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On Fixed Point Theorems In Rectangular S_b-Metric Spaces*

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Abstract

In this paper, we introduce the notion of a rectangular S_b-metric space which extends the rectangular metric space given by Branciari [3]. We establish some fixed point theorems in this new structure. We provide some applications of our results in differential and integral equations.

1 Introduction

Fréchet [6] introduced the concept of metric spaces, laying the groundwork for a pivotal area of mathematical research. Banach [2] further advanced this field by establishing the fixed point theorem, which is regarded as one of the most significant results in analysis and the primary foundation of metric fixed point theory. This theorem has been generalized in numerous directions, demonstrating its extensive applicability. Since then, metric spaces have become integral to various branches of mathematics, including functional analysis, nonlinear analysis, and topology. The structure of metric spaces has been extensively generalized by many researchers.

The concept of the b-metric space was introduced by Czerwik [4], while Sedghi et al. [13] developed the notion of the S-metric space. Various fixed point theorems for different contractive mappings have been established within these spaces. Dhanraj et al. [5] proved fixed point theorems using an orthogonal Geraghty-type contraction in Branciari b-metric spaces. Gnanaprakasam et al. [8] introduced orthogonal α-almost Istraţscu contractions and proved fixed point results in b-metric spaces. Gholidahneh et al. [7] extended modular b-metric spaces and derived fixed point results for $\alpha\hat{\nu}$-Meir-Keeler contractions. Iqbal et al. [9] introduced a generalized multivalued (α, L)-almost contraction in b-metric spaces and proved the existence and uniqueness of a fixed point. Iqbal et al. [10] introduced generalized weak contractions and established fixed point results in b-metric spaces. Mani et al. [11] obtained fixed point results in bicomplex valued b-metric spaces. Prakasam et al. [12] proved fixed point theorems for O-generalized contractions, generalizing known results and demonstrating the existence of solutions to integral equations.

Building on this foundation, Souayah and Mlaiki [14] introduced a novel structure known as the S_b-metric space, which is defined using b-metric and S-metric spaces. Motivated by these generalizations, this paper introduces the concept of a rectangular S_b-metric space, which extends the idea of a rectangular metric space. It also presents several fixed point theorems for various contractive mappings within a complete rectangular S_b-metric space. Branciari [3] defined a rectangular metric space as follows.

Definition 1 ([3]) *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping satisfying the following properties:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *for* $x, y \in X$ *and distinct* $u, v \in X \backslash \{x, y\}$*.*

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Then d is called a rectangular metric on X and (X, d) is called a rectangular metric space.

Czerwik [4] defined b-metric space as follows.

Definition 2 ([4]) *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping satisfying the following properties:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) there exists a real number $s \geq 1$ *such that*

$$d(x, y) \leq s[d(x, z) + d(z, y)] \quad \text{for all } x, y, z \in X.$$

Then d is called a b-metric on X and the ordered pair (X, d) is called a b-metric space with coefficient s.

Sedghi et al. [13] introduced the notion of an S-metric space which is defined as follows.

Definition 3 ([13]) *Let X be a non-empty set and $\overline{S} : X \times X \times X \to [0, \infty)$ be a mapping satisfying the following properties;*

(i) $\overline{S}(x, y, z) = 0$ *if and only if* $x = y = z$*;*

(ii) $\overline{S}(x, y, z) \leq \overline{S}(x, x, a) + \overline{S}(y, y, a) + \overline{S}(z, z, a)$, *for all* $a, x, y, z \in X$ *(rectangle inequality).*

Then $(X, \overline{S})$ is called a S-metric space.

Adewale and Iluno [1] introduced the notion of rectangular S-metric space as follows.

Definition 4 ([1]) *Let X be a non-empty set and $\underline{S} : X \times X \times X \to [0, \infty)$ be a mapping satisfying the following properties;*

(i) $\underline{S}(x, y, z) = 0$ *if and only if* $x = y = z$*;*

(ii) $\underline{S}(x, y, z) \leq \underline{S}(x, x, a) + \underline{S}(y, y, a) + \underline{S}(z, z, a)$ *for* $x, y, z \in X$ *and distinct* $a \in X \backslash \{x, y, z\}$*.*

Then $(X, \underline{S})$ is called a rectangular S-metric space.

Example 1 ([1]) *Let $X = \mathbb{N} \cup \{0\}$ and define $\underline{S} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by*

$$\underline{S}(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ xyz, & \text{otherwise.} \end{cases}$$

Then $(X, \underline{S})$ is called a rectangular S-metric space.

Souayah et al. [14] combined the concept of a b-metric space and an S-metric space, and introduced a new metric space, called an S_b-metric space, as follows.

Definition 5 ([14]) *Let X be a non-empty set and $s \geq 1$ be a given real number. Then a mapping $\overline{S_b}$: $X \times X \times X \to [0, \infty)$ is said to be S_b-metric on X, if following properties are satisfied;*

(i) $\overline{S_b}(x, y, z) = 0$ *if and only if* $x = y = z$*;*

(ii) $\overline{S_b}(x, x, y) = \overline{S_b}(y, y, x)$ *for all* $x, y \in X$*;*

(iii) $\overline{S_b}(x, y, z) \leq s\left[\, \overline{S_b}(x, x, a) + \overline{S_b}(y, y, a) + \overline{S_b}(z, z, a)\right]$ *for all* $x, y, z, a \in X$*.*

Then $(X, \overline{S_b})$ is called a S_b-metric space.

Definition 6 *Let X be a non-empty set and $s \geq 1$ be a given real number. A function $\underline{S_b} : X \times X \times X \to [0, \infty)$ is said to be a rectangular S_b-metric on X, if*

(i) $\underline{S_b}(x, y, z) = 0$ *if and only if* $x = y = z$*;*

(ii) $\underline{S_b}(x, x, y) = \underline{S_b}(y, y, x)$ *for all* $x, y \in X$*;*

(iii) $\underline{S_b}(x, y, z) \leq s\left[\underline{S_b}(x, x, a) + \underline{S_b}(y, y, a) + \underline{S_b}(z, z, a)\right]$ *for all* $x, y, z \in X$ *and distinct* $a \in X \backslash \{x, y, z\}$*.*

A pair $(X, \underline{S_b})$ is called a rectangular S_b-metric space.

Example 2 *Let X be a non-empty set and card $(X) \geq 5$. Let $P = X_1, X_2$ be a partition of X such that card $(X_1) \geq 4$. Let $s \geq 1$. Define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by*

$$\underline{S_b}(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ 4s, & \text{if } (x, y, z) \in X_1^3, \\ 2, & \text{if } (x, y, z) \notin X_1^3. \end{cases}$$

Then $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \geq 1$.

Proof. The conditions (i) and (ii) in the Definition 6 are obviously satisfied. We now prove only the triangle inequality,

$$\underline{S_b}(x, y, z) \leq s\left[\underline{S_b}(x, x, a) + \underline{S_b}(y, y, a) + \underline{S_b}(z, z, a)\right], \tag{1}$$

for all $x, y, z \in X$ and distinct $a \in X \backslash \{x, y, z\}$.

