**Applied Mathematics E-Notes, 26(2026), 59-69** © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

# Minimal Semi Completion Of $I \times J$ Doubly Substochastic Matrices*

Mohammad Dadkhah†, Ali Bayati Eshkaftaki‡

Received 20 July 2024

**Abstract**

In this study, we aim to convert a given doubly substochastic matrix $A$ into a semi doubly stochastic matrix $D$ by adding some columns. We show that by adding a minimum (cardinal) number of columns, a doubly substochastic matrix $A$ is transformed into a semi doubly stochastic matrix $D$. Such a minimum cardinality is called the semi sub-defect of $A$. Additionally, we obtain a general formula for the semi sub-defect of $I \times J$ doubly substochastic matrices. Our findings also demonstrate that for any increasable matrix $A$, the semi sub-defects of $A$ and its transpose, $A^t$, are equal.

## 1 Introduction

An $n \times n$ non-negative matrix $A = [a_{ij}]$ is called doubly stochastic if

$$\sum_{i=1}^{n} a_{ij} = 1, \text{ and } \sum_{j=1}^{n} a_{ij} = 1, \ \forall i, j = 1, \ldots, n. \tag{1}$$

Doubly substochastic matrices are defined by replacing the equalities in (1) by the inequalities $\sum_{i=1}^{n} a_{ij} \leq 1$ and $\sum_{j=1}^{n} a_{ij} \leq 1$.

Let $\{x_i : i \in I\}$ be a class of non-negative real numbers. The sum of the $x_i$ is defined by

$$\sum_{i \in I} x_i = \sup \left\{ \sum_{i \in F} x_i : F \text{ is a finite subset of } I \right\}.$$

This allows us to extend doubly (sub) stochastic matrices to infinity as the following definition states.

**Definition 1** ([5, 6, 11, 13]) *Let $I, J$ be two non-empty sets. An $I \times J$ non-negative matrix $A = [a_{ij}]_{i \in I, j \in J}$ is called*

*(i) doubly stochastic if $\sum_{j \in J} a_{ij} = 1$ for all $i \in I$, and $\sum_{i \in I} a_{ij} = 1$ for all $j \in J$,*

*(ii) doubly substochastic if $\sum_{j \in J} a_{ij} \leq 1$ for all $i \in I$, and $\sum_{i \in I} a_{ij} \leq 1$ for all $j \in J$,*

*(iii) semi doubly stochastic if $\sum_{j \in J} a_{ij} = 1$ for all $i \in I$ and $\sum_{i \in I} a_{ij} \leq 1$ for all $j \in J$.*

The sets of all $I \times J$ doubly stochastic, doubly substochastic, and semi doubly stochastic matrices are denoted respectively by $\mathcal{DS}(I, J)$, $\mathcal{DSS}(I, J)$ and $\mathcal{SDS}(I, J)$. It is clear that

$$\mathcal{DS}(I, J) \subseteq \mathcal{SDS}(I, J) \subseteq \mathcal{DSS}(I, J).$$

*Mathematics Subject Classifications: 15A03, 15A83, 15A04, 47L05.

†Department of Mathematics, Faculty of Mathematical Sciences, Shahrekord University, IRAN
‡Department of Mathematics, Faculty of Mathematical Sciences, Shahrekord University, IRAN

In the remainder of this paper, we will briefly write d.s. and d.s.s. respectively for doubly stochastic and doubly substochastic matrices.

We note that d.s. matrices are all square. More precisely, if $I$ and $J$ are arbitrary sets and $\mathcal{DS}(I,J) \neq \emptyset$, then $\operatorname{card}(I) = \operatorname{card}(J)$; ([5, Theorem 2.2] and [1]). Therefore, in the study of d.s. matrices we can limit ourselves to $\mathcal{DS}(I,I) := \mathcal{DS}(I,I)$, the set of all d.s. square matrices. Nevertheless, it is clear that d.s.s. matrices are not necessarily square and more precisely, there is no relationship between the (cardinal) number of rows and columns of a d.s.s. matrix, because the zero $I \times J$ matrix is a d.s.s. matrix. The next theorem guarantees that every (finitely) d.s.s. matrix is a sub-matrix of a d.s. matrix.

