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Analyzing The Existence Theorem: Implications For Weakly Singular Integral Equations*

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Abstract

This article discusses the solvability of a weakly singular integral equation with a logarithm kernel. These equations are defined within the function space $C[0,\ell]$, which consists of real-valued functions. The primary methodological framework employed in our proofs is the concept of a measure of noncompactness in conjunction with the theorem of Petryshyn. Furthermore, we illustrate the practical significance of our findings by presenting a series of applications related to nonlinear singular integral equations. These examples serve to demonstrate the efficacy and applicability of our theoretical results, thereby contributing to the broader understanding of such integral equations in mathematical analysis.

Keywords: Measure of noncompactness; weakly singular integral equation; condensing map; existence results; Petryshyn's theorem.

1 Introduction

Functional integral equations ($\mathsf{F}\mathbb{IE}$s) are powerful tools used in a range of fields, such as mathematics [20, 27, 31], engineering [4, 20], bio-engineering [24, 33, 39], and applied sciences [15, 37]. The $\mathbb{IE}$s are increasingly popular because they can accurately model complex phenomena. In recent years, there has been significant interest in both applied and pure analysis in $\mathsf{F}\mathbb{IE}$s, including ordinary and partial equations [1, 10, 19, 35].

In 2002, Butzer et al. [6] presented a novel approach to differentiation and integration on $(0,\infty)$ of the form

$$\mathbb{J}^{\sigma}_{0^+;c}v(r)=\int_0^r\left(\tfrac{\tau}{r}\right)^c\left(\log\tfrac{r}{\tau}\right)^{\sigma-1}\tfrac{v(\tau)}{\tau\gamma(\sigma)}\,\mathrm{d}\tau,\quad \sigma>0,\ c\in\mathbb{R},$$

in terms of classical Mellin transform $(\mathbb{T}_{\mathcal{M}})\mathcal{M}$ of function $v:\mathbb{R}^{>0}\to\mathbb{C}$ given by

$$\mathbb{T}_{\mathcal{M}}(v(r))=\int_0^\infty\tau^{r-1}v(\tau)\mathrm{d}\tau,\quad r=c+\mathrm{i}t,\ c,t\in\mathbb{R},$$

which was a generalization of Hadamard's article [14], for more instances, see [18, 34]. Das et al. studied the existence of solution of generalized $\mathbb{IE}$s of fractional order with two variables and proved a new fixed point theorem applying the measure of noncompactness (`M.N.C`) and a new contraction operator which generalized the Darbo's fixed point theorem (D-theorem) [10]. Paul et al. investigated the existence, uniqueness, stabilities of Hyers-Ulam-Rassias and Hyers-Ulam, and local stability of the solutions of the following nonlinear Volterra-Fredholm $\mathbb{IE}$ involving the Erdélyi-Kober fractional integral operator, for $r\in\mathbb{R}^{>0}$,

$$v(r)=w(r)+y_1(r)\int_0^r\tfrac{\mu^\gamma(r^\rho-\tau^\rho)}{\gamma(\delta)}\mathbb{f}_1\left(r,\mu,v(\tau)\right)\mathrm{d}\tau+y_2(r)\int_0^s\tfrac{\mu^\gamma(r^\rho-\tau^\rho)}{\gamma(\delta)}\mathbb{f}_2\left(r,\mu,v(\tau)\right)\mathrm{d}\tau,$$

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with $s \in \mathbb{R}^{>0}$, the function $w, y_1, y_2 : \mathbb{R}^{\geq 0} \to \mathbb{R}$, where $r, \mu, \rho, \delta \in \mathbb{R}^{\geq 0}$ and $\mathbb{f}_i \in C\big(\mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}\big)$ [28]. They established the existence and uniqueness, and analyzed stabilities of Hyers-Ulam-Rassias and Hyers-Ulam of the solution for the nonlinear $\mathbb{IE}$ in the case of the Riemann-Liouville fractional operator of the form

$$v(r) = w(r) + y(r) \int_0^r \tfrac{(r-\tau)^\sigma}{\gamma(\sigma)} \eta(\tau) \mathbb{f}\,(v(\tau))\,\mathrm{d}\tau, \quad r \in \Delta_\ell := [0, \ell],$$

where $\sigma \in \mathbb{R}^{>0}$ with the continuous functions $\mathbb{f} : C(\Delta_\ell) \to \mathbb{R}$ and , $w, y, \eta : \Delta_\ell \to \mathbb{R}$, based on the Leray-Schauder alternative and Banach's fixed point theorem [29]. Researchers have focused on studying the existence of solutions for $\mathsf{F}\mathbb{IE}$s, particularly those involving weakly singular $\mathbb{IE}$s ($\mathsf{WS}\mathbb{IE}$s). One of the most commonly used methods is the concept of `M.N.C`. The root of this concept goes back to the famous work of Kuratowski [21]. This method plays a vital role in the publications of research [3].

In 1955, Darbo [8] established a ground-breaking theorem that proves the existence of fixed points for condensing operators using the concept of `M.N.C`. This theorem has found wide application in various scientific fields, particularly in finding solutions for both $\mathsf{F}\mathbb{IE}$s and differential equations ($\mathbb{DE}$s) [5, 7, 22, 36, 38, 40]. Bhat et al. discussed the numerical solutions and studied the existence of the unique solutions of weakly singular Volterra and Fredholm $\mathbb{IE}$s, which were used to demonstrate the problems like heat conduction in engineering and the electrostatic potential theory, based on the modified Lagrange polynomial interpolation technique combined with the biconjugate gradient stabilized method [5]. Authors [12] obtained results to existence of solutions for the following nonlinear $\mathsf{F}\mathbb{IE}$s,