Case 1: Let $(x, y, z) \in X_1^3$. Then $x, y, z \in X_1$. This implies that

$$\underline{S_b}(x, y, z) = 4s. \tag{2}$$

Since $a \in X \backslash \{x, y, z\}$, two sub-cases arise:

Case (i) If $a \in X_1$, then $x, y, z, a \in X_1$, which implies that (x, x, a), (y, y, a) and $(z, z, a) \in X_1^3$. Hence, by the definition of $\underline{S_b}$, we have

$$\underline{S_b}(x, x, a) = \underline{S_b}(y, y, a) = \underline{S_b}(z, z, a) = 4s. \tag{3}$$

Using (2) and (3), the condition (1) is satisfied ($4s \leq 12s^2$, $s \geq 1$).

Case (ii) If $a \notin X_1$, then $x, y, z \in X_1$ and $a \notin X_1$, which implies that (x, x, a), (y, y, a) and $(z, z, a) \notin X_1^3$. Hence, by the definition of $\underline{S_b}$, we have

$$\underline{S_b}(x, x, a) = \underline{S_b}(y, y, a) = \underline{S_b}(z, z, a) = 2. \tag{4}$$

Using (2) and (4), the condition (1) is satisfied ($4s \leq 6s$, $s \geq 1$).

Case 2: Let $(x, y, z) \notin X_1^3$. Then at least one of $x, y, z \notin X_1$. Without loss of generality, suppose that $x \notin X_1$ and $y, z \in X_1$. By the definition of $\underline{S_b}$, we have

$$\underline{S_b}(x, y, z) = 2. \tag{5}$$

Since $a \in X \backslash \{x, y, z\}$, two sub-cases arise:

Case (i) If $a \in X_1$, then $(x, x, a) \notin X_1^3$ but (y, y, a) and $(z, z, a) \in X_1^3$. By the definition of $\underline{S_b}$, we obtain that

$$\underline{S_b}(x, x, a) = 2, \ \underline{S_b}(y, y, a) = \underline{S_b}(z, z, a) = 4s. \tag{6}$$

Using (5) and (6), the condition (1) is satisfied ($2 \leq 2s + 8s^2$, $s \geq 1$).

Case (ii) If $a \notin X_1$, then $(x,x,a),(y,y,a),(z,z,z) \notin X_1^3$. By the definition of $\underline{S_b}$, we have that

$$\underline{S_b}(x,x,a) = \underline{S_b}(y,y,a) = \underline{S_b}(z,z,a) = 2. \tag{7}$$

Using (5) and (7), the condition (1) is satisfied $(2 \leq 6s,\ s \geq 1)$.

■

Example 3 *Let $X = \mathbb{N} \cup \{0\}$. Define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by*

$$\underline{S_b}(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ 2t,\ \forall t \in \mathbb{N}, & \text{otherwise.} \end{cases}$$

Then $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \geq 1$.

Proof. The conditions (i) and (ii) in the Definition 6 are obviously satisfied. We now prove only the triangle inequality (1). We have, for all $x,y,z \in X$ with at least one of $x,y,z \neq 0$,

$$\underline{S_b}(x,y,z) = 2a. \tag{8}$$

Let $a \in X \backslash \{x,y,z\}$. Then

$$\underline{S_b}(x,x,a) = \underline{S_b}(y,y,a) = \underline{S_b}(z,z,a) = 2t. \tag{9}$$

Using (8) and (9), the condition (1) is satisfied $(2t \leq 6st,\ s \geq 1)$. ■

Example 4 *Let $X = \mathbb{R}$ (the set of real numbers) and define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by*

$$\underline{S_b}(x,y,z) = |x-y| + |y-z| + |z-x|.$$

Then $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \geq 1$.

Proof. Here

(i) $\underline{S_b}(x,y,z) = 0$ if and only if $x = y = z$.

(ii) $\underline{S_b}(x,x,y) = |x-x| + |x-y| + |y-x| = 2|x-y| = \underline{S_b}(y,y,x)$.

(iii) Let $a \in \mathbb{R} \backslash \{x,y,z\}$. Then

$$\begin{aligned} \underline{S_b}(x,y,z) &= |x-y| + |y-z| + |z-x| \\ &= |(x-a)+(a-y)| + |(y-a)+(a-z)| + |(z-a)+(a-x)| \\ &\leq |(x-a)| + |(a-y)| + |(y-a)| + (a-z)| + |(z-a)| + |(a-x)| \\ &= 2|x-a| + 2|y-a| + 2|z-a| \\ &= \underline{S_b}(x,x,a) + \underline{S_b}(y,y,a) + \underline{S_b}(z,z,a). \end{aligned}$$

Therefore

$$\underline{S_b}(x,y,z) \leq s\left[\underline{S_b}(x,x,a) + \underline{S_b}(y,y,a) + \underline{S_b}(z,z,a)\right], \text{ since } s \geq 1.$$

Thus, $\underline{S_b}$ satisfies all the conditions in the Definition 6. Hence, $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \geq 1$. ■

Example 5 *Let $X = [0,1]$ (the closed interval of real numbers between 0 and 1) and define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by*

$$\underline{S_b}(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}.$$

Then $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \geq 2$.

Proof. Here

(i) $\underline{S_b}(x,y,z) = 0$ if and only if $x = y = z$.

(ii) $\underline{S_b}(x,x,y) = \max\{|x-x|, |x-y|, |y-x|\} = |x-y| = \underline{S_b}(y,y,x)$.

(iii) Let $a \in X \backslash \{x,y,z\}$. Then

$$\begin{aligned}\underline{S_b}(x,y,z) &= \max\{|x-y|, |y-z|, |z-x|\} \\ &= \max\{|(x-a)+(a-y)|, |(y-a)+(a-z)|, |(z-a)+(a-x)|\} \\ &\leq \max\{|x-a|+|a-y|, |y-a|+|a-z|, |z-a|+|a-x|\} \\ &\leq \max\{|x-a|, |y-a|, |z-a|\} + \max\{|a-y|, |a-z|, |a-x|\} \\ &= 2\max\{|x-a|, |y-a|, |z-a|\}.\end{aligned}$$

Now, three cases arise here:

Case 1: If $\max\{|x-a|, |y-a|, |z-a|\} = |x-a|$, then

$$\underline{S_b}(x,y,z) \leq 2|x-a| \leq 2\left[|x-a|+|y-a|+|z-a|\right] = 2\left[\underline{S_b}(x,x,a)+\underline{S_b}(y,y,a)+\underline{S_b}(z,z,a)\right].$$

Case 2: If $\max\{|x-a|, |y-a|, |z-a|\} = |y-a|$, then

$$\underline{S_b}(x,y,z) \leq 2|y-a| \leq 2\left[|x-a|+|y-a|+|z-a|\right] = 2\left[\underline{S_b}(x,x,a)+\underline{S_b}(y,y,a)+\underline{S_b}(z,z,a)\right].$$