**Theorem 1 ([10])** *Let $A \in M_n(\mathbb{R})$ be a given matrix with non-negative entries. Then $A$ is d.s.s. if and only if $A$ has a completion, that is, $A$ is an upper left principal sub-matrix of a d.s. matrix.*

In [10], for a given d.s.s. matrix $A$, a question was discussed regarding the minimum size of a d.s. matrix which contains $A$ as a sub-matrix. This minimum size is known as the sub-defect of $A$ and is denoted by $\operatorname{sd}(A)$. In fact, the sub-defect of $A$ is the smallest integer $k$ so that by adding $k$ rows and columns, $A$ becomes a d.s. matrix.

**Definition 2 (b1, subdefect of product of)** *Let $A = [a_{ij}]$ be an $I \times J$ non-negative (real) matrix.*

*(i) The summation of $A$, denoted by $\operatorname{sum}(A)$, is defined as $\operatorname{sum}(A) = \sum_{i\in I, j\in J} a_{ij}$. $A$ is called summable if $\operatorname{sum}(A) < \infty$.*

*(ii) The row summation of $A$ is the function $\mathbf{r}_A : I \to [0,\infty]$ which is defined by $\mathbf{r}_A(i) = \sum_{j\in J} a_{ij}$;*

*(iii) The column summation of $A$ is the function $\mathbf{c}_A : J \to [0,\infty]$ which is defined by $\mathbf{c}_A(j) = \sum_{i\in I} a_{ij}$.*

The following theorem gives us the value of the sub-defect for all $n \times n$ d.s.s. matrices.

**Theorem 2 ([10])** *Let $A = [a_{ij}]$ be an $n \times n$ d.s.s. matrix. Then, the sub-defect of $A$ is $\operatorname{sd}(A) = \lceil n - \operatorname{sum}(A)\rceil$, where $\lceil x \rceil$ is the ceiling of $x$, that is the smallest integer greater than or equal to $x$.*

The notion of sub-defect has been generalized to arbitrary (finite or infinite) $I \times I$ d.s.s. matrices. For an $I \times I$ d.s.s. matrix $A$, the sub-defect of $A$, denoted by $\operatorname{sd}(A)$, is defined as the minimum cardinal number $\alpha$ such that $A$ has a completion $D \in \mathcal{DS}(I \cup J)$, where $J$ is a set of cardinality $\alpha$ which is disjoint from $I$. We refer to [3, 7, 8, 9], for more details about the sub-defect of d.s.s. matrices. In the remainder of this paper, by using cardinal numbers, we establish the *semi sub-defect* of arbitrary $I \times J$ d.s.s. matrices. For this purpose, the following two theorems are required.

**Theorem 3 (Theorem 11, Ch. 8, [12])** *Let $\alpha$ be an arbitrary ordinal number. Then the set of all ordinal numbers $\beta$ such that $\beta < \alpha$ is a well-ordered set whose ordinal number is $\alpha$.*

**Theorem 4 (Theorem 12, Ch. 8, [12])** *Any set of ordinal numbers is well-ordered.*

We organize this paper as follows. Section 2 discusses the existence of a semi completion, minimal semi completion and semi sub-defect of an arbitrary d.s.s. matrix. In Section 3, for any arbitrary d.s.s. matrix $A$ we obtain a formula for $\operatorname{ssd}(A)$, the semi sub-defect of $A$. Then, based on this formula, we get some relevant results about the semi sub-defect of d.s.s. matrices.

## 2 Minimal Semi Completion of D.S.S. Matrices

In this section, we first show any arbitrary (square or non-square, finite or infinite) d.s.s. matrix can be converted into a semi d.s. matrix by adding a (cardinal) number of columns. We then show the number of added columns can be considered minimal.

**Definition 3** *Suppose that $A = [a_{ij}]_{i\in I, j\in J}$ is an $I\times J$ d.s.s. matrix. A semi d.s. matrix $D$ is called a semi completion of $A$ if there is an index set (of columns) $J_0$ with $J_0 \cap J = \emptyset$ and an $I \times J_0$ d.s.s. matrix $B = [a_{ij}]_{i\in I, j\in J_0}$ such that $D = [a_{ij}]_{i\in I, j\in J\cup J_0}$. The cardinality of $J_0$ is called the order of the semi completion $D$.*

**Theorem 5** *Every $I \times J$ d.s.s. matrix $A = [a_{ij}]_{i\in I, j\in J}$ has a semi completion of order $\alpha = \mathrm{card}\,(I)$.*