$$\begin{aligned} v(r) = &\left[y_1\big(r, v(r)\big) + \mathbb{G}_1\left(r, \int_0^r \mathbb{f}_1\big(r, \tau, v(\tau)\big)\mathrm{d}\tau, \int_0^r \mathbb{h}_1\big(r, \tau, v(\tau)\big), v(\tau)\right)\right] \\ &\times \left[y_2\big(r, v(r)\big) + \mathbb{G}_2\left(r, \int_0^r \mathbb{f}_2\big(r, \tau, v(\tau)\big)\mathrm{d}\tau, \int_0^r \mathbb{h}_2\big(r, \tau, v(\tau)\big), v(\tau)\right)\right], \end{aligned}$$

for $r \in \Delta_\ell$ where $y_i \in C\big(\Delta_\ell \times \mathbb{R}^2\big)$, $\mathbb{G}_i \in C\big(\Delta_\ell \times \mathbb{R}^3\big)$, $i = 1, 2$. Metwali and Mishra in [23], proved a new compactness criterion in the Lebesgue spaces $L_p\big(\mathbb{R}^{>0}\big)$, $1 \leq p < \infty$ and used such criteria to construct a `M.N.C` in the mentioned spaces and applied such `M.N.C` with a modified version of D-theorem in proving the existence of monotonic integrable solutions for a product of n-Hammerstein $\mathbb{IE}$s, $n \geq 2$, is given by,

$$v(r) = y\big(r, v(r)\big) + \prod_{i=1}^{n} \left(\mathbb{f}_i\big(r, v(r)\big) + \mathbb{h}_i\big(r, v(r)\big) \cdot |v(r)|^{p/q_i} \int_0^\infty \mathbb{K}_i(r, \tau) \mathbb{u}_i\big(\tau, v(\tau)\big)\mathrm{d}\tau\right),$$

where $p < q_i < \infty$. Nashine et al. investigated the solutions of a system of $\mathsf{F}\mathbb{IE}$ is expressed by,

$$\begin{cases} v(r) = \mathbb{G}\left(r, v(r), q(r), \int_0^r \mathbb{f}\big(r, \tau, v(\tau), q(\tau)\big)\mathrm{d}\tau\right), \\ q(r) = \mathbb{G}\left(r, q(r), v(r), \int_0^r \mathbb{f}\big(r, \tau, q(\tau), v(\tau)\big)\mathrm{d}\tau\right), \end{cases}$$

in the setting of `M.N.C` on real-valued bounded and continuous Banach space by using Darbo type fixed and coupled fixed point results for μ-set (ω, ϑ)-contraction operator and arbitrary `M.N.C` in this spaces, where $\mathbb{G} \in C\big(\mathbb{R}^{\geq 0} \times \mathbb{R}^3\big)$ along with the some conditions [25]. The D-theorem offers a significant advantage over Schauder's theorem by relaxing the requirement of compactness for the operator's domain [2]. This flexibility makes it suitable for studying solvability in different types of equations, including implicit $\mathbb{DE}$s, $\mathbb{I}$-$\mathbb{DE}$s, and equations arising in controllability problems for dynamical systems [9, 11].

Although D-theorem is highly effective in this field, the method is still criticized for its tendency to impose many conditions on the problem, which is seen as a significant weakness. In 2016, by the assistance of `M.N.C` and fixed point theorem of Petryshyn (P-theorem), Kazemi et al. established that the sub-linear conditions in D-theorem is an additional condition [16, 32].

This article analyzes the existence result for the following $\mathsf{WS}\mathbb{IE}$,

$$v(r) = \mathbb{g}\left(r, \mho_{i=1}^{s} v(\beta_i(r))\right) + \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) \int_0^r \ln|\tau - r| \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} v(\gamma_k(r))\right) \mathrm{d}\tau, \tag{1}$$

for $r, \tau \in \Delta_\ell$, with $\mho_{p=1}^{q} v(\hat{\eta}_p(\cdot)) := \big(v(\hat{\eta}_1(\cdot)), v(\hat{\eta}_2(\cdot)), \ldots, v(\hat{\eta}_q(\cdot)) \big)$, where

W1) functions β_i, α_j and $\gamma_k \in C(\Delta_\ell)$ with $1 \le i \le s$, $1 \le j \le m$ and $1 \le k \le n$.

In the following, we consider the outline of the paper. In Section 2, we collect some definitions, lemmas and theorems, which are essential to prove our main results. In Section 3, we establish and prove a new existence theorem by utilizing the P-theorem for WSIE (1). In Section 4, we also give few examples to support our main theorem. Finally, In Section 5, concludes the paper.

2 Auxiliary Facts and Notations

In this section, we will review several definitions and theorems, providing additional facts to enhance understanding.

Definition 1 ([13, 21]) *Let B be a bounded subset of a Banach space $\mathbb{F}$. The M.N.C types of Kuratowski and Hausdroff are expressed by,*

$$\begin{aligned} \mathfrak{m}_{\text{K}}(B) &= \inf \Big\{ \backepsilon > 0 : A \textit{ may be covered by finitely multiple sets of diameter} \le \backepsilon \Big\}, \\ \mathfrak{m}_{\text{H}}(B) &= \inf \Big\{ \backepsilon > 0 : \textit{there is a finite } \backepsilon \textit{-net for } B \in \mathbb{F} \Big\}. \end{aligned} \tag{2}$$

Theorem 1 ([30]) *Let B, $\tilde{B} \in \mathbb{F}$ and $\lambda \in \mathbb{R}$. Then*

i) $\mathfrak{m}_{\text{H}}(B) = 0$ *iff B is relatively-compact;*

ii) $B \subseteq \tilde{B} \implies \mathfrak{m}_{\text{H}} \le \mathfrak{m}_{\text{H}}(\tilde{B})$;

iii) $\mathfrak{m}_{\text{H}}(\bar{B}) = \mathfrak{m}_{\text{H}}(conv\ B) = \mathfrak{m}_{\text{H}}(B)$;

iv) $\mathfrak{m}_{\text{H}}(B \cup \tilde{B}) = \max \{\mathfrak{m}_{\text{H}}(B), \mathfrak{m}_{\text{H}}(\tilde{B})\}$;

v) $\mathfrak{m}_{\text{H}}(\lambda B) = |\lambda| \mathfrak{m}_{\text{H}}(B)$;

vi) $\mathfrak{m}_{\text{H}}(B + \tilde{B}) \le \mathfrak{m}_{\text{H}}(B) + \mathfrak{m}_{\text{H}}(\tilde{B})$.