Case 3: If $\max\{|x-a|, |y-a|, |z-a|\} = |z-a|$, then

$$\underline{S_b}(x,y,z) \leq 2|z-a| \leq 2\left[|x-a|+|y-a|+|z-a|\right] = 2\left[\underline{S_b}(x,x,a)+\underline{S_b}(y,y,a)+\underline{S_b}(z,z,a)\right].$$

By Cases 1–3, we have

$$\underline{S_b}(x,y,z) \leq 2\left[\underline{S_b}(x,x,a)+\underline{S_b}(y,y,a)+\underline{S_b}(z,z,a)\right].$$

Thus, $\underline{S_b}$ satisfies all the conditions in the Definition 6. Hence, $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \geq 2$. ■

Example 6 *Let $X = C([0,T])$, the space of continuous functions on $[0,T]$, and define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by*

$$\underline{S_b}(x,y,z) = \sup_{t\in[0,T]} \left(|x(t)-y(t)| + |y(t)-z(t)| + |z(t)-x(t)|\right).$$

Then $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \geq 1$.

Proof. Here

(i) $\underline{S_b}(x,y,z) = 0$ if and only if $x = y = z$.

(ii)

$$\begin{aligned}\underline{S_b}(x,x,y) &= \sup_{t\in[0,T]} \{|x(t)-x(t)| + |x(t)-y(t)| + |y(t)-x(t)|\} \\ &= 2\sup_{t\in[0,T]} |x(t)-y(t)| = \underline{S_b}(y,y,x).\end{aligned}$$

(iii) Let $a \in X\backslash\{x,y,z\}$. Then

$$\begin{aligned}
\underline{S_b}(x,y,z) &= \sup_{t\in[0,T]} \left(|x(t)-y(t)| + |y(t)-z(t)| + |z(t)-x(t)|\right) \\
&= \sup_{t\in[0,T]} \big(|(x(t)-a(t)) + (a(t)-y(t))| + |(y(t)-a(t)) + (a(t)-z(t))| \\
&\quad +|(z(t)-a(t)) + (a(t)-x(t))|\big) \\
&\le 2\sup_{t\in[0,T]} (|x(t)-a(t)|) + 2\sup_{t\in[0,T]} (|y(t)-a(t)|) + 2\sup_{t\in[0,T]} (|z(t)-a(t)|) \\
&= \underline{S_b}(x,x,a) + \underline{S_b}(y,y,a) + \underline{S_b}(z,z,a) \\
&\le s\left[\underline{S_b}(x,x,a) + \underline{S_b}(y,y,a) + \underline{S_b}(z,z,a)\right] \ (\because s \ge 1).
\end{aligned}$$

Thus, $\underline{S_b}$ satisfies all the conditions in the Definition 6.

Hence, $\underline{S_b}$ is a rectangular S_b-metric on X with coefficient $s \ge 1$. ■

Definition 7 *Let $(X, \underline{S_b})$ be a rectangular S_b-metric space and $\{x_n\}$ be a sequence in X. Then*

(i) A sequence $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $\underline{S_b}(x_n, x_n, z) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = z$.

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if and only if $\underline{S_b}(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

(iii) $(X, \underline{S_b})$ is said to be a complete rectangular S_b-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$.

2 Main Results

Theorem 1 *Let X be a complete rectangular S_b-metric space and $T : X \to X$ be a self map. If there exists a real number k satisfying $0 \le k < \frac{1}{2s}$, where $s \ge 1$ is a real number, such that for every $x, y, z \in X$,*

$$\underline{S_b}(Tx, Ty, Tz) \le k\underline{S_b}(x, y, z). \tag{10}$$

Then T has a unique fixed point in X.

Proof. From inequality (10), we have

$$\underline{S_b}(Tx, Ty, Ty) \le k\underline{S_b}(x, y, y). \tag{11}$$

Suppose that T satisfies condition (11) and for an arbitrary point $x_0 \in X$, define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then

$$\underline{S_b}(x_n, x_n, x_{n+1}) = \underline{S_b}(Tx_{n-1}, Tx_{n-1}, Tx_n) \le k\underline{S_b}(x_{n-1}, x_{n-1}, x_n).$$

Setting $S_n = \underline{S_b}(x_n, x_n, x_{n+1})$, we have, $S_n \le kS_{n-1}$ and hence, we obtain $S_n \le k^n S_0$ for all n. Suppose that there exists $n \in \mathbb{N}$ such that $x_0 = x_n$. Then

$$\begin{aligned}
\underline{S_b}(x_0, x_0, Tx_0) = \underline{S_b}(x_n, x_n, Tx_n) = \underline{S_b}(x_n, x_n, x_{n+1}), \\
S_0 = S_n \le k^n S_0,
\end{aligned}$$

which is a contradiction. Since $k < \frac{1}{2s}$, $s \ge 1$ implies that $k < 1$. Hence, for all $n \in \mathbb{N}$, we have $x_0 \ne x_n$. Using the same argument, we have $x_n \ne x_m$ for all $n, m \in \mathbb{N}$ with $n \ne m$. Therefore, the terms of $\{x_n\}$ are distinct. Using condition (iii) from Definition 6, we have for all $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
\underline{S_b}(x_n, x_m, x_m) &\leq s\left[\underline{S_b}(x_n, x_n, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1})\right] \\
&= s\Bigg[\underline{S_b}(x_n, x_n, x_{n+1}) + 2\underline{S_b}(x_m, x_m, x_{n+1})\Bigg] \\
&\leq s\Bigg[S_n + 2\Bigg(s\big(\underline{S_b}(x_m, x_m, x_{n+2}) + \underline{S_b}(x_m, x_m, x_{n+2}) \\
&\quad + \underline{S_b}(x_{n+1}, x_{n+1}, x_{n+2})\big)\Bigg)\Bigg] \\
&= s\Bigg[S_n + 2s \cdot S_{n+1} + 2^2 \cdot s\underline{S_b}(x_m, x_m, x_{n+2})\Bigg] \\
&\leq s\Bigg[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots + (2s)^{m-n-1} S_{m-1}\Bigg] \\
&\leq s\Bigg[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots\Bigg] \\
&\leq s\Bigg[k^n S_0 + (2S)k^{n+1} S_0 + (2s)^2 k^{n+2} S_0 + \cdots\Bigg] \ (\because S_n \leq k^n S_0, \forall\, n \in \mathbb{N}) \\
&= sk^n \Bigg[1 + (2sk) + (2sk)^2 + \cdots\Bigg] S_0 \\
&= sk^n \left(\frac{1}{1 - 2sk}\right) S_0 \ (\because 0 \leq k < \frac{1}{2s} \text{ implies that } 2sk < 1).
\end{aligned}$$

Taking the limit as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \underline{S_b}(x_n, x_m, x_m) = 0 \quad \left(\because k < 1 \text{ implies that } \lim_{n\to\infty} k^n = 0\right).$$