**Proof.** Assume that $J_0$ is a set disjoint to $J$ and $\mathrm{card}\,(J_0) = \mathrm{card}\,(I)$. Let $\theta : I \to J_0$ be a bijection and $a_{ij}$ is defined for each $i \in I$ and $j \in J_0$ by

$$a_{ij} = (1 - \mathbf{r}_A(i))\delta_{\theta(i),j},$$

where $\delta_{r,s}$ denotes the Kronecker delta, which is equal to one if $r = s$ and zero otherwise. Now if we set $B = [a_{ij}]_{i\in I, j\in J_0}$, and $D = [a_{ij}]_{i\in I, j\in J\cup J_0}$, then $D = [a_{ij}]_{i\in I, j\in J\cup J_0}$ is a semi completion of $A$ which is of order $\mathrm{card}\,(J_0) = \mathrm{card}\,(I)$. ■

**Example 1** *Let $A$ be a d.s.s. matrix*

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & 1 & \cdots \\ 0 & 0 & \frac{1}{4} & 0 & \cdots \\ 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

*Then both of the matrices*

$$D_1 = \left[\begin{array}{ccccccc|ccccc} \cdots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ \cdots & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

*and*

$$D_2 = \left[\begin{array}{ccccccc|ccccc} \cdots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ \cdots & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \cdots & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

*are semi completions of $A$. As can be seen, the matrix $D_2$ is obtained by the construction presented in the proof of Theorem 5, while the matrix $D_1$ is not. This example shows that the semi completion of a doubly substochastic matrix is not necessarily unique.*

**Example 2** *Let $D = [a_{ij}]$ be an $n \times n$ d.s.s. matrix and $A = \mathrm{diag}(D, 1, 1, 1, \ldots)$. Then both of the following* *matrices*

$$D_1 = \left[\begin{array}{cccccc|cccccc} \cdots & 0 & 0 & \cdots & 0 & 1-\mathbf{r}_D(1) & a_{11} & \cdots & a_{1n} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \cdots & 1-\mathbf{r}_D(2) & 0 & a_{21} & \cdots & a_{2n} & 0 & 0 & \cdots \\ & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ \cdots & 0 & 1-\mathbf{r}_D(n) & \cdots & 0 & 0 & a_{n1} & \cdots & a_{nn} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots \\ \cdot^{\cdot^{\cdot}} & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{array}\right],$$

*and*

$$D_2 = \left[\begin{array}{cccc|cccccc} 0 & \cdots & 0 & 1-\mathbf{r}_D(1) & a_{11} & \cdots & a_{1n} & 0 & 0 & \cdots \\ 0 & \cdots & 1-\mathbf{r}_D(2) & 0 & a_{21} & \cdots & a_{2n} & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & 0 & \cdots \\ 1-\mathbf{r}_D(n) & \cdots & 0 & 0 & a_{n1} & \cdots & a_{nn} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{array}\right],$$

*are semi completions of $A$. Moreover, the semi completion $D_1$ is of order $\aleph_0$, while $D_2$ is of order $n$.*

In the remainder of this section, we will show that every $I \times J$ d.s.s matrix has a minimal semi completion.

**Theorem 6** *Let $A \in \mathcal{DSS}(I, J)$. Then there exists a (unique) minimum cardinal number $\alpha$ such that $A$ has a semi completion of order $\alpha$. Furthermore, $\alpha \leq \mathrm{card}(I)$.*

**Proof.** According to Theorem 5, by adding some columns to $A$ (that is $\mathrm{card}(I)$), it becomes a semi d.s. matrix. Now let $\daleth$ be the class of all cardinal numbers $\beta$ with $\beta \leq \mathrm{card}(I)$ and such that $A$ has a semi completion of order $\beta$. Since every cardinal number is an ordinal number, it follows from Theorems 3 and 4 that $\daleth$ is a (non-empty) well-ordered set. Therefore, $\daleth$ has a unique minimum element $\alpha$. ∎

**Definition 4** *Let $A \in \mathcal{DSS}(I, J)$. The semi sub-defect of $A$, denoted by $\mathrm{ssd}(A)$, is defined as the minimum cardinal number $\alpha$ such that $A$ has a semi completion of order $\alpha$; i.e.,*

$$\mathrm{ssd}(A) = \min\{\mathrm{card}(K) : D = [d_{ij}]_{I \times (J \cup K)} \quad \textit{is a semi completion of } A\}.$$

*We say the semi sub-defect of $A$ is finite (infinite) if $\mathrm{ssd}(A) < (\geq)\aleph_0$. A semi completion of order $\alpha = \mathrm{ssd}(A)$ is called a minimal semi completion of $A$.*