In the following, we will operate in the space $C(\Delta_\ell)$ endowed with the norm $\|v\| = \sup\{|v(r)| : r \in \Delta_\ell\}$. Recall that the modulus of continuity of a function $v \in C(\Delta_\ell)$ is defined as,

$$\omega(v, \backepsilon) = \sup \Big\{ |v(r) - v(\tilde{r})| \ : \ |r - \tilde{r}| \le \backepsilon \Big\}.$$

Theorem 2 ([17]) *For all bounded sets $B \subset C(\Delta_\ell)$, the M.N.C (2) is equivalent to,*

$$\mathfrak{m}_{\text{H}}(B) = \lim_{\backepsilon \to 0} \sup_{v \in B} \omega(v, \backepsilon).$$

Definition 2 ([26]) *Assume that $\mathfrak{W} : \mathbb{F} \to \mathbb{F}$ be a continuous mapping of $\mathbb{F}$. $\mathfrak{W}$ is called a k-set contraction if for all bounded subset $B \subset \mathbb{F}$, $\mathfrak{W}(B)$ is bounded and $\mathfrak{m}_{\text{K}}(\mathfrak{W}(B)) \le k \mathfrak{m}_{\text{K}}(B)$, $0 < k < 1$. If $\mathfrak{m}_{\text{K}}(\mathfrak{W}(B) < \mathfrak{m}_{\text{K}}(B)$, for each $\mathfrak{m}_{\text{K}}(H) > 0$, then $\mathfrak{W}$ is called densifying (or condensing) map.*

Theorem 3 (P-theorem [30]) *Assume that $\mathfrak{W} : \bar{B}_{r_\circ} \to \mathbb{F}$ be a condensing mapping which satisfies the boundary condition, $\mathfrak{W}(v) = kv$, for some v in $\partial B_{r_\circ}$ with $k \le 1$, then the set of fixed points of $\mathfrak{W}$ in $\bar{B}_{r_\circ}$ is non-empty, where $\bar{B}_{r_\circ}$ and $\partial \bar{B}_{r_\circ}$ are closed ball at center 0 and sphere in $\mathbb{F}$ around 0, respectively with radius $r_\circ > 0$.*

3 An Existence Theorem Based on P-theorem

We consider the essential assumptions to verify the existence of a solution for FWS$\mathbb{IE}$ (1):

M1) $\mathbb{g} \in C(\Delta_\ell \times \mathbb{R}^s)$ and $\mathbb{f} \in C(\Delta_\ell \times \mathbb{R}^m)$ and there exist q_i, λ_j for $1 \le i \le s$, $1 \le j \le m$ such that

$$\left|\mathbb{g}(r, v_1, v_2, \ldots, v_s) - \mathbb{g}(r, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_s)\right| \le \sum_{i=1}^{s} q_i \left|v_i - \tilde{v}_i\right|$$

$$\left|\mathbb{f}(r, v_1, v_2, \ldots, v_m) - \mathbb{f}(r, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m)\right| \le \sum_{j=1}^{m} \lambda_j \left|v_j - \tilde{v}_j\right|,$$

M2) There exists $r_\circ > 0$ such that

$$\sup \left| \mathbb{g}\left(r, \mho_{i=1}^{s} v(\beta_i(r))\right) + \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) \int_0^r \ln|\tau - r| \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} v(\gamma_k(r))\right) \mathrm{d}\tau \right| \le r_\circ,$$

with $\sum_{i=1}^{s} q_i + \Omega \sum_{j=1}^{m} \lambda_j < 1$, where

$$\Omega = \sup \left\{ \left|\mathbb{u}(r, \tau, v_1, v_2, \ldots, v_n)\right| : r, \tau \in \Delta_\ell,\ v_i \in [-r_\circ, r_\circ],\ 1 \le i \le n \right\}.$$

Theorem 4 *Assume that (M1) and (M2) hold. Then* FWS$\mathbb{IE}$ (1) *has at least one solution in* $C(\Delta_\ell)$.

Proof. Define the operator $\mathfrak{O} : B_{r_\circ} \to C(\Delta_\ell)$ as follows

$$(\mathfrak{O}v)(r) = \mathbb{g}\left(r, \mho_{i=1}^{s} v(\beta_i(r))\right) + \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) \int_0^r \ln|\tau - r| \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} v(\gamma_k(r))\right) \mathrm{d}\tau,$$

where $B_{r_\circ} = \{v \in C(\Delta_\ell) : \|v\| \le r_\circ\}$. We divide the proof into several steps.

