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Fixed Points Of Generalized Interpolative Meir-Keeler Type Contractions*

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Abstract

In this paper, we have established fixed point theorem employing the Meir-Keeler type contractions in a metric space endowed with a generalized interpolative. Our findings have also led to the deduction of certain related fixed point results. We also illustrate our results by an example.

1 Introduction/Preliminaries

The most important discovery in fixed point theory was developed in 1922 by Polish mathematician Banach [4], known as the Banach contraction principle. It states that every contraction mapping on a complete metric space has a single fixed point. In 1976, Jungck [8] introduced the coincidence point and common fixed point theorems as a generalization of the Banach contraction principle. In recent years, researchers have introduced weaker versions of commuting maps, resulting in exciting common fixed point outcomes (see [5, 14]). On the other hand, generalizations of the underlying space have been trending since some decades. One of such important generalizations was initiated by Turinici [17, 18] in 1986, where he proved fixed point results in a partial ordered set. In this continuation, Alam and Imdad [1, 2] generalized the Banach contraction principle using a binary relation. Since then, many relation-theoretic fixed point theorems are being studied regularly, see [2, 6] and references therein. Several researchers reported numerous fixed point results employing relatively more generalized contractions. One of the most important contributions to metric fixed point theory after Banach's well-known fixed point theorem [4] was made by Kannan [9, 10]. Later on, it was discovered that the Kannan and Banach contractions are separate entities [4]. However, Meir-Keeler [13] presented an intriguing contraction inequality that is referred to as a uniform contraction in another context. The Meir-Keeler fixed point theorem proved in 1969 [13], is a significant result in the fixed point literature. This theorem has far-reaching implications in various fields including:

- differential equations- for finding the existence and uniqueness of solutions,
- integral equations- for finding the existence and uniqueness of solutions,
- optimization problems- for checking the existence of optimal solutions.

The Meir-keeler fixed point theorem is a powerful tool for establishing the existence and uniqueness of fixed points in various mathematical contexts. Many researchers have been attracted to the direction of the Meir-Keeler fixed point theorem and its generalized forms [3, 7, 15, 16, 12]. This work explores a modified variant of the Meir-Keeler type contraction, which involves merging generalized interpolative contractions.

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2 Preliminaries

We first recall the basic definitions and results.

Definition 1 ([13]) *Let (X, d) be a complete metric space. A mapping $\mathcal{T}: X \to X$ is said to be a Meir-Keeler contraction on X, if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq d(a, b) < \epsilon + \delta \implies d(\mathcal{T}a, \mathcal{T}b) < \epsilon, \ \forall a, b \in X. \tag{1}$$

We call (1) the MK-contraction.

Theorem 1 ([13]) *On a complete metric space (X, d), any MK-contraction $\mathcal{T}: X \to X$ has a unique fixed point.*

Definition 2 ([11]) *Let (X, d) be a complete metric space. A mapping $\mathcal{T}: X \to X$ is said to be an interpolative Kannan type contraction on X, if there exists $\mu \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(\mathcal{T}a, \mathcal{T}b) \leq \mu[d(a, \mathcal{T}a)]^{\alpha}[d(b, \mathcal{T}b)]^{1-\alpha}, \tag{2}$$

for every $a, b \in X \setminus Fix(\mathcal{T})$, where $Fix(\mathcal{T}) = \{a \in X | \mathcal{T}a = a\}$.

Theorem 2 ([11]) *On a complete metric space (X, d), any interpolative Kannan-contraction $\mathcal{T}: X \to X$ has a fixed point.*

Definition 3 ([12]) *Let (X, d) be a complete metric space. A mapping $\mathcal{T}: X \to X$ is said to be an interpolative Kannan-Meir-Keeler type contraction on X, if there exists $\mu \in [0, 1)$ such that for every $a, b \in X \setminus Fix(\mathcal{T})$ we have*

(1) for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon < [d(a, \mathcal{T}a)]^{\alpha}[d(b, \mathcal{T}b)]^{1-\alpha} < \epsilon + \delta \implies d(\mathcal{T}a, \mathcal{T}a) \leq \epsilon, \tag{3}$$

(2)

$$d(\mathcal{T}a, \mathcal{T}b) \leq \mu[d(a, \mathcal{T}a)]^{\alpha}[d(b, \mathcal{T}b)]^{1-\alpha}. \tag{4}$$

We call this the KMK-interpolative contraction condition. Now, we define a generalized interpolative condition in the following way;

Definition 4 *Let (X, d) be a complete metric space. A mapping $\mathcal{T}: X \to X$ is said to be a generalized interpolative type contraction on X, if there exist $\mu \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that*

$$d(\mathcal{T}a, \mathcal{T}b) \leq \mu[d(a, \mathcal{T}a)]^{\alpha}[d(b, \mathcal{T}b)]^{\beta}, \tag{5}$$

for every $a, b \in X \setminus Fix(\mathcal{T})$, where $Fix(\mathcal{T}) = \{a \in X | \mathcal{T}a = a\}$.