**Remark 1** *It is clear that for $A, B \in \mathcal{DSS}(I, J)$ and $\lambda \in [0, 1]$ the following assertions hold:*

- $\mathrm{ssd}(A)$ *is finite, if and only if $A$ has a semi completion of order $n \in \mathbb{N} \cup \{0\}$.*
- *If $A$ has a semi completion of order $\alpha$ and $\beta > \alpha$, then $A$ has a semi completion of order $\beta$.*
- $\mathrm{ssd}(\lambda A + (1 - \lambda)B) \leq \max\{\mathrm{ssd}(A), \mathrm{ssd}(B)\}$.
- *If $I = J$, then $\mathrm{ssd}(A) \leq \mathrm{sd}(A)$.*

From Theorem 6 the minimal completion always exists, however as we will show in Example 3, it is not necessarily unique.

**Theorem 7** *If $A$ is an $I \times J$ d.s.s. matrix, then $\mathrm{ssd}(A) \leq \mathrm{card}(I)$.*

**Proof.** The proof is obtained by using Theorem 6. ∎

# 3 Semi Sub-Defect of D.S.S. Matrices

In this section, we want to obtain the value of $\mathrm{ssd}(A)$, the semi sub-defect of $A$, for an arbitrary d.s.s matrix $A$. We also show that for all square $n \times n$ d.s.s. matrices, $\mathrm{ssd}(A)$ is exactly equal to $\mathrm{sd}(A)$, the sub-defect of $A$.

**Theorem 8** *Let $A = [a_{ij}] \in \mathcal{DSS}(I, J)$. If $\sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) < \infty$, then*

$$\mathrm{ssd}(A) = \lceil \sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) \rceil < \infty. \tag{2}$$

**Proof.** To prove the claim, it is sufficient to show the following assertions:

(i) Every semi completion of $A$ is of order $\alpha \geq \lceil \sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) \rceil$.

(ii) There exists a semi completion of $A$ which is of order $\alpha = \lceil \sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) \rceil$.

(Proof of (i)): Let $J_1$ be an arbitrary set which is disjoint from $J$ and $\alpha = \mathrm{card}\,(J_1)$. If $D = [a_{ij}]_{i\in I, j\in J\cup J_1}$ is an arbitrary semi completion of $A$, then we consider the two cases:

Case 1. If $\alpha$ is infinite, then (i) clearly holds.

Case 2. If $\alpha = k \geq 0$ is an integer number, then we define $B = [a_{ij}]_{i\in I, j\in J_1}$ and therefore, because $D$ is a semi d.s. matrix, we have

$$k = \sum_{j\in J_1} 1 \geq \sum_{j\in J_1} \sum_{i\in I} a_{ij} = \mathrm{sum}(B) = \sum_{i\in I} \sum_{j\in J_1} a_{ij} = \sum_{i\in I}(1 - \sum_{j\in J} a_{ij}),$$

which follows clearly that $k \geq \lceil \sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) \rceil$.

(Proof of (ii)). If $\alpha = k = \lceil \sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) \rceil$, then without loss of generality, we can assume that $J_1 = \{1, \ldots, k\}$ and $J \cap J_1 = \emptyset$. Now, if $a_{ij}$ is defined for each $i \in I$ and $j \in J_1 = \{1, \ldots, k\}$ by

$$a_{ij} = \frac{1 - \mathbf{r}_A(i)}{k},$$

then it is clear that $D = [a_{ij}]_{i\in I, j\in J\cup J_1}$ is a semi d.s. completion of $A$ which is of order $\alpha = k$. ■

We note that Theorem 8 contains a constructive proof that provides a minimal semi completion of each $A \in \mathcal{DSS}(I, J)$ which satisfies the condition $\sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) < \infty$. The following corollary follows directly from the previous theorem and it can be used to obtain the semi sub-defect of $I \times J$ d.s.s. matrices that contain a finite number of rows.

**Corollary 1** *Let $A$ be an $I \times J$ d.s.s. matrix. If $I$ is assumed a finite set, then*

$$\mathrm{ssd}(A) = \lceil \mathrm{card}(I) - \mathrm{sum}(A) \rceil.$$

Using the previous corollary, for every $m \times n$ d.s.s matrix $A$ we have $\mathrm{ssd}(A) = \lceil m - \mathrm{sum}(A) \rceil$. Moreover, Theorem 2 together with Corollary 1 imply that for finitely square d.s.s. matrices, the values of both sub-defect and semi sub-defect are the same. So, we have the following corollary.