`Step 1`. Consider arbitrary $v, \tilde{v} \in B_{r_\circ}$ and $\backepsilon > 0$ such that $\|v - \tilde{v}\| < \backepsilon$. Then,

$$\begin{aligned}
& |(\mathfrak{O}v)(r) - (\mathfrak{O}\tilde{v})(r)| \\
= & \Bigg| \mathbb{g}\left(r, \mho_{i=1}^{s} v(\beta_i(r))\right) + \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) \int_0^r \ln|\tau - r| \mathbb{u}\big(r, \tau, \mho_{k=1}^{n} v(\gamma_k(r))\big) \mathrm{d}\tau \\
& - \mathbb{g}\left(r, \mho_{i=1}^{s} \tilde{v}(\beta_i(r))\right) - \mathbb{f}\big(r, \mho_{j=1}^{m} \tilde{v}(\alpha_j(r))\big) \int_0^r \ln|\tau - r| \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} \tilde{v}(\gamma_k(r))\right) \mathrm{d}\tau \Bigg| \\
\le & \left| \mathbb{g}\left(r, \mho_{i=1}^{s} v(\beta_i(r))\right) - \mathbb{g}\left(r, \mho_{i=1}^{s} \tilde{v}(\beta_i(r))\right) \right| \\
& + \left| \left[\mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) - \mathbb{f}\left(r, \mho_{j=1}^{m} \tilde{v}(\alpha_j(r))\right) \right] \int_0^r \ln|\tau - r| \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} \tilde{v}(\gamma_k(r))\right) \mathrm{d}\tau \right| \\
& + \left| \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) \int_0^r \ln|\tau - r| \left[\mathbb{u}\left(r, \tau, \mho_{k=1}^{n} v(\gamma_k(r))\right) - \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} \tilde{v}(\gamma_k(r))\right) \right] \mathrm{d}\tau \right| \\
\le & \left| \mathbb{g}\left(r, \mho_{i=1}^{s} v(\beta_i(r))\right) - \mathbb{g}\left(r, \mho_{i=1}^{s} \tilde{v}(\beta_i(r))\right) \right| \\
& + \left| \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) - \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) \right| \int_0^r \ln|\tau - r| \left| \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} \tilde{v}(\gamma_k(r))\right) \right| \mathrm{d}\tau \\
& + \left| \mathbb{f}\left(r, \mho_{j=1}^{m} v(\alpha_j(r))\right) \right| \int_0^r \ln|\tau - r| \left| \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} v(\gamma_k(r))\right) - \mathbb{u}\left(r, \tau, \mho_{k=1}^{n} \tilde{v}(\gamma_k(r))\right) \right| \mathrm{d}\tau \Bigg| \\
\le & \sum_{i=1}^{s} q_i \left| v(\beta_i(r)) - \tilde{v}(\beta_i(r)) \right| + \sum_{j=1}^{m} \lambda_j \left| v(\alpha_j(r)) - \tilde{v}(\alpha_j(r)) \right| \Omega \left| \ell(\ln \ell - 1) \right| + \sup(\mathbb{f}) \left| \ell(\ln \ell - 1) \right| \omega(v, \backepsilon)
\end{aligned}$$

$$\leq \quad \left[\sum_{i=1}^{s} q_i + \Omega\big|\ell(\ln\ell-1)\big|\sum_{j=1}^{m}\lambda_j\right]\|v-\tilde{v}\| + \sup(\mathbb{f})\big|\ell(\ln\ell-1)\big|\omega(v,\ni),$$

where

$$\sup(\mathbb{f}) = \Big\{\big|\mathbb{f}(r,v_1,v_2,\dots,v_m)\big| : r\in\Delta_\ell, v_i\in[-r_\circ,r_\circ],\ 1\leq i\leq n\Big\}$$

and

$$\omega(v,\quad\ni\quad) = \sup\Big\{\big|\mathbb{u}(r,\tau,v_1,v_2,\dots,v_n)-\mathbb{u}(r,\tau,\tilde{v}_1,\tilde{v}_2,\dots,\tilde{v}_n)\big| : r,\tau\in\Delta_\ell,$$
$$v_i,\tilde{v}_i\in[-r_\circ,r_\circ],\ 1\leq i\leq n,\ \|v-\tilde{v}\|\leq\ni\Big\}.$$

The uniform continuity of $\widehat{\mathbb{u}} = \mathbb{u}(r,\tau,v_1,v_2,\dots,v_n)$ on $\Delta_\ell^2\times[-r_\circ,r_\circ]^n$ yields $\omega(v,\ni)\to 0$ as $\ni\to 0$. Then, the operator $\mathfrak{O}$ is continuous on $B_{r_\circ}$.

`Step 2`. We show that the operator $\mathfrak{O}$ is a condensing map in view of measure $\mathfrak{m}_{\text{H}}$. For arbitrary $\ni>0$, v in bounded subset $\Psi\subset\mathbb{F}$, and $r_1,r_2\in\Delta_\ell$ with $|r_2-r_1|\leq\ni$, we obtain