3 Main Results

We start this section with the definition of generalized interpolative Meir-Keeler type contraction.

Definition 5 *Let (X, d) be a complete metric space. A mapping $\mathcal{T}: X \to X$ is said to be a generalized interpolative Meir-Keeler type contraction on X, if there exists $\mu \in [0, 1)$ such that for every $a, b \in X \setminus Fix(\mathcal{T})$ we have*

(1) for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon < [d(a, \mathcal{T}a)]^{\alpha}[d(b, \mathcal{T}b)]^{\beta} < \epsilon + \delta \implies d(\mathcal{T}a, \mathcal{T}a) \leq \epsilon, \tag{6}$$

(2) for $\alpha, \beta \in (0,1)$ *with* $\alpha + \beta < 1$,

$$d(\mathcal{T}a, \mathcal{T}b) \leq \mu[d(a, \mathcal{T}a)]^{\alpha}[d(b, \mathcal{T}b)]^{\beta}. \tag{7}$$

Theorem 3 *On a complete metric space* (X, d)*, any generalized interpolative Meir-Keeler type contraction* $\mathcal{T} : X \to X$ *has a fixed point.*

Proof. Let $a_0 \in X$ be an arbitrary point in X. We build the sequence $\{a_n\}$ by the following rule: $\mathcal{T}a_n = a_{n+1}$, for all $n \in \mathbb{N}$. Thus, by (7), we have $a = a_n$, $b = a_{n-1}$ and

$$\begin{aligned} d(a_{n+1}, a_n) &= d(\mathcal{T}a_n, \mathcal{T}a_{n-1}) \leq \mu[d(a_n, \mathcal{T}a_n)]^{\alpha}[d(a_{n-1}, \mathcal{T}a_{n-1})]^{\beta} \\ &\leq \mu[d(a_n, a_{n+1})]^{\alpha}[d(a_{n-1}, a_n)]^{\beta}. \end{aligned}$$

This implies that

$$[d(a_{n+1}, a_n)]^{(1-\alpha)} \leq \mu[d(a_{n-1}, a_n)]^{\beta},$$

i.e.,

$$d(a_{n+1}, a_n) \leq \mu^{\frac{1}{1-\alpha}}[d(a_{n-1}, a_n)]^{\frac{\beta}{1-\alpha}},$$

provided $\frac{\beta}{1-\alpha} < 1$. i.e., $\beta + \alpha < 1$. Then, the sequence $\{d(a_n, a_{n+1})\}$ is strictly decreasing and since $d(a_n, a_{n+1}) > 0$, for every $n \in \mathbb{N} \cup \{0\}$, it follows that the sequence $\{d(a_n, a_{n+1})\}$ converges to some $\Omega \leq 0$. We claim that $\Omega = 0$. Indeed, if we suppose that $\Omega > 0$, we can find $n \in \mathbb{N}$, such that

$$\Omega < d(a_n, a_{n+1}) < \Omega + \delta(\Omega),$$

for any $n \leq \mathbb{N}$. Then, since

$$\Omega < d(a_n, a_{n+1}) < [d(a_{n-1}, a_n)]^{\alpha}[d(a_n, a_{n+1})]^{\beta},$$

keeping in mind (6), it follows that $d(a_n, a_{n+1}) \leq \Omega$, for any $n \leq \mathbb{N}$. This is a contradiction, and that's why we get $\Omega = 0$. In order to show that $\{a_n\}$ is a Cauchy sequence, let $\epsilon > 0$ be fixed and we can consider that $\delta(\epsilon)$ can be chosen such that $\delta(\epsilon) < \epsilon$. Since $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$, we can find $l \in \mathbb{N}$ such that $d(a_n, a_{n+1}) < \frac{\epsilon}{2}$, for $n \geq l$, and we claim that

$$d(a_n, a_{n+p}) < \epsilon, \tag{8}$$

for any $p \in \mathbb{N}$. Of course, the above inequality (8) holds for $p = 1$. Suppose that for some p, (8) holds, we will prove it for $p + 1$. Indeed, using the triangle inequality, together with (7) we have

$$\begin{aligned} d(a_n, a_{n+p+1}) &\leq d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+p+1}) \\ &= d(a_n, a_{n+1}) + d(\mathcal{T}a_n, \mathcal{T}a_{n+p}) \\ &< d(a_n, a_{n+1}) + [d(a_n, a_{n+1})]^{\alpha}[d(a_{n+p}, a_{n+p+1})]^{\beta} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, the sequence $\{a_n\}$ is Cauchy and by the completeness of the space X it follows that there exists $\rho \in X$ such that

$$\lim_{n\to\infty} a_n = \rho. \tag{9}$$

We shall show that $\rho = \mathcal{T}\rho$. Supposing on the contrary, that $\rho \neq \mathcal{T}\rho$, by (7) we have