**Corollary 2** *Let $A$ be an $m \times n$ d.s.s. matrix. Then the following statements hold.*

(i) $\mathrm{ssd}(A) = \lceil m - \mathrm{sum}(A) \rceil$;

(ii) *If $m = n$, then* $\mathrm{ssd}(A) = \mathrm{sd}(A)$.

**Example 3** *Suppose that*

$$A=\begin{bmatrix}\frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\\ 0 & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\\ 0 & 0 & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\end{bmatrix}.$$

*Then the matrix $A$ satisfies* $\mathrm{sum}(A)=\frac{7}{8}$. *Thus, by using Theorem 1 we have* $\mathrm{ssd}(A)=\lceil 3-\frac{7}{8}\rceil=\lceil\frac{17}{8}\rceil=3$. *The following two semi d.s. matrices*

$$D_1=\left[\begin{array}{ccc|c}0 & 0 & \frac{1}{2} & \\ 0 & \frac{1}{2} & \frac{1}{4} & A\\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \end{array}\right]=\left[\begin{array}{ccc|cccccc}0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\end{array}\right]$$

*and*

$$D_2=\left[\begin{array}{ccc|c}\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & A\\ \frac{7}{24} & \frac{7}{24} & \frac{7}{24} & \end{array}\right]=\left[\begin{array}{ccc|cccccc}\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\\ \frac{7}{24} & \frac{7}{24} & \frac{7}{24} & 0 & 0 & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots\end{array}\right]$$

*are both minimal semi completions of $A$. It is worth noting that*

$$\begin{aligned}\sum_{i=1}^{3}\Big(1-\sum_{j=1}^{\infty}a_{ij}\Big) &= \Big(1-(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots)\Big)+\Big(1-(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots)\Big)\\ &\quad+\Big(1-(\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots)\Big)\\ &= \frac{17}{8}<\infty,\end{aligned}$$

*it follows that the matrix $A$ satisfies the condition of Theorem 8. In fact, $D_2$ is obtained by the construction presented in the proof of Theorem 8(ii) while $D_1$ is not. This example shows the minimal semi completion is not necessarily unique.*

**Theorem 9** *If $A$ is an $I\times J$ semi d.s. matrix, then* $\mathrm{card}(I)\le\mathrm{card}(J)$.

**Proof.** Suppose that $A=[a_{ij}]\in\mathcal{SDS}(I,J)$. We consider the following two cases:

*Case 1.* $J$ is finite. Then we have

$$\sum_{i\in I}1=\sum_{i\in I}\sum_{j\in J}a_{ij}=\sum_{j\in J}\sum_{i\in I}a_{ij}\le\sum_{j\in J}1=\mathrm{card}\,(J),$$

which implies that $I$ is a finite set and $\mathrm{card}\,(I)\le\mathrm{card}\,(J)$.

*Case 2.* $J$ is infinite. In this case, if $I$ is a finite set, then clearly $\mathrm{card}\,(I)\le\mathrm{card}\,(J)$. Otherwise, if $I$ is assumed to be an infinite set, then we set

$$X=\{(i,j)|a_{ij}>0\},$$

$$X_i=\{(i,j)|j\in J,a_{ij}>0\},\quad\text{for all } i\in I,$$

and

$$X^j=\{(i,j)|i\in I,a_{ij}>0\},\quad\text{for all } j\in J.$$

It is clear that each of the sets $X_i$ and $X^j$ is countable, $X_i\neq\emptyset$, and we have

$$\bigsqcup_{i\in I}X_i=X=\bigsqcup_{j\in J}X^j,\tag{3}$$

where $\bigsqcup$ denotes the disjoint union of sets.

Now let $\theta : I \to X = \bigsqcup_{i\in I} X_i$ be any choice function; that is a function which satisfies $\theta(i) \in X_i$, for all $i \in I$. It is clear that $\theta$ is one-to-one, which implies that

$$\text{card}\,(I) \le \text{card}(X). \tag{4}$$

On the other hand, by using (3), we obtain that

$$\text{card}(X) \le \aleph_0\,\text{card}(J) \le \text{card}(J). \tag{5}$$

From the relations (4) and (5) we obtain that card$(I) \le$ card$(J)$. ■

The next lemma shows (in the sense of cardinal numbers) that the number of rows and columns of a d.s.s. matrix are equal under certain conditions. Note that the notation supp$(f)$ is used briefly for $\{x \in I : f(x) \neq 0\}$, the support of a function $f : I \to \mathbb{R}$ and for an $I \times J$ matrix $A = [a_{ij}]$, the notation supp$(A)$ is used when we consider $A$ as the function on $I \times J$ with $(i, j) \mapsto a_{ij}$.