Therefore, for $n, m, l \in \mathbb{N}$ with $n > m > l$, we have

$$\underline{S_b}(x_n, x_m, x_l) \leq s\Bigg[\underline{S_b}(x_n, x_n, x_{n-1}) + \underline{S_b}(x_m, x_m, x_{n-1}) + \underline{S_b}(x_l, x_l, x_{n-1})\Bigg].$$

Taking the limit as $n, m, l \to \infty$, we get

$$\lim_{n,m,l\to\infty} \underline{S_b}(x_n, x_m, x_l) = 0.$$

This shows that $\{x_n\}$ is an $\underline{S_b}$-Cauchy sequence in X. Since X is a complete rectangular S_b-metric space, there exists $u \in X$ such that $\{x_n\}$ converges to u. Now, we will show that u is a fixed point of T, i.e. $Tu = u$. Consider

$$\underline{S_b}(x_n, Tu, Tu) \leq k\underline{S_b}(x_{n-1}, u, u).$$

Taking the limit as $n \to \infty$, we get

$$\underline{S_b}(u, Tu, Tu) \leq k\underline{S_b}(u, u, u) \implies \underline{S_b}(u, Tu, Tu) \leq 0.$$

Since $\underline{S_b}(u, Tu, Tu) \geq 0$, $\underline{S_b}(u, Tu, Tu) = 0$ which implies $Tu = u$. Thus, u is a fixed point of T. Now, to show uniqueness, suppose there exists $v \in X$ such that $v \neq u$ and $Tv = v$. Then, $\underline{S_b}(Tu, Tv, Tv) \leq k\underline{S_b}(u, v, v)$ gives that $\underline{S_b}(u, v, v) \leq k\underline{S_b}(u, v, v)$ which is a contradiction unless $\underline{S_b}(u, v, v) = 0$, since $k < 1$. Therefore, we have $u = v$. This completes the uniqueness. ∎

Theorem 2 *Let X be a complete rectangular S_b-metric space and $T : X \to X$ be a mapping for which there exists a real number b satisfying $0 \le b < \frac{1}{4s+1}$, where $s \ge 1$ is a real number, such that for all $x, y, z \in X$,*

$$\underline{S_b}(Tx, Ty, Tz) \le b\Big[\underline{S_b}(x, Tx, Tx) + \underline{S_b}(y, Ty, Ty) + \underline{S_b}(z, Tz, Tz)\Big]. \tag{12}$$

Then T has a unique fixed point in X.

Proof. From inequality (12), we have

$$\underline{S_b}(Tx, Ty, Ty) \le b\Big[\underline{S_b}(x, Tx, Tx) + \underline{S_b}(y, Ty, Ty) + \underline{S_b}(z, Tz, Tz)\Big]. \tag{13}$$

Suppose T satisfies (13). Let $x_0 \in X$ be an arbitrary point in X. We define a sequence $\{x_n\}$ by $x_n = T^n(x_0)$. Consider

$$\underline{S_b}(x_n, x_n, x_{n+1}) \le b\Big[\underline{S_b}(x_{n-1}, x_{n-1}, x_n) + \underline{S_b}(x_{n-1}, x_{n-1}, x_n) + \underline{S_b}(x_n, x_n, x_{n+1})\Big].$$

It gives that

$$\underline{S_b}(x_n, x_n, x_{n+1}) \le \frac{2b}{1-b}\underline{S_b}(x_{n-1}, x_{n-1}, x_n).$$

Put $p = \frac{2b}{1-b}$. Since $b < \frac{1}{4s+1}$, we see that $p < \frac{1}{2}$. Then, we have

$$\underline{S_b}(x_n, x_n, x_{n+1}) \le p\underline{S_b}(x_{n-1}, x_{n-1}, x_n).$$

Continuing this, we obtain

$$\underline{S_b}(x_n, x_n, x_{n+1}) \le p^n\underline{S_b}(x_0, x_0, x_1).$$

Let $S_n = \underline{S_b}(x_n, x_n, x_{n+1})$. Then the above inequality implies

$$S_n \le p^n S_0. \tag{14}$$

Suppose there exists $n \in \mathbb{N}$ such that $x_0 = x_n$. Then

$$\begin{aligned} \underline{S_b}(x_0, x_0, Tx_0) &= \underline{S_b}(x_n, x_n, Tx_n), \\ \underline{S_b}(x_0, x_0, Tx_0) &= \underline{S_b}(x_n, x_n, x_{n+1}), \\ S_0 &= S_n. \end{aligned}$$

So, we see that

$$S_0 \le p^n S_0,$$

which is a contradiction since $p < \frac{1}{2}$. Hence, for all $n \in \mathbb{N}$, we have $x_0 \neq x_n$. Repeating the same argument, for all $n, m \in \mathbb{N}$ with $n \neq m$, we have $x_n \neq x_m$. Thus, the terms of $\{x_n\}$ are distinct. By repeated use of

(iii) in the Definition 6 for all distinct $x_{n+1}, x_{n+2}, \cdots, x_{m-1}$, with $m > n$, we have

$$\begin{aligned}
\underline{S_b}(x_n, x_m, x_m) &\leq s\Big[\underline{S_b}(x_n, x_n, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1})\Big] \\
&= s\Big[\underline{S_b}(x_n, x_n, x_{n+1}) + 2\underline{S_b}(x_m, x_m, x_{n+1})\Big] \\
&\leq s\Big[S_n + 2\Big(s\big(\underline{S_b}(x_m, x_m, x_{n+2}) + \underline{S_b}(x_m, x_m, x_{n+2}) + \underline{S_b}(x_{n+1}, x_{n+1}, x_{n+2})\big)\Big)\Big] \\
&= s\Big[S_n + 2s \cdot S_{n+1} + 2^2 \cdot s\underline{S_b}(x_m, x_m, x_{n+2})\Big] \\
&\leq s\Big[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots + (2s)^{m-n-1} S_{m-1}\Big] \\
&\leq s\Big[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots\Big] \\
&\leq s\Big[p^n S_0 + (2S)p^{n+1} S_0 + (2s)^2 p^{n+2} S_0 + \cdots\Big] \ \big(\text{Since } S_n \leq p^n S_0, \forall n \in \mathbb{N}\big) \\
&= sp^n\Big[1 + (2sp) + (2sp^2) + \cdots\Big] S_0 \\
&= sp^n (1 - 2sp)^{-1} S_0. \ \left(\text{Since } b < \frac{1}{4s+1} \text{ implies that } 2sp < 1\right).
\end{aligned}$$

Taking the limit as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \underline{S_b}(x_n, x_m, x_m) = 0. \tag{15}$$

For $n, m, l \in \mathbb{N}$, with $n > m > l$, we have

$$\underline{S_b}(x_n, x_m, x_l) \leq s\Big[\underline{S_b}(x_n, x_n, x_{n-1}) + \underline{S_b}(x_m, x_m, x_{n-1} + \underline{S_b}(x_l, x_l, x_{n-1})\Big].$$