$$\begin{aligned}
&\Big|(\mathfrak{O}v)(r_2)-(\mathfrak{O}v)(r_1)\Big|\\
=\ &\Bigg|\mathbb{g}\left(r_2,\mho_{i=1}^{s}v(\beta_i(r_2))\right)\\
&+\mathbb{f}\left(r_2,\mho_{j=1}^{m}v(\alpha_j(r_2))\right)\int_0^{r_2}\ln|\tau-r_2|\mathbb{u}\left(r_2,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\mathrm{d}\tau-\mathbb{g}\left(r_1,\mho_{i=1}^{s}v(\beta_i(r_1))\right)\\
&-\mathbb{f}\left(r_1,\mho_{j=1}^{m}v(\alpha_j(r_1))\right)\int_0^{r_1}\ln|\tau-r_1|\mathbb{u}\left(r_1,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\mathrm{d}\tau\Bigg|\\
\leq\ &\Bigg|\mathbb{g}\left(r_2,\mho_{i=1}^{s}v(\beta_i(r_2))\right)-\mathbb{g}\left(r_2,\mho_{i=1}^{s}v(\beta_i(r_1))\right)\Bigg|\\
&+\Bigg|\mathbb{g}\left(r_2,\mho_{i=1}^{s}v(\beta_i(r_1))\right)-\mathbb{g}\left(r_1,\mho_{i=1}^{s}v(\beta_i(r_1))\right)\Bigg|\\
&+\Bigg|\mathbb{f}\left(r_2,\mho_{j=1}^{m}v(\alpha_j(r_2))\right)\int_0^{r_2}\ln|\tau-r_2|\mathrm{u}\left(r_2,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\mathrm{d}\tau\\
&-\mathbb{f}\left(r_1,\mho_{j=1}^{m}v(\alpha_j(r_1))\right)\int_0^{r_1}\ln|\tau-r_1|\mathbb{u}\left(r_1,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\mathrm{d}\tau\Bigg|\\
\leq\ &\Bigg|\mathbb{g}\left(r_2,\mho_{i=1}^{s}v(\beta_i(t_2))\right)-\mathbb{g}\left(r_2,\mho_{i=1}^{s}v(\beta_i(r_1))\right)\Bigg|\\
&+\Bigg|\mathbb{g}\left(r_2,\mho_{i=1}^{s}v(\beta_i(r_1))\right)-\mathbb{g}\left(r_1,\mho_{i=1}^{s}v(\beta_i(r_1))\right)\Bigg|\\
&+\Bigg\{\Bigg|\mathbb{f}\left(r_2,\mho_{j=1}^{m}v(\alpha_j(r_2))\right)-\mathbb{f}\left(r_2,\mho_{j=1}^{m}v(\alpha_j(r_1))\right)\Bigg|\\
&+\Bigg|\mathbb{f}\left(r_2,\mho_{j=1}^{m}v(\alpha_j(t_1))\right)-\mathbb{f}\left(r_1,\mho_{j=1}^{m}v(\alpha_j(r_1))\right)\Bigg|\Bigg\}\int_0^{r_2}\ln|\tau-r_2|\mathbb{u}\left(r_2,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\mathrm{d}\tau\\
&+\Bigg|\mathbb{f}\left(r_1,\mho_{j=1}^{m}v(\alpha_j(r_1))\right)\int_0^{r_2}\ln|\tau-r_2|\Bigg[\mathbb{u}\left(r_2,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\\
&-\mathbb{u}\left(r_1,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\Bigg]\mathrm{d}\tau\Bigg|\\
&+\Bigg|\mathbb{f}\left(r_1,\mho_{j=1}^{m}v(\alpha_j(r_1))\right)\int_0^{r_2}\mathbb{u}\left(r_1,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\big(\ln(\tau-r_2)-\ln(\tau-r_1)\big)\mathrm{d}\tau\Bigg|\\
&+\Bigg|\mathbb{f}\left(r_1,\mho_{j=1}^{m}v(\alpha_j(r_1))\right)\int_{r_1}^{r_2}\ln|\tau-r_1|\mathrm{u}\left(r_1,\tau,\mho_{k=1}^{n}v(\gamma_k(\tau))\right)\mathrm{d}\tau\Bigg|
\end{aligned}$$

$$\leq \sum_{i=1}^{s} q_i \big| v(\beta_i(r_2)) - v(\beta_i(r_1)) \big| + \omega_{\mathbb{g}}(\ni) + \left[\sum_{j=1}^{m} \lambda_j \big| v(\alpha_j(r_2)) - v(\alpha_j(r_1)) \big| + \omega_{\mathbb{f}}(\ni) \right] \big| \ell(\ln \ell - 1) \big| \Omega$$
$$+ \sup(\mathbb{f}) \big| r_2(\ln r_2 - 1) \big| \omega_{\mathbb{u}}(\ni) + 2 \sup(\mathbb{f}) \Omega \, \big| \ell(\ln \ell - 1) \big| + \sup(\mathbb{f}) \Omega \big| r_2 - r_1 \big|,$$

where for $r_2, r_1 \in \Delta_\ell$, $v_i \in [-r_\circ, r_\circ]$, $1 \leq i \leq m$, $|r_2 - r_1| \leq \ni$,

$$\begin{cases} \omega_{\mathbb{g}}(\ni) = \sup \Big| \mathbb{g}(r_2, v_1, v_2, \ldots, v_m) - \mathbb{g}(r_1, v_1, v_2, \ldots, v_m) \Big|, \\ \omega_{\mathbb{f}}(\ni) = \sup \Big| \mathbb{f}(r_2, v_1, v_2, \ldots, v_m) - \mathbb{f}(r_1, v_1, v_2, \ldots, v_m) \Big|, \\ \omega_{\mathbb{u}}(\ni) = \sup \Big| \mathbb{u}(r_2, v_1, v_2, \ldots, v_m) - \mathbb{u}(r_1, v_1, v_2, \ldots, v_m) \Big|. \end{cases} \tag{3}$$

From Ineqs. (3), we obtain

$$\omega(\mathfrak{O}v, \quad \ni \quad) \leq \sum_{i=1}^{s} q_i \omega(v, \omega(\beta_j, \ni)) + \omega_{\mathbb{g}}(\varepsilon) + \left[\sum_{j=1}^{m} \lambda_j \omega(v, \omega(\alpha_i, \ni)) + \omega_{\mathbb{f}}(\ni) \right] \big| \ell(\ln \ell - 1) \big| \Omega$$
$$+ \sup(\mathbb{f}) \big| r_2(\ln r_2 - 1) \big| \omega_{\mathbb{u}}(\ni) + 2 \sup(\mathbb{f}) \Omega \big| \ell(\ln \ell - 1) \big| + \sup(\mathbb{f}) \Omega \ni .$$

By taking the supremum over Ψ and $\ni \to 0$, we

$$\mathfrak{m}_{\mathrm{H}}(\mathfrak{O}\Psi) \leq \left[\sum_{i=1}^{s} q_i + \Omega \sum_{j=1}^{m} \lambda_j \right] \mathfrak{m}_{\mathrm{H}}(\Psi).$$

Assumption (M2) together with Definition 2 imply that $\mathfrak{O}$ is a condensing map.