**Lemma 1** *Suppose that $I$ and $J$ are arbitrary infinite sets and $X = [x_{ij}] \in \mathcal{DSS}(I, J)$ is a d.s.s. matrix so that $X$ does not have any rows or columns that are completely zero. Then* card$\,(I) =$ card$\,(\text{supp}(X)) =$ card$\,(J)$.

**Proof.** For each $i \in I$, suppose that $\theta(i) \in J$ is such that $x_{i\theta(i)} \neq 0$. Then the function $\eta : I \to \text{supp}(X)$ which is defined by $\eta(i) = (i, \theta(i))$, is one-to-one. Therefore, we have

$$\text{card}\,(I) \le \text{card}\,(\text{supp}(X)). \tag{6}$$

On the other hand, from the assumptions, $X \in \mathcal{DSS}(I, J)$, and hence,

$$R(i) = \{j \in J | (i, j) \in \text{supp}(X)\}$$

is a (non-empty) countable subset of $J$. Also, supp$(X) = \bigsqcup_{i\in I}\{i\} \times R(i)$. Thus we have

$$\text{card}\,(\text{supp}(X)) \le \text{card}\,(I). \tag{7}$$

Using the above relations (6) and (7), we see that card$\,(I) =$ card$\,(\text{supp}(X))$. In the previous argument if $X^T$, the transpose of $X$, is replaced by $X$, then we obtain that

$$\text{card}\,(J) = \text{card}\,(\text{supp}(X^T)) = \text{card}\,(\text{supp}(X)),$$

which completes the proof. ■

We note that the condition that $I$ and $J$ are both infinite cannot be removed from Lemma 1. As an example, for each $m \in \mathbb{N}$, the following matrix

$$X = \begin{bmatrix} \frac{1}{2^m} & \frac{1}{2^{m+1}} & \cdots \\ \vdots & \vdots & \\ \frac{1}{2^m} & \frac{1}{2^{m+1}} & \cdots \end{bmatrix}_{m\times\mathbb{N}}$$

is an extremely positive d.s.s. $m \times \mathbb{N}$ matrix. However, it is clear that $m \neq \text{card}\,(\mathbb{N}) \neq \aleph_0$.

**Theorem 10** *Suppose that $A = [a_{ij}] \in \mathcal{DSS}(I, J)$. If $\sum_{i\in I}\big(1-\sum_{j\in I} a_{ij}\big) = \infty$, then* ssd$(A) =$ card$\,(\text{supp}(1-\mathbf{r}_A))$.

**Proof.** Suppose that for some non-negative $I \times J_1$ matrix $B = [b_{ij}]$, $D = [A|B]$ is a minimal semi completion of $A$. Using the assumption

$$\sum_{i\in I}\big(1-\sum_{j\in I} a_{ij}\big) = \infty,$$

we conclude that $I$ is infinite. Define $I_1 := \operatorname{supp}(1 - \mathbf{r}_A)$ and $R := [b_{ij}]_{I_1 \times J_1}$. Because $D = [A|B]$ is a minimal semi completion of $A$, it follows that $B$ does not contain a zero column. It follows that $R$ also has no zero column. On the other hand, it is clear from the definition of $I_1$ that $R$ also does not contain any zero row. Moreover, the two sets $I_1$ and $J_1$ are both infinite. So, Using Lemma 1 it follows that

$$\operatorname{ssd}(A) = \operatorname{card}(J_1) = \operatorname{card}(I_1) = \operatorname{card}(\operatorname{supp}(1 - \mathbf{r}_A)).$$

■

Now, by combining Theorems 8 and 10, we reach our main result in the following theorem.

**Theorem 11** *Let $A \in \mathcal{DSS}(I, J)$. Then*

(i) $\operatorname{ssd}(A)$ *is finite if and only if* $\sum_{i\in I}(1 - \sum_{j\in J} a_{ij})$ *converges. In this case we have* $\operatorname{ssd}(A) = \lceil \sum_{i\in I}(1 - \sum_{j\in J} a_{ij})\rceil$.