Taking the limit as $n, m, l \to \infty$ and using equation (15), we get

$$\lim_{n,m,l\to\infty} \underline{S_b}(x_n, x_m, x_l) = 0.$$

Therefore, $\{x_n\}$ is an $\underline{S_b}$-Cauchy sequence in X. Since $(X, \underline{S_b})$ is complete, there exists $u \in X$ such that $\{x_n\}$ is $\underline{S_b}$-convergent to u. Now, we will show that u is a fixed point of T, i.e. $Tu = u$. Consider

$$\begin{aligned}
\underline{S_b}(x_n, Tu, Tu) &\leq \Big[\underline{S_b}(x_{n-1}, x_n, x_n) + \underline{S_b}(u, Tu, Tu) + \underline{S_b}(u, Tu, Tu)\Big] \\
&= \Big[\underline{S_b}(x_{n-1}, x_n, x_n) + 2\underline{S_b}(u, Tu, Tu)\Big].
\end{aligned}$$

Taking the limit as $n \to \infty$ and since T is $\underline{S_b}$-continuous in its variables, we get

$$\underline{S_b}(u, Tu, Tu) \leq b\Big[\underline{S_b}(u, u, u) + 2\underline{S_b}(u, Tu, Tu)\Big].$$

So $\underline{S_b}(u, Tu, Tu) \leq 2b\underline{S_b}(u, Tu, Tu)$. It implies that

$$\underline{S_b}(u, Tu, Tu) \leq 0.$$

Since $\underline{S_b}(u,Tu,Tu) \geq 0$, $\underline{S_b}(u,Tu,Tu) = 0$ which implies that $Tu = u$. Therefore, u is a fixed point of T. Now, to show uniqueness, suppose there exists $v \in X$ such that $v \neq u$ and $Tv = v$. Then

$$\underline{S_b}(Tu,Tv,Tv) \leq b\Big[\underline{S_b}(u,Tu,Tu) + \underline{S_b}(v,Tv,Tv) + \underline{S_b}(v,Tv,Tv)\Big].$$

Since $Tu = u$ and $Tv = v$, we obtain

$$\underline{S_b}(u,v,v) \leq b\Big[\underline{S_b}(u,u,u) + \underline{S_b}(v,v,v) + \underline{S_b}(v,v,v)\Big].$$

It implies that

$$\underline{S_b}(u,v,v) \leq 0,$$

which is a contradiction. Hence, we have $u = v$. This proves the uniqueness. ■

Theorem 3 *Let $(X,\underline{S_b})$ be a complete rectangular S_b-metric space and $T : X \to X$ be a mapping for which there exist real numbers a, b, c satisfying $0 \leq a < \frac{1}{2s}$, $0 \leq b < \frac{1}{2s}$ and $0 \leq c < \frac{1}{2s}$, where $s \geq 1$ is a real number with*

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} \quad \text{and} \quad \phi(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{t}{3}, & \text{if } t \neq 0, \end{cases}$$

such that for each x, y, $z \in X$,

$$\underline{S_b}(Tx,Ty,Tz) \leq \phi\big(\delta\underline{S_b}(x,y,z) + 2\delta\underline{S_b}(x,x,Tx)\big). \tag{16}$$

Then T has a unique fixed point in X.

Proof. From inequality (16), we have

$$\underline{S_b}(Tx,Ty,Ty) \leq \phi\big(\delta\underline{S_b}(x,y,y) + 2\delta\underline{S_b}(x,x,Tx)\big). \tag{17}$$

Suppose that T satisfies (17) and for an arbitrary point $x_0 \in X$, define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then

$$\begin{aligned} \underline{S_b}(x_n,x_n,x_{n+1}) &= \underline{S_b}(Tx_{n-1},Tx_{n-1},Tx_n) \\ &\leq \phi\big(\delta\underline{S_b}(x_{n-1},x_{n-1},x_n) + 2\delta\underline{S_b}(x_{n-1},x_{n-1},x_n)\big) \\ &= \phi\big(3\delta\underline{S_b}(x_{n-1},x_{n-1},x_n)\big) \\ &= \delta\underline{S_b}(x_{n-1},x_{n-1},x_n). \end{aligned}$$

Therefore

$$\underline{S_b}(x_n,x_n,x_{n+1}) \leq \delta\underline{S_b}(x_{n-1},x_{n-1},x_n).$$

Let $S_n = \underline{S_b}(x_n,x_n,x_{n+1})$. We get $S_n \leq \delta S_{n-1}$ and deduce that

$$S_n \leq \delta^n S_0, \ \forall n \in \mathbb{N}.$$

Suppose that there exists $n \in \mathbb{N}$ such that $x_0 = x_n$. Then

$$\underline{S_b}(x_0,x_0,Tx_0) = \underline{S_b}(x_n,x_n,Tx_n)$$

and

$$\underline{S_b}(x_0,x_0,x_1) = \underline{S_b}(x_n,x_n,x_{n+1}).$$

So $S_0 = S_n$. It implies that

$$S_0 \leq \delta^n S_0,$$

which is a contradiction since $\delta < \frac{1}{2}$. Hence, $x_0 \neq x_n$ for $n, m \in \mathbb{N}$. Using the same argument for all $n, m \in \mathbb{N} \cup \{0\}$ with $n \neq m$ and $x_n \neq x_m$, the terms of $\{x_n\}$ are distinct. By repeated use of (iii) in the Definition 6 for all distinct $x_{n+1}, x_{n+2}, \cdots, x_{m-1}$, with $m > n$, we have

$$\begin{aligned}
\underline{S_b}(x_n, x_m, x_m) &\leq s\Big[\underline{S_b}(x_n, x_n, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1})\Big] \\
&= s\Big[\underline{S_b}(x_n, x_n, x_{n+1}) + 2\underline{S_b}(x_m, x_m, x_{n+1})\Big] \\
&\leq s\Big[S_n + 2\Big(s\big(\underline{S_b}(x_m, x_m, x_{n+2}) + \underline{S_b}(x_m, x_m, x_{n+2}) + \underline{S_b}(x_{n+1}, x_{n+1}, x_{n+2})\big)\Big)\Big] \\
&= s\Big[S_n + 2s \cdot S_{n+1} + 2^2 \cdot s\underline{S_b}(x_m, x_m, x_{n+2})\Big] \\
&= s\Big[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots + (2s)^{m-1} S_{m-1}\Big] \\
&\leq s\Big[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots\Big] \\
&\leq s\Big[\delta^n S_0 + (2s)\delta^{n+1} S_0 + (2s)^2 \delta^{n+2} S_0 + \cdots\Big]\big(\text{ Since } S_n \leq \delta^n S_0, \forall n \in \mathbb{N} \cup \{0\}\big) \\
&= s\delta^n\Big[1 + (2s\delta) + (2s\delta)^2 + \cdots\Big] S_0 \\
&= s\delta^n (1 - 2s\delta)^{-1} S_0.
\end{aligned}$$