`Step 3`. Suppose that $v \in \partial B_{r_\circ} = \{ v \in C(\Delta_\ell) : \|v\| = r_\circ \}$ and $\mathfrak{O}v = \zeta v$. Then we get $\|\mathfrak{O}v\| = \zeta \|v\| = \zeta r_\circ$. Thus, (W3) implies that,

$$\big| (\mathfrak{O}v)(r) \big| = \left| \mathbb{g}\left(r, \overset{s}{\underset{i=1}{\mho}} v(\beta_i(r)) \right) + \mathbb{f}\left(r, \overset{m}{\underset{j=1}{\mho}} v(\alpha_j(r)) \right) \int_0^r \ln|\tau - r| \mathbb{u}\left(r, \tau, \overset{n}{\underset{k=1}{\mho}} v(\gamma_k(r)) \right) \mathrm{d}\tau \right| \leq r_\circ, \tag{4}$$

and so, $\|\mathfrak{O}v\| \leq r_\circ$ which implies that $\zeta \leq 1$. Hence, under the assumptions (M1) and (M2), the FWS𝕀𝔼 (1) has at least one solution $C(\Delta_\ell)$. ■

Corollary 1 *Under the control conditions as follows:*

M3) Let $\mathbb{g}, \mathbb{f} \in C(\Delta_\ell \times \mathbb{R})$ and there exists q, $\lambda \in \mathbb{R}^{>0}$ such that,

$$\big| \mathbb{g}(r, v) - \mathbb{g}(r, \tilde{v}) \big| \leq q \big| v - \tilde{y} \big|, \; \big| \mathbb{f}(r, v) - \mathbb{f}(r, \tilde{v}) \big| \leq \lambda \big| v - \tilde{y} \big|.$$

M4) There exists $r_\circ > 0$ such that

$$\sup \left| \mathbb{g}\big(r, v(r)\big) + \mathbb{f}\big(r, v(r)\big) \int_0^r \ln|\tau - r| \mathbb{u}\big(r, \tau, v(r)\big) \mathrm{d}\tau \right| \leq r_\circ,$$

with $q + \Omega\lambda < 1$ where

$$\Omega = \sup \Big\{ \big| \mathbb{u}(r, \tau, v) \big| \; : v, \; \tau \in \Delta_\ell, v \; \in [-r_\circ, r_\circ] \Big\},$$

Then the FWS𝕀𝔼,

$$v(r) = \mathbb{g}\big(r, v(r)\big) + \mathbb{f}\big(r, v(r)\big) \int_0^r \ln|\tau - r| \mathbb{u}\big(r, \tau, v(r)\big) \mathrm{d}\tau,$$

has at least one solution in $C(\Delta_\ell)$.

Proof. The proof is relevant to Theorem 4, so we can exclude the irrelevant parts. ■

4 Applications with Numerical Examples

In this part, we give some examples of FWSIE to explain the advantage of our results.

Example 1 *Consider the following* FWSIE *in* $C(\Delta_\ell)$, $\ell = 1$,

$$v(r) = \frac{r^4}{3(1+r^4)}\left(v(\sqrt{r}) + v(r^2)\right) + \frac{r^2+v\left(\sqrt{\sin r}\right)}{12(1+r)}\int_0^1 \ln|\tau - r|\left(\sqrt[3]{v(\tau)} + \ln\left(1+\left|v\left(\tau^2\right)\right|\right)\right)\mathrm{d}\tau. \tag{5}$$

Clearly,

- $\beta_1(r) = \sqrt{r}$, $\beta_2(r) = r^2$, $\alpha_1(r) = \sqrt{\sin r}$, $\gamma_1(r) = r$, $\gamma_2(r) = r^2$,
- $\mathbb{g}(r, v_1, v_2) = \frac{r^4}{3(1+r^4)}v_1 + \frac{r^4}{3(1+r^4)}v_2$, $\mathbb{f}(r, v_1) = \frac{r^2+v_1}{12(1+r)}$,
- $\mathbb{u}(r, \tau, v_1, v_2) = \sqrt[3]{v_1} + \ln(1 + |v_2|)$.

By taking these data, we have

$$\begin{aligned}\left|\mathbb{g}(r, v_1, v_2) - \mathbb{g}(v, \tilde{v}_1, \tilde{v}_2)\right| &= \left|\tfrac{r^4}{3(1+r^4)}v_1 + \tfrac{r^4}{3(1+r^4)}v_2 - \tfrac{r^4}{3(1+r^4)}\tilde{v}_1 - \tfrac{r^4}{3(1+r^4)}\tilde{v}_2\right| \\ &\leq \tfrac{1}{3}\left|v_1 - \tilde{v}_1\right| + \tfrac{1}{3}\left|v_2 - \tilde{v}_2\right|,\end{aligned} \tag{6}$$

and

$$\left|\mathbb{f}(r, v) - \mathbb{f}(r, \tilde{v})\right| = \left|\tfrac{r^2+\tilde{v}}{12(1+r)} - \tfrac{r^2+\tilde{v}}{12(1+r)}\right| \leq \tfrac{1}{12}\left|v - \tilde{v}\right|. \tag{7}$$

Based on Ineqs. (6) *and* (7), *the assumption (M1) holds when* $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{3}$ *and* $\lambda_1 = \frac{1}{12}$. *In order to verify assumption (M2) observe that the inequality appearing in this assumption has the form,*

$$\tfrac{2}{3}r_\circ + \tfrac{r_\circ+1}{12}\left(\sqrt[3]{r_\circ} + \ln(1 + |r_\circ|)\right) \leq r_\circ. \tag{8}$$

It is easy to verify that the number $r_\circ \in [0.3103, 4.01835]$ *satisfies Ineq.* (8). *Further, for* $r_\circ = 1$, *we have* $\Omega = 1 + \ln 2 \approx 1.6931$ *and* $q_1 + q_2 + \Omega\lambda_1 = 0.8077583 < 1$. *Therefore, all conditions of Theorem 4 are satisfied. This implies that the* FWSIE (5) *has at least one solution in* $C(\Delta_\ell)$.