(ii) $\operatorname{ssd}(A)$ *is infinite if and only if* $\sum_{i\in I}(1 - \sum_{j\in J} a_{ij})$ *diverges. In this case we have* $\operatorname{ssd}(A) = \operatorname{card}(\operatorname{supp}(1 - \mathbf{r}_A))$.

The next result is easily obtained based on the previous theorem.

**Corollary 3** *The function* $\operatorname{ssd}(\cdot)$ *which takes every* $A \in \mathcal{DSS}(I, J)$ *to* $\operatorname{ssd}(A)$*, is order-decreasing, i.e.;* $\operatorname{ssd}(A) \geq \operatorname{ssd}(B)$*, for all* $A, B \in \mathcal{DSS}(I, J)$ *with* $A \leq B$.

**Proof.** Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $I \times J$ d.s.s matrices such that $A \leq B$. Then the next two relations follow from the fact $0 \leq a_{ij} \leq b_{ij} \leq 1$:

$$\operatorname{supp}(1 - \mathbf{r}_A) \supseteq \operatorname{supp}(1 - \mathbf{r}_B), \tag{8}$$

and

$$\sum_{i\in I}(1 - \sum_{j\in J} a_{ij}) \geq \sum_{i\in I}(1 - \sum_{j\in J} b_{ij}). \tag{9}$$

Now, we consider the following two cases:

Case 1. $\operatorname{ssd}(A)$ is finite. Then, using Theorem 11 the series $\sum_{i\in I}(1 - \sum_{j\in J} a_{ij})$ converges, and then, according to (9), $\sum_{i\in I}(1 - \sum_{j\in J} b_{ij})$ is also a convergent series. So, we conclude that

$$\operatorname{ssd}(A) = \lceil \sum_{i\in I}(1 - \sum_{j\in J} a_{ij})\rceil \geq \lceil \sum_{i\in I}(1 - \sum_{j\in J} b_{ij})\rceil = \operatorname{ssd}(B).$$

Case 2. $\operatorname{ssd}(A)$ is infinite. If $\operatorname{ssd}(B)$ is finite, then it is clear that $\operatorname{ssd}(A) \geq \aleph_0 > \operatorname{ssd}(B)$. Otherwise, we assume that $\operatorname{ssd}(B)$ is also infinite. Therefore, by using (8) we have

$$\operatorname{ssd}(A) = \operatorname{card}(\operatorname{supp}(1 - \mathbf{r}_A)) \geq \operatorname{card}(\operatorname{supp}(1 - \mathbf{r}_B)) = \operatorname{ssd}(B).$$

■

**Corollary 4** *If* $A, B \in \mathcal{DSS}(I, J)$ *and* $0 \leq \lambda \leq 1$*, then*

$$\min\{\operatorname{ssd}(A), \operatorname{ssd}(B)\} \leq \operatorname{ssd}\big(\lambda A + (1 - \lambda)B\big) \leq \max\{\operatorname{ssd}(A), \operatorname{ssd}(B)\}. \tag{10}$$

**Proof.** In the two cases $\lambda = 0$ or $\lambda = 1$ the claim is obvious. Otherwise, we have $0 < \lambda < 1$, and then

$$\operatorname{supp}(1 - \mathbf{r}_{\lambda A+(1-\lambda)B}) = \operatorname{supp}(1 - \mathbf{r}_A) \cup \operatorname{supp}(1 - \mathbf{r}_B), \tag{11}$$

and

$$\sum_{i\in I}\left(1 - \sum_{j\in J}(\lambda a_{ij} + (1 - \lambda)b_{ij})\right) = \lambda \sum_{i\in I}\left(1 - \sum_{j\in J} a_{ij}\right) + (1 - \lambda)\sum_{i\in I}\left(1 - \sum_{j\in J} b_{ij}\right). \tag{12}$$

Now, if both of the series

$$\sum_{i\in I}\left(1-\sum_{j\in J}a_{ij}\right) \quad \text{and} \quad \sum_{i\in I}\left(1-\sum_{j\in J}b_{ij}\right)$$

are convergent, then using Theorem 11 and the relation (12) we can obtain the claim. Otherwise, if at least one of the series $\sum_{i\in I}\left(1-\sum_{j\in J}a_{ij}\right)$ and $\sum_{i\in I}\left(1-\sum_{j\in J}b_{ij}\right)$ diverges, then the series

$$\sum_{i\in I}\left(1-\sum_{j\in J}(\lambda a_{ij}+(1-\lambda)b_{ij})\right)$$

also diverges, which shows, again by Theorem 11, that $\mathrm{ssd}(\lambda A+(1-\lambda)B)$ is infinite and moreover, by (11) we have