$$\begin{aligned} 0 &< d(\rho, \mathcal{T}\rho) \leq d(\rho, a_{n+1}) + d(a_{n+1}, \mathcal{T}\rho) = d(\rho, a_{n+1}) + d(\mathcal{T}a_n, \mathcal{T}\rho) \\ &< d(\rho, a_{n+1}) + [d(a_n, \mathcal{T}a_n]^{\alpha}[d(\rho, \mathcal{T}\rho)]^{\beta} \textit{ as } \mu < 1, \\ &= [d(a_n, \mathcal{T}a_n]^{\alpha}[d(\rho, \mathcal{T}\rho)]^{\beta} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $d(\rho, \mathcal{T}\rho) = 0$, that is, ρ is a fixed point of the mapping $\mathcal{T}$. ■

4 Numerical Example

Example 1 *Let* $X = \mathbb{R}^2$ *and* $\mathfrak{A} = \{P, Q, R, S\}$*, where* $P = (2, 0)$*,* $Q = (0, 1)$*,* $R = (3, 0)$*,* $S = (3, 1)$*. Let* $d : X \times X \to \mathbb{R}^+$ *be defined by*

$$d(A, B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

for any $A, B \in X$*,* $A = (a_1, a_2)$*,* $B = (b_1, b_2)$*, with* $a_1, a_2, b_1, b_2 \in \mathbb{R}$*. Define the mapping* $\mathcal{T} : X \to X$ *as follows* $\mathcal{T}P = \mathcal{T}R = \mathcal{T}S = P$*,* $\mathcal{T}Q = S$ *and* $\mathcal{T}\rho = \rho$ *for* $\rho \in X \setminus \mathfrak{A}$*.*

For $A, B \in X$, we evaluate the metric d in the following table. We show that a suitable μ exists for which (6) and (7) are satisfied for all $A, B \in X \setminus Fix(\mathcal{T})$.

A	$\mathcal{T}A$	B	$\mathcal{T}B$	$d(A, \mathcal{T}A)$	$d(B, \mathcal{T}B)$	d(A,B)	$d(A, \mathcal{T}A)^\alpha.d(B, \mathcal{T}B)^\beta$
Q	S	R	P	3	1	$\sqrt{10}$	3^α
Q	S	S	P	3	$\sqrt{2}$	2	$3^\alpha 2^{\beta/2}$
R	P	S	P	1	$\sqrt{2}$	1	$2^{\beta/2}$

For first pair $Q, R \in X \setminus Fix(\mathcal{T})$ and for given $\epsilon > 0$, we have (6)

$$\epsilon < 3^\alpha(\sqrt{2})^\beta < \epsilon + \delta \implies d(\mathcal{T}Q, \mathcal{T}R) = d(S, P) = \sqrt{2} \leq \epsilon$$

and (7) yields $d(\mathcal{T}Q, \mathcal{T}R) = \sqrt{2} \leq \mu 3^\alpha(\sqrt{2})^\beta$, which is valid for all $\mu > \frac{\sqrt{2}}{3}$ and $\mu \in (\frac{\sqrt{2}}{3}, 1)$.

For second pair $Q, S \in X \setminus Fix(\mathcal{T})$ and for given $\epsilon > 0$, we have (7)

$$\epsilon < 3^\alpha < \epsilon + \delta \implies d(\mathcal{T}Q, \mathcal{T}S) = d(S, P) = \sqrt{2} \leq \epsilon$$

and (7) yields $d(\mathcal{T}Q, \mathcal{T}S) = \sqrt{2} \leq \mu 3^\alpha(\sqrt{2})^\beta$, which is valid for all values $\mu \geq \frac{\sqrt{2}}{3^\alpha}$ and $\mu \in (\frac{\sqrt{2}}{3}, \sqrt{2})$.

For third pair $R, S \in X \setminus Fix(\mathcal{T})$ and for given $\epsilon > 0$, we have (6)

$$\epsilon < 2^{\frac{\beta}{2}} < \epsilon + \delta \implies d(\mathcal{T}R, \mathcal{T}S) = d(P, P) = 0 \leq \epsilon$$

and (7) yields $d(\mathcal{T}R, \mathcal{T}S) = 0 \leq \mu(\sqrt{2})^\beta$ which is valid for all values $\mu \geq 0$.

Thus for all values of μ, for which

$$\mu \in (\frac{\sqrt{2}}{3}, 1) \cap (\frac{\sqrt{2}}{3}, \sqrt{2}) \cap (0, 1) = (\frac{\sqrt{2}}{3}, 1).$$

Example 1 satisfies all the conditions (6) and (7) of Theorem 3 with suitable values of $\mu \in (\frac{\sqrt{2}}{3}, 1) \subset (0, 1)$. This validates our Theorem 3. Note that $P = (2, 0) \in \mathbb{R}^2$ is the unique fixed point of X.

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