Example 2 *Consider* FWSIE *as form,*

$$v(r) = \frac{r^2}{2(1+r^2)} + \frac{v\left(\sqrt{r}\right)}{10} + \frac{v(\sin r)}{15} + \frac{\sqrt[3]{r}+v\left(\cos r^2\right)}{10}\int_0^1 \ln|\tau - r|\left(\sqrt{v(\tau)} + \frac{\tau^3}{8} + \frac{v^2(\tau)}{6}\right)\mathrm{d}\tau, \tag{9}$$

in $C(\Delta_\ell)$, $\ell = 1$. *It can be seen that* $\beta_1(r) = \sqrt{r}$, $\beta_2(r) = \sin r$, $\alpha_1(r) = \cos r$, $\gamma_1(r) = \gamma_2(r) = \tau$, $\alpha_1 = \cos r$, $\mathbb{g}(r, v_1, v_2) = \frac{r^2}{2(1+r^2)} + \frac{v_1}{10} + \frac{v_2}{15}$, $\mathbb{f}(r, v_1) = \frac{\sqrt[3]{r}+v_1}{10}$ *and* $\mathbb{u}(r, \tau, v_1, v_2) = \sqrt{v_1} + \frac{r^3}{8} + \frac{r_2^2}{6}$. *It is evident that*

$$\left|\mathbb{g}(r, v_1, v_2) - \mathbb{g}(r, \tilde{v}_1, \tilde{v}_2)\right| = \left|\tfrac{v_1}{10} + \tfrac{v_2}{15} - \tfrac{\tilde{v}_1}{10} - \tfrac{\tilde{v}_2}{15}\right| \leq \tfrac{1}{10}\left|v_1 - \tilde{v}_1\right| + \tfrac{1}{15}\left|v_2 - \tilde{v}_2\right|, \tag{10}$$

and

$$\left|\mathbb{f}(r, v) - \mathbb{f}(r, \tilde{v})\right| = \left|\tfrac{\sqrt[3]{r}+v}{10} - \tfrac{\sqrt[3]{r}+\tilde{v}}{10}\right| \leq \tfrac{1}{10}\left|v - \tilde{v}\right|. \tag{11}$$

Thus, Ineqs. (10) *and* (11) *confirm the assumption (M1) in Theorem 4 satisfies when* $q_1 = \frac{1}{10}$, $q_2 = \frac{1}{15}$ *and* $\lambda_1 = \frac{1}{10}$. *In order to verify assumption (M2) observe that the inequality appearing in this assumption has the form,*

$$\tfrac{1}{2} + \tfrac{r_\circ}{10} + \tfrac{1}{15} + \tfrac{r_\circ+1}{10}\left(\sqrt{r_\circ} + \tfrac{1}{8} + \tfrac{r_\circ^2}{6}\right) \leq r_\circ.$$

It is easy to verify that the number $r_\circ \in [0.87808, 5.01871]$ *satisfies the last inequality inequality. Also, for* $r_\circ = 2.114$, *we have* $\Omega = \sqrt{2} + \frac{1}{8} + \frac{2}{3} \approx 2.20588$ *and* $q_1 + q_2 + \Omega\lambda_1 = 0.38725 < 1$. *Therefore, all conditions of Theorem 4 are satisfied. This implies that the* FWSIE (5) *has at least one solution in* $C(\Delta_\ell)$.

5 Conclusion

The central aim in the theory of integral equations revolves around the existence and uniqueness of solutions. Therefore, several researchers have shared their findings and methodologies in this field. In alignment with this, the authors of this paper present a new approach using $\mathtt{M.N.C}$ and P-theorem for a nonlinear $\mathsf{FWS}\mathbb{IE}$. This method offers several advantages over similar techniques, including fewer conditions and no need to confirm the operator's mapping of a closed convex subset onto itself. The outcomes of this research are diverse and noteworthy, making it intriguing and deserving of further investigation in subsequent studies.

References

[1] S. Abbas, M. Benchohra and G. M. N'Guérékata, Advanced Fractional Differential and Integral Equations, Nova Science Pub Inc, New York, 2015.

[2] N. Adjimi, A. Boutiara, M. E. Samei, S. Etemad and S. Rezapour, On solutions of a hybrid generalized Caputo-type problem via the measure of non-compactness in the generalized version of Darbo's theorem, J. Inequal. Appl., 2023(2023), 23pp.

[3] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhauser, Basel, Switzerland, 1992.

[4] J. Alzabut, S. R. Grace, S. S. Santra, and M. E. Samei, Oscillation of even-order nonlinear dynamic equations with sublinear and superlinear neutral terms on time scales, Qual. Theory Dyn. Syst., 23(2024), 16pp.

[5] I. A. Bhat, L. N. Mishra, V. N. Mishra, C. Tunç and O. Tunç, Precision and efficiency of an interpolation approach to weakly singular integral equations, Internat. J. Numer. Methods Heat Fluid Flow, 34(2024), 1479–1499.

[6] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, Journal of Mathematical Analysis and Applications, 269(2002), 1–27.

[7] H. V. S. Chauhan, B. Singh, C. Tunç and O. Tunç, On the existence of solutions of non-linear 2D Volterra integral equations in a Banach space, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 116(2022), 11pp.

[8] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, Rend. Semin. Mat. Univ. Padova, 24(1955), 84–92.

[9] A. Das, B. Hazarika, S. K. Panda and V. Vijayakumar, An existence result for an infinite system of implicit fractional integral equations via generalized Darbo's fixed point theorem, Comput. Appl. Math., 40(2021), 143.

[10] A. Das, B. Hazarika, V. Parvaneh and M. Mursaleen, Solvability of generalized fractional order integral equations via measures of noncompactness, Math. Sci., 15(2021), 241–251.

[11] A. Deep, S. Abbas, B. Sing, M. R. Alharthi and K. S. Nisar, Solvability of functional stochastic integral equations via Darbo's fixed point theorem, Alexandria Engineering Journal, 60(2021), 5631–5636.

