Applied Mathematics E-Notes, 26(2026), 148-151 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Generalization Of Bolzano's And Darboux's Intermediate Value Theorems*

Lazhar Bougoffa†

Received 20 September 2024

Abstract

The purpose of this short note is to present a new generalization of Bolzano's and Darboux's Intermediate Value Theorems.

1 Introduction

We begin by presenting the well-known Bolzano's and Darboux's Intermediate Value Theorems [1]:

Theorem 1 (Bolzano's Intermediate Value Theorem) *Let $I \subset \mathbb{R}$ be an interval and let f be continuous on I. If K is a number between $f(a)$ and $f(b)$, then there exists at least one point $c \in (a, b)$ such that*

$$f(c) = K.$$

Theorem 1 is a stronger generalization of the Intermediate Value Theorem, which specifically addresses continuous functions and asserts that such functions on a closed interval $[a, b]$ take on (at least once) any number that lies between two of their values.

Darboux extends this idea to derivatives and demonstrates that every derivative possesses the intermediate value property, meaning that if a derivative takes on two values at different points, it must also take on all values in between those two points. He also provided examples of functions that are differentiable but have derivatives that are discontinuous.

More precisely, Darboux's theorem says that

Theorem 2 (Darboux's Intermediate Value Theorem) *If f is differentiable on $[a, b]$ and K is a number between $f'(a)$ and $f'(b)$ with $a < b$, then there exists at least one point $c \in (a, b)$ such that*

$$f'(c) = K.$$

Darboux's theorem can be viewed as a generalization of Bolzano's theorem. The proofs of these theorems can be found in [1]. A new proof of Darboux's Theorem, relying solely on the Mean Value Theorem and the Intermediate Value Theorem, is also presented in [2] and [3]. Related theorems, such as those by Flett and Wayment, which are similar to these well-known results, are discussed in [4].

A key question arises: Can the constant K in the hypotheses of Theorems 1 and 2 be replaced, for instance, by a continuous function g? The more general form of this result can indeed be derived as an extension of these theorems, and the following results address this question.

---

*Mathematics Subject Classifications: Primary 26A24.

†Imam Mohammad Ibn Saud Islamic University, Faculty of Science, Department of Mathematics, Riyadh, Saudi Arabia

2 Main Results

New generalizations of these well-known theorems can be stated as follows:

Theorem 3 *Let f, g and h be continuous functions on I. If $a, b \in I$ with $a < b$ such that*

$$h(a) < g(x) < f(b) \quad \text{for all } x \in [a, b],$$

then there exists at least one point $c \in (a, b)$ such that

$$\int_a^b g(x)dx = (c-a)f(c) + (b-c)h(c).$$

Proof. Suppose that $h(a) < g(x) < f(b)$ for all $x \in [a, b]$. Thus

$$h(a)(b-a) < \int_a^b g(x)dx < f(b)(b-a).$$

Let

$$F(x) = (x-a)f(x) + (b-x)h(x) - \int_a^b g(x)dx.$$

Then $F(a) < 0 < F(b)$. By the Location of Roots Theorem, there exists at least one point $c \in (a, b)$ such that $F(c) = 0$. The required result then follows. ■

Corollary 1 *Let f and h be continuous functions on I. If $a, b \in I$ with $a < b$ such that*

$$h(a) < K < f(b),$$

then there exists at least one point $c \in (a, b)$ such that

$$K = \frac{(c-a)f(c) + (b-c)h(c)}{b-a}.$$

Proof. The desired result is obtained by applying Theorem 3 to the function $g(x) = K$. ■

Remark 1 *Note that if $h(x) = f(x)$ for all $x \in [a, b]$, then this Corollary reduces to Bolzano's Theorem.*

We also present a new variant of the Location of Roots Theorem:

Corollary 2 *Let f and h be continuous functions on I. If $a, b \in I$ with $a < b$ such that*

$$h(a) < 0 < f(b),$$

then there exists at least one point $c \in (a, b)$ such that

$$(c-a)f(c) + (b-c)h(c) = 0.$$

Theorem 4 *Let f and h be differentiable on I, where $I \subset \mathbb{R}$ and let g be a continuous function on I. If $a, b \in I$ with $a < b$ such that*

$$h'(a) < g(x) < f'(b) \quad \text{for all } x \in [a, b],$$

then there exists at least one point $c \in (a, b)$ such that

$$\int_a^b g(x)dx = (c-a)f'(c) + (b-c)h'(c).$$

Proof. Suppose that $h'(a) < g(x) < f'(b)$ for all $x \in [a,b]$. Thus,

$$h'(a)(b-a) < \int_a^b g(x)dx \equiv K_1 < f'(b)(b-a).$$

We define F on $[a,b]$ by

$$F(x) = K_1(x-a) - (x-a)f(x) - (b-x)h(x) + \int_a^x f(t)dt - \int_a^x h(t)dt.$$

Since F is continuous, it attains a maximum value on $[a,b]$. We have

$$F'(x) = K_1 - (x-a)f'(x) - (b-x)h'(x).$$

Since $F'(a) = K_1 - (b-a)h'(a) > 0$, the maximum of F does not occur at $x = a$. Similarly, the maximum does not occur at $x = b$, as $F'(b) = K_1 - (b-a)f'(b) < 0$. Therefore, F attains its maximum at some point $c \in (a,b)$. It follows that $F'(c) = 0$ and

$$K_1 = (c-a)f'(c) + (b-c)h'(c).$$

This completes the proof. ■

Corollary 3 *Let f and h be continuous functions on I. If $a, b \in I$ with $a < b$ such that*

$$h'(a) < K < f'(b),$$

then there exists at least one point $c \in (a,b)$ such that

$$K = \frac{(c-a)f'(c) + (b-c)h'(c)}{b-a}.$$

Proof. The desired result is obtained by applying Theorem 4 to the function $g(x) = K$. ■