Therefore

$$\underline{S_b}(x_n, x_m, x_m) \leq s\delta^n (1 - 2s\delta)^{-1} S_0.$$

Taking the limit as $n, m \to \infty$, we get

$$\lim_{n,m \to \infty} \underline{S_b}(x_n, x_m, x_m) = 0. \tag{18}$$

For $n, m, l \in \mathbb{N} \cup \{0\}$ with $n > m > l$, we have

$$\underline{S_b}(x_n, x_m, x_l) \leq s\Big[\underline{S_b}(x_n, x_n, x_{n-1}) + \underline{S_b}(x_m, x_m, x_{n-1}) + \underline{S_b}(x_l, x_l, x_{n-1})\Big].$$

Taking the limit as $n, m, l \to \infty$ and using equation (18), we get

$$\lim_{n,m,l \to \infty} \underline{S_b}(x_n, x_m, x_l) = 0.$$

Therefore, $\{x_n\}$ is an $\underline{S_b}$-Cauchy sequence in X. Since $(X, \underline{S_b})$ is complete, there exists $u \in X$ such that $\{x_n\}$ is $\underline{S_b}$ convergent to u. Now, we show that u is a fixed point of T. Suppose, on the contrary, that $Tu \neq u$. Then

$$\underline{S_b}(x_n, Tu, Tu) \leq \phi\big(\delta\underline{S_b}(x_{n-1}, u, u) + 2\delta\underline{S_b}(x_{n-1}, x_{n-1}, x_n)\big).$$

Taking the limit as $n \to \infty$ and since T is $\underline{S_b}$-continuous in its variables, we get

$$\underline{S_b}(u, Tu, Tu) \leq \phi\big(\delta\underline{S_b}(u, u, u) + 2\delta\underline{S_b}(u, u, u)\big) = \phi(0) = 0.$$

Then $\underline{S_b}(u, Tu, Tu) \leq 0$. Since $\underline{S_b}(u, Tu, Tu) \geq 0$, we obtain that $\underline{S_b}(u, Tu, Tu) = 0$, i.e. $Tu = u$. Thus, u is a fixed point of T. Now, to show uniqueness, suppose there exists $v \in X$ such that $v \neq u$ and $Tv = v$. Then

$$\underline{S_b}(Tu, Tv, Tv) \leq \phi\big(\delta\underline{S_b}(u, v, v) + 2\delta\underline{S_b}(u, u, Tu)\big).$$

Using $Tu = u$ and $Tv = v$, we get $\underline{S_b}(u, v, v) \leq 0$. Thus $\underline{S_b}(u, v, v) = 0$ since $\underline{S_b}(u, v, v) \geq 0$. Hence, we have $u = v$. This proves the uniqueness. ■

Theorem 4 *Let* $(X, \underline{S_b})$ *be a complete rectangular* S_b*-metric space and* $T : X \to X$ *be a mapping for which there exist real numbers* a, b, c *satisfying* $0 \le a < \frac{1}{2s}$, $0 \le b < \frac{1}{2s}$ *and* $0 \le c < \frac{1}{2s}$, *where* $s \ge 1$ *is a real number with*

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$$

such that for each $x, y, z \in X$,

$$\underline{S_b}(Tx, Ty, Tz) \le \phi\big(\delta \underline{S_b}(x, y, z)\big) + \psi\big(2\delta \underline{S_b}(x, x, Tx)\big), \tag{19}$$

where the functions $\phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(t) = \frac{t}{2}$ *and* $\psi(t) = \frac{t}{4}$, *monotone increasing functions. Then* T *has a unique fixed point.*

Proof. From (19), we have

$$\underline{S_b}(Tx, Ty, Ty) \le \phi\big(\delta \underline{S_b}(x, y, y)\big) + \psi\big(2\delta \underline{S_b}(x, x, Tx)\big). \tag{20}$$

Suppose T satisfies condition (20) and for any arbitrary point $x_0 \in X$, define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then

$$\begin{aligned}\underline{S_b}(x_n, x_n, x_{n+1}) &= \underline{S_b}(Tx_{n-1}, Tx_{n-1}, Tx_n)\\ &\le \phi\big(\delta \underline{S_b}(x_{n-1}, x_{n-1}, x_n)\big) + \psi\big(2\delta \underline{S_b}(x_{n-1}, x_{n-1}, x_n)\big).\end{aligned}$$

Thus

$$\underline{S_b}(x_n, x_n, x_{n+1}) \le \delta \underline{S_b}(x_{n-1}, x_{n-1}, x_n).$$

Let $S_n = \underline{S_b}(x_n, x_n, x_{n+1})$. We get

$$S_n \le \delta S_{n-1}.$$

Continuing this, we obtain

$$S_n \le \delta^n S_0, \ \forall n \in \mathbb{N}.$$

Suppose that there exists $n \in \mathbb{N}$ such that $x_0 = x_n$. Then

$$\underline{S_b}(x_0, x_0, Tx_0) = \underline{S_b}(x_n, x_n, Tx_n) \ \text{ and } \ \underline{S_b}(x_0, x_0, x_1) = \underline{S_b}(x_n, x_n, x_{n+1}).$$

So $S_0 = S_n$. It implies that

$$S_0 \le \delta^n S_0,$$

which is a contradiction since $\delta < \frac{1}{2}$. Hence, $x_0 \neq x_n$ for $n, m \in \mathbb{N}$. Using the same argument for all $n, m \in \mathbb{N} \cup \{0\}$ with $n \neq m$ and $x_n \neq x_m$, the terms of $\{x_n\}$ are distinct. By repeated use of (iii) in the

Definition 6 for all distinct $x_{n+1}, x_{n+2}, \cdots, x_{m-1}$ with $m > n$, we have

$$\begin{aligned}
\underline{S_b}(x_n, x_m, x_m) &\leq s\Big[\underline{S_b}(x_n, x_n, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1}) + \underline{S_b}(x_m, x_m, x_{n+1})\Big] \\
&= s\Big[\underline{S_b}(x_n, x_n, x_{n+1}) + 2\underline{S_b}(x_m, x_m, x_{n+1})\Big] \\
&\leq s\Big[S_n + 2\Big(s\big(\underline{S_b}(x_m, x_m, x_{n+2}) + \underline{S_b}(x_m, x_m, x_{n+2}) + \underline{S_b}(x_{n+1}, x_{n+1}, x_{n+2})\big)\Big)\Big] \\
&= s\Big[S_n + 2s \cdot S_{n+1} + 2^2 \cdot s\underline{S_b}(x_m, x_m, x_{n+2})\Big] \\
&\leq s\Big[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots + (2s)^{m-n-1} S_{m-1}\Big] \\
&\leq s\Big[S_n + (2s)S_{n+1} + (2s)^2 S_{n+2} + \cdots\Big] \\
&\leq s\Big[\delta^n S_0 + (2s)\delta^{n+1} S_0 + (2s)^2 \delta^{n+2} S_0 + \cdots\Big] \cdots (\because S_n \leq \delta^n S_0, \forall\ n \in \mathbb{N} \cup \{0\}) \\
&= s\delta^n \Big[1 + (2s\delta) + (2s\delta)^2 + \cdots\Big] S_0 \\
&= s\delta^n (1 - 2s\delta)^{-1} S_0.
\end{aligned}$$