[12] D. Dhiman, L. N. Mishra and V. N. Mishra, Solvability of some non-linear functional integral equations via measure of noncompactness, Adv. Stud. Contemp. Math., 32(2022), 157–171.

[13] L. S. Goldenstein and A. S. Markus, On the measure of non-compactness of bounded sets and of linear operators, Studies in Algebra and Math. Anal. (Russian), (1965), 45–54.

[14] J. Hadamard, Essai sur l'étude des fonctions données par leur développement de Taylor, J. Math. Pures Appl., 4(1892), 101–186.

[15] H. A. Hammad, R. A. Rashwan, A. Nafea, M. E. Samei and S. Noeiaghdam, Stability analysis for a tripled system of fractional pantograph differential equations with nonlocal conditions, J. Vib. Control, 30(2024), 632–647.

[16] M. Kazemi and R. Ezzati, Existence of solutions for some nonlinear two dimensional Volterra integral equations via measures of noncompactness, Appl. Math. Comput., 275(2016), 165–171.

[17] M. Kazemi, R. Ezzati and A. Deep, On the solvability of non-linear fractional integral equations of product type, J. Pseudo-Differ. Oper. Appl., 39(2023), 165–171.

[18] A. A. Kilbas, Hadamard-type fractional calculus, J. Korean Math. Soc., 38(2001), 1191–1204

[19] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B. V., Amsterdam, 2006.

[20] D. Kumar, A. Yildirim, M. K. A. Kaabar, H. Rezazadeh and M. E. Samei, Exploration of some novel solutions to a coupled Schrödinger-KdV equations in the interactions of capillary-gravity waves, Math. Sci. (Springer), 18(2024), 291–303.

[21] K. Kuratowski, Sur les espaces completes, Fund. Math., 15(1934), 301–335.

[22] M. M. A. Metwali and K. Cichoń, On solutions of some delay Volterra integral problems on a half line, Nonlinear Anal. Model. Control, 26(2021), 661–677.

[23] M. M. A. Metwali and V. N. Mishra, On the measure of noncompactness in $L_p(\mathbb{R}^+)$ and applications to a product of n-integral equations, Turkish J. Math., 47(2023), 372–386.

[24] B. Mohammadaliee, V. Roomi and M. E. Samei, SEIARS model for analyzing COVID-19 pandemic process via ψ-Caputo fractional derivative and numerical simulation, Scientific Reports, 14(2024), 723.

[25] H. Nashine, R. Arab and R. Agarwal, Existence of solutions of system of functional integral equations using measure of noncompactness, Int. J. Nonlinear Anal. Appl., 12(2021), 583–595.

[26] R. Nussbaum, The Fixed Point Index and Fixed Point Theorems for k-Set Contractions, The University of ChicagoProQuest Dissertations & Theses, Chicago, 1969.

[27] V. K. Pathak, L. N. Mishra, V. N. Mishra and D. Baleanu, On the solvability of mixed-type fractional-order non-linear functional integral equations in the Banach space $c(i)$, Fractal and Fractional, 6(2022), 744.

[28] S. K. Paul, L. N. Mishra, V. N. Mishra and D. Baleanu, Analysis of mixed type nonlinear Volterra-Fredholm integral equations involving the Erdélyi-Kober fractional operator, J. King Saud Univ. Sci., 35(2023), 102949.

[29] S. K. Paul, L. N. Mishra, V. N. Mishra and D. Baleanu, An effective method for solving nonlinear integral equations involving the Riemann-Liouville fractional operator, AIMS Math., 8(2023), 17448–17469.

[30] W. V. Petryshyn, Structure of the fixed points sets of k-set contractions, Arch. Rational Mech. Anal., 40(1971), 312–328.

[31] M. Sadaf, S. Arshad, G. Akram, M. A. Bin Iqbal and M. E. Samei, Solitary wave solutions of Camassa-Holm nonlinear Schrödinger and $(3+1)$-dimensional boussinesq equations, Optical and Quantum Electronics, 56(2024), 720.

[32] H. R. Sahebi, M. Kazemi and M. E. Samei, Some existence results for a nonlinear q-integral equations via M.N.C and fixed point theorem Petryshyn, Bound. Value Probl., 2024(2024), 110.

[33] M. E. Samei, L. Karimi, M. K. A. Kaabar, R. Raeisi, J. Alzabut and F. Martínez González, Efficiency of vaccines for Covid-19 and stability analysis with fractional derivative, Comput. Methods Differ. Equ., 12(2024), 454–470.

[34] A. A. Samko, S. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives (Theorie and Applications), Yverdon, Gordan and Breach Science Publishers, 1993.

[35] A. G. Sanatee, L. Rathour, V. N. Mishra and V. Dewangan, Some fixed point theorems in regular modular metric spaces and application to Caratheodory's type anti-periodic boundary value problem, J. Anal., 31(2023), 619–632.

[36] K. Shah, B. Abdalla and T. Abdeljawad, Analysis of multipoint impulsive problem of fractional-order differential equations, Bound. Value Probl., 1(2023), 17pp.

[37] K. Shah and T. Abdeljawad, Study of radioactive decay process of uranium atoms via fractals-fractional analysis, South African Journal of Chemical Engineering, 48(2024), 63–70.

[38] K. Shah, T. Abdeljawad and B.Abdalla, On a coupled system under coupled integral boundary conditions involving non-singular differential operator, AIMS Math., 8(2023), 9890–9910.

[39] K. Shah, S. Ahmad, A. Ullah and T. Abdeljawad, Study of Chronic Myeloid Leukemia with T-cell under fractal-fractional order model, Open Physics, 22(2024), 16pp.

[40] A. Ullah, Z. Ullah, T. Abdeljawad, Z. Hammouch and K. Shah, A hybrid method for solving fuzzy Volterra integral equations of separable type kernels, J. King Saud Univ. Sci., 33(2021), 101246.