}.$$

At present, we have proved that there exists a generalized birandom matrix $Q = (q_{ij})$ such that $\|a^{(k)}Q - a^{(k+1)}\| = 0$. It is also necessary to prove that Q is a nonnegative matrix, and this step may not be easy.

5 Conclusions

Majorization theory and Schur-convexity are often used to establish and prove symmetry inequalities. In this work, we use this theory to establish some cyclic inequalities as well.

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