That is $\underline{S_b}(x_n, x_m, x_m) \leq s\delta^n (1 - 2s\delta)^{-1} S_0$. Taking the limit as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \underline{S_b}(x_n, x_m, x_m) = 0. \tag{21}$$

For $n, m, l \in \mathbb{N}$, with $n > m > l$, we have

$$\underline{S_b}(x_n, x_m, x_l) \leq s\Big[\underline{S_b}(x_n, x_n, x_{n-1}) + \underline{S_b}(x_m, x_m, x_{n-1}) + \underline{S_b}(x_l, x_l, x_{n-1})\Big].$$

Taking the limit as $n, m, l \to \infty$ and using equation (21), we get

$$\lim_{n,m,l\to\infty} \underline{S_b}(x_n, x_m, x_l) = 0.$$

Therefore, $\{x_n\}$ is an $\underline{S_b}$-Cauchy sequence in X. Since $(X, \underline{S_b})$ is complete, there exists $u \in X$ such that $\{x_n\}$ is $\underline{S_b}$-convergent to u. Now, we will show that u is a fixed point of T. Consider

$$\underline{S_b}(x_n, Tu, Tu) \leq \phi\big(\delta \underline{S_b}(x_{n-1}, u, u)\big) + \psi\big(2\delta \underline{S_b}(x_{n-1}, x_{n-1}, x_n)\big).$$

Taking the limit as $n \to \infty$ and since T is $\underline{S_b}$-continuous in its variables, we get

$$\underline{S_b}(u, Tu, Tu) \leq \phi\big(\delta \underline{S_b}(u, u, u)\big) + \psi\big(2\delta \underline{S_b}(u, u, u)\big) = \phi(0) + \psi(0) = 0.$$

Then $\underline{S_b}(u, Tu, Tu) \leq 0$. Since $\underline{S_b}(u, Tu, Tu) \geq 0$, we see that $\underline{S_b}(u, Tu, Tu) = 0$. Hence, we have $Tu = u$. Thus, u is a fixed point of T. Now, to show uniqueness, suppose there exists $v \in X$ such that $v \neq u$ and $Tv = v$. Then

$$\underline{S_b}(Tu, Tv, Tv) \leq \phi\big(\delta \underline{S_b}(u, v, v)\big) + \psi\big(2\delta \underline{S_b}(u, u, Tu)\big).$$

Using $Tu = u$ and $Tv = v$, we get $\underline{S_b}(u, v, v) \leq 0$ which implies that $\underline{S_b}(u, v, v) = 0$ as $\underline{S_b}(u, v, v) \geq 0$. Hence, we have $u = v$. This proves the uniqueness. ■

3 Some Examples for the Main Results

Example 7 *Let* $X = \mathbb{R}$ *and define* $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ *as follows*

$$\underline{S_b}(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Then $(X, \underline{S_b})$ *is a complete rectangular* S_b*-metric space. Define a map* $T : X \to X$ *by* $Tx = \frac{x}{3}$*. Then*

$$\underline{S_b}(Tx, Ty, Tz) = \underline{S_b}\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) = \left|\frac{x}{3} - \frac{y}{3}\right| + \left|\frac{y}{3} - \frac{z}{3}\right| + \left|\frac{z}{3} - \frac{x}{3}\right| = \frac{1}{3}\left(|x - y| + |y - z| + |z - x|\right).$$

Therefore, $\underline{S_b}(Tx, Ty, Tz) = \frac{1}{3}\underline{S_b}(x, y, z)$*. Let* $k = \frac{1}{3}$*. Since* $0 \leq k < \frac{1}{2s}(s \geq 1)$*, all the conditions of Theorem 1 are satisfied. Hence, by Theorem 1,* T *has a unique fixed point in* X*. A fixed point* x^* *satisfies* $Tx^* = x^*$*. Solving this, we get* $x^* = 0$*. Thus, the function* $Tx = \frac{x}{3}$ *in the complete rectangular* S_b*-metric space* $(X, \underline{S_b})$ *has a unique fixed point* $x = 0$*.*

Example 8 *Let* $X = [0, 1]$ *and define* $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ *as follows:*

$$\underline{S_b}(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}.$$

Then $(X, \underline{S_b})$ *is a complete rectangular-*S_b *metric space. Define a map* $T : X \to X$ *by* $Tx = \frac{x}{5}$*. Then*

$$\underline{S_b}(Tx, Ty, Tz) = \underline{S_b}\left(\frac{x}{5}, \frac{y}{5}, \frac{z}{5}\right) = \max\left\{\left|\frac{x}{5} - \frac{y}{5}\right|, \left|\frac{y}{5} - \frac{z}{5}\right|, \left|\frac{z}{5} - \frac{x}{5}\right|\right\} = \frac{1}{5}\max\{(|x - y|, |y - z|, |z - x|)\}.$$

Therefore,

$$\underline{S_b}(Tx, Ty, Tz) = \frac{1}{5}\underline{S_b}(x, y, z).$$

Let $k = \frac{1}{5}$*. Since* $0 \leq k < \frac{1}{2s}(s \geq 1)$*, all the conditions of Theorem 1 are satisfied. Hence, by Theorem 1,* T *has a unique fixed point in* X*. A fixed point* x^* *satisfies* $Tx^* = x^*$*. Solving this, we get* $x^* = 0$*. Thus, the function* $Tx = \frac{x}{5}$ *in the complete rectangular* S_b*-metric space* $(X, \underline{S_b})$ *has a unique fixed point* $x = 0$*.*

4 Application to Differential Equations

Consider the differential equation

$$x'(t) = -\lambda x(t) + g(t),$$

with initial condition $x(0) = x_0$, where λ is a positive constant and g is a continuous function. The equivalent integral form is

$$x(t) = x_0 + \int_0^t (-\lambda x(s) + g(s))\, ds.$$

Define the operator T by

$$(Tx)(t) = x_0 + \int_0^t (-\lambda x(s) + g(s))\, ds.$$

Let $X = C([0, T])$, the space of continuous functions on $[0, T]$, and define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$\underline{S_b}(x, y, z) = \sup_{t \in [0,T]} \left(|x(t) - y(t)| + |y(t) - z(t)| + |z(t) - x(t)|\right).$$