Remark 2 *Note that if $g(x) = K$ and $h'(x) = f'(x)$ for all $x \in [a,b]$, then Theorem 4 reduces to Darboux's Theorem.*

3 Examples

Example 1 *This example is given to obtain the conclusion of Theorem 3. Let $f(x) = x$, $g(x) = \sin x$ and $h(x) = -x,\ x \in [\frac{\pi}{2}, \pi]$. The hypothesis of Theorem 3 holds. This can be seen from*

$$h(\frac{\pi}{2}) = -\frac{\pi}{2} < \sin x < f(\pi) = \pi,\ \ x \in [\frac{\pi}{2}, \pi].$$

It follows that there exists $\frac{\pi}{2} < c < \pi$ such that

$$\int_{\frac{\pi}{2}}^{\pi} \sin x dx = c(c - \frac{\pi}{2}) - c(\pi - c).$$

Hence $2c^2 - \frac{3\pi}{2}c - 1 = 0$. We can find $c = \frac{1}{8}\left(3\pi + \sqrt{39 + 9\pi^2}\right) \approx 2.5913516$.

Example 2 *Consider*

$$f(x) = \begin{cases} x^3 + 3x, & x \le 0, \\ x^3 + 2x, & x > 0 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} x^2, & x \le 0, \\ 2x + x^3, & x > 0. \end{cases}$$

The functions f and h are differentiable on $\mathbb{R}$ but their derivatives

$$f'(x) = \begin{cases} 3x^2+3, & x \le 0, \\ 3x^2+2, & x > 0 \end{cases} \quad \text{and} \quad h'(x) = \begin{cases} 2x, & x \le 0, \\ 2+3x^2, & x > 0 \end{cases}$$

are not continuous at $x = 0$.

The hypothesis of Corollary 3 (Theorem 4) holds, for example, on $[-1, 1]$. Consequently, Corollary 3 guarantees that the expression involving $f'(x)$ and $h'(x)$

$$\frac{(x+1)f'(x) + (1-x)h'(x)}{b-a} \tag{1}$$

takes every value between $h'(-1) = -2$ and $f'(1) = 5$ at some point in $[-1, 1]$.

In other words, this example demonstrates the power of Theorem 4, ensuring that the expression of the derivatives of differentiable functions (1) takes every intermediate value between two given points, even if the derivatives are not continuous.

Example 3 *Let f and h be differentiable in (a, b) with $a < b$ such that $f'(x) > 0$ and $h'(x) < 0$ for all $x \in (a, b)$. Define the function F by*

$$F(x) = (x-a)f'(x) + (b-x)h'(x) \quad \text{for all } x \in (a, b).$$

F is not necessarily continuous. Assume that $F(x) \neq 0$ for all $x \in (a, b)$, then F retains the same sign, positive or negative, for all $x \in (a, b)$.

If possible, suppose $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ such that $F(x_1)$ and $F(x_2)$ have opposite signs, that is,

$$F(x_1) = (x_1 - a)f'(x_1) + (b - x_1)h'(x_1) < 0$$

and

$$F(x_2) = (x_2 - a)f'(x_2) + (b - x_2)h'(x_2) > 0.$$

Hence

$$h'(x_1) < -\frac{x_1 - a}{b - x_1} f'(x_1) < 0$$

and

$$f'(x_2) > -\frac{b - x_2}{x_2 - a} h'(x_2) > 0.$$

It follows that

$$h'(x_1) < 0 < f'(x_2).$$

We now apply Theorem 4 with $g(x) = 0$ or Corollary 3 with $K = 0$ to obtain $c \in (x_1, x_2)$ such that

$$(c-a)f'(c) + (b-c)h'(c) = 0.$$

This means that $F(c) = 0$ for some number $c \in (x_1, x_2)$, which is a contradiction. Hence, F retains the same sign for all $x \in (a, b)$.

Acknowledgment. The author would like to thank the anonymous referee who provided useful and detailed comments on an earlier version of the manuscript.

References

[1] R. Bartle and D. Sherbert, Introduction to Real Analysis, New York: John Wiley and Sons, 1992.

[2] L. Olsen, A new proof of Darboux's theorem, Amer. Math. Monthly, 111(2004), 713–715.

[3] Jr. S. B. Nadler, A proof of Darboux's theorem, Amer. Math. Monthly, 117(2010), 174–175.

[4] L. Bougoffa, A generalization of Wayment's mean value theorem for integrals, Amer. Math. Monthly, 131(2024), 624–626.