Then $\underline{S_b}$ is a rectangular S_b-metric on X. For $x, y, z \in X$,

$$|(Tx)(t) - (Ty)(t)| \leq \int_0^t |-\lambda x(s) + \lambda y(s)|\, ds \leq \lambda \int_0^t |x(s) - y(s)| ds,$$

and similarly,

$$|(Ty)(t) - (Tz)(t)| \leq \lambda \int_0^t |y(s) - z(s)| ds$$

and

$$|(Tz)(t) - (Tx)(t)| \leq \lambda \int_0^t |z(s) - x(s)| ds.$$

Thus,

$$\begin{aligned}\underline{S_b}(Tx, Ty, Tz) &= \sup_{t\in[0,T]} (|x(t) - y(t)| + |y(t) - z(t)| + |z(t) - x(t)|) \\ &\leq \sup_{t\in[0,T]} \left(\lambda \int_0^t |x(s) - y(s)| ds + \lambda \int_0^t |y(s) - z(s)| ds + \lambda \int_0^t |z(s) - x(s)| ds\right) \\ &\leq \lambda \int_0^T \left(\sup_{t\in[0,T]} |x(t) - y(t)| + \sup_{t\in[0,T]} |y(t) - z(t)| + \sup_{t\in[0,T]} |z(t) - x(t)|\right) ds.\end{aligned}$$

Since

$$\sup_{t\in[0,T]} |x(t) - y(t)| + \sup_{t\in[0,T]} |y(t) - z(t)| + \sup_{t\in[0,T]} |z(t) - x(t)| = \underline{S_b}(x, y, z),$$

we have

$$\underline{S_b}(Tx, Ty, Tz) \leq \lambda T \underline{S_b}(x, y, z).$$

If $\lambda T < \frac{1}{2s}$, then by Theorem 1, T has a unique fixed point in X. Consequently, the differential equation

$$x'(t) = -\lambda x(t) + g(t)$$

with initial condition $x(0) = x_0$, has a unique solution in the space of continuous functions.

5 Application to Integral Equations

5.1 Application to Fredholm Integral Equations

Consider the Fredholm integral equation

$$x(t) = \lambda \int_a^b K(t, s) f(s, x(s))\, ds,$$

where λ is a constant, $K(t, s)$ is a given kernel, and f is a continuous function. Define the operator T on a suitable function space X by

$$(Tx)(t) = \lambda \int_a^b K(t, s) f(s, x(s))\, ds.$$

Let $X = C([a, b])$, the space of continuous functions on $[a, b]$, and define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$\underline{S_b}(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Then $\underline{S_b}$ is a rectangular S_b-metric on X. Assume that $K(t, s)$ and $f(s, x)$ are such that for some $k \in \left[0, \frac{1}{2s}\right)$ (with $s \geq 1$),

$$|K(t, s)| \leq M \quad \text{and} \quad |f(s, x) - f(s, y)| \leq L|x - y|$$

for some constants M and L. Then, for $x, y, z \in X$,

$$|(Tx)(t) - (Ty)(t)| \leq \lambda ML \int_a^b |x(s) - y(s)| ds,$$

and similarly,

$$|(Ty)(t) - (Tz)(t)| \leq \lambda ML \int_a^b |y(s) - z(s)| ds$$

and

$$|(Tz)(t) - (Tx)(t)| \leq \lambda ML \int_a^b |z(s) - x(s)| ds.$$

Thus,

$$\begin{aligned} \underline{S_b}(Tx, Ty, Tz) &= |Tx - Ty| + |Ty - Tz| + Tz - Tx| \\ &\leq |\lambda| ML \left(\int_a^b (|x(t) - y(t)| + |y(t) - z(t)| + |z(t) - x(t)|) \, ds \right). \end{aligned}$$

Since

$$|x(t) - y(t)| + |y(t) - z(t)| + |z(t) - x(t)| = \underline{S_b}(x, y, z),$$

we have

$$\underline{S_b}(Tx, Ty, Tz) \leq |\lambda| ML \int_a^b \underline{S_b}(x, y, z) ds = |\lambda| ML(b - a) \underline{S_b}(x, y, z).$$

If $|\lambda| ML(b - a) < \frac{1}{2s}$, then by the Theorem 1, T has a unique fixed point in X. Consequently, the integral equation

$$x(t) = \lambda \int_a^b K(t, s) f(s, x(s)) ds$$

has a unique solution in the space of continuous functions.

5.2 Application to Volterra Integral Equations

Consider the Volterra integral equation of the second kind

$$x(t) = g(t) + \lambda \int_0^t K(t, s) f(s, x(s)) ds,$$

where λ is a constant, $K(t, s)$ is a given kernel, $g(t)$ is a known function, and f is a continuous function. Define the operator T on a suitable function space X by

$$(Tx)(t) = g(t) + \lambda \int_0^t K(t, s) f(s, x(s)) ds.$$

Let $X = C([0, T])$, the space of continuous functions on $[0, T]$, and define $\underline{S_b} : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$\underline{S_b}(x, y, z) = \sup_{t \in [0, T]} (|x(t) - y(t)| + |y(t) - z(t)| + |z(t) - x(t)|).$$

Assume that $K(t, s)$ and $f(s, x)$ are such that for some $k \in [0, \frac{1}{2s})$ (with $s \geq 1$),

$$|K(t, s)| \leq M \text{ and } |f(s, x) - f(s, y)| \leq L|x - y|,$$

for some constants M and L. Then, for $x, y, z \in X$,

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \left| \lambda \int_0^t K(t, s) (f(s, x(s)) - f(s, y(s))) \, ds \right| \\ &\leq \lambda \int_0^t |K(t, s)||f(s, x(s)) - f(s, y(s))| ds \leq \lambda ML \int_0^t |x(s) - y(s)| ds. \end{aligned}$$

Since $\sup_{t\in[0,T]} |x(t) - y(t)| \leq \underline{S_b}(x,y,z)$,

$$|(Tx)(t) - (Ty)(t)| \leq \lambda MLT \underline{S_b}(x,y,z).$$

Similarly,

$$|(Ty)(t) - (Tz)(t)| \leq \lambda MLT \underline{S_b}(x,y,z)$$

and

$$|(Tz)(t) - (Tx)(t)| \leq \lambda MLT \underline{S_b}(x,y,z).$$

Thus,

$$\underline{S_b}(Tx,Ty,Tz) = \sup_{t\in[0,T]} \left(|(Tx)(t) - (Ty)(t)| + |(Ty)(t) - (Tz)(t)| + |(Tz)(t) - (Tx)(t)|\right).$$

It implies that

$$\underline{S_b}(Tx,Ty,Tz) \leq 3\lambda MLT \underline{S_b}(x,y,z).$$

If $3\lambda MLT < \frac{1}{2s}$, then by the Theorem 1, T has a unique fixed point in X. Consequently, the Volterra integral equation

$$x(t) = g(t) + \lambda \int_0^t K(t,s) f(s,x(s)) ds$$

has a unique solution in the space of continuous functions.

6 Conclusions and Future Works

In this study, we established a fixed-point theorems for self-maps satisfying Banach-type contractive conditions in a complete rectangular S_b-metric space, which extends the traditional metric space framework. Additionally, we demonstrated the applicability of our results to differential and integral equations.

For future research, a broader exploration of applications in various mathematical and real-world problems can be pursued. Moreover, there is potential for studying common fixed-point theorems under different contractive conditions. Extending our results to multivalued mappings and further investigating their applications will provide new insights and directions for future study.

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