$$\begin{aligned}
\min\{\mathrm{ssd}(A),\mathrm{ssd}(B)\} &\le \mathrm{card}\left(\mathrm{supp}(1-\mathbf{r}_{\lambda A+(1-\lambda)B})\right)\\
&= \mathrm{ssd}(\lambda A+(1-\lambda)B)\\
&= \mathrm{card}\left(\mathrm{supp}(1-\mathbf{r}_A)\cup\mathrm{supp}(1-\mathbf{r}_B)\right)\\
&= \max\{\mathrm{card}\,(\mathrm{supp}(1-\mathbf{r}_A)),\mathrm{card}\,(\mathrm{supp}(1-\mathbf{r}_B))\}\\
&= \max\{\mathrm{ssd}(A),\mathrm{ssd}(B)\}. \qquad (13)
\end{aligned}$$

■

**Remark 2** *Let $A,B\in\mathcal{DSS}(I,J)$. If $0<\lambda<1$ and at least one of the cardinal numbers $\mathrm{ssd}(A)$ and $\mathrm{ssd}(B)$ is infinite, then*

$$\mathrm{ssd}\big(\lambda A+(1-\lambda)B\big)=\max\{\mathrm{ssd}(A),\mathrm{ssd}(B)\}.$$

*This assertion can be proved by using the relation (13) in the proof of Corollary 4.*

Based on Corollary 4, it can be shown that the set of all $I\times J$ d.s.s matrices $A$ whose $\mathrm{ssd}(A)$ is a fixed cardinal number is convex. Thus, we have the following result.

**Corollary 5** *Let $\alpha$ be a cardinal number with $0\le\alpha\le\mathrm{card}\,(I)$. Then $\mathcal{C}_\alpha:=\{A\in\mathcal{DSS}(I,J)|\mathrm{ssd}(A)=\alpha\}$ is a non-empty convex subset of $\mathcal{DSS}(I,J)$, and $\{\mathcal{C}_\alpha|0\le\alpha\le\mathrm{card}\,(I)\}$ is a partition of $\mathcal{DSS}(I,J)$ into convex sets.*

For a square d.s.s. matrix $A\in\mathcal{DSS}(I)$, it is natural to ask what is the relationship between $\mathrm{ssd}(A)$ and $\mathrm{ssd}(A^t)$. As the next example shows, $\mathrm{ssd}(A)$ and $\mathrm{ssd}(A^t)$ are not necessarily equal.

**Example 4** *Let $L:\ell^2\to\ell^2$ be the left shift operator, which is defined by $Lx=(x_2,x_3,x_4,\ldots)$, for all $x=(x_n)\in\ell^2$. The matrix form of $L$ is equal to*

$$A=\begin{bmatrix}0&1&0&0&\cdots\\0&0&1&0&\cdots\\0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}.$$

*The adjoint of $L$ is the right shift operator $R:\ell^2\to\ell^2$, with $Rx=(0,x_1,x_2,x_3,\ldots)$ for all $x=(x_n)\in\ell^2$. It is clear that the matrix form of $R$ is equal to*

$$B=A^t=\begin{bmatrix}0&0&0&\cdots\\1&0&0&\cdots\\0&1&0&\cdots\\0&0&1&\cdots\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}.$$

*It is clear that both of $A$ and $B$ are square $\mathbb{N}\times\mathbb{N}$ d.s.s matrices. However, according to Theorem 11, we obtain that* $\mathrm{ssd}(A) = 0 \neq 1 = \mathrm{ssd}(B) = \mathrm{ssd}(A^t)$.

**Definition 5 ([2])** *Let $A = [a_{ij}]$ be an $I \times I$ doubly substochastic matrix. Then $A$ is called increasable if there exists an $I \times I$ doubly stochastic matrix $D$ such that $A \leq D$.*

**Theorem 12 (Characterization of increasable d.s.s. matrices, [2])** *Let $A = [a_{ij}] \in \mathcal{DSS}(I)$. Then the following conditions are equivalent.*

*(i)* *$A$ is increasable,*

*(ii)* $\|1-\mathbf{r}_A\|_1 = \|1-\mathbf{c}_A\|_1$ $\big($*i.e.*, $\sum_{i\in I}(1-\sum_{j\in I} a_{ij}) = \sum_{j\in I}(1-\sum_{i\in I} a_{ij})\big)$ *and if* $s := \|1-\mathbf{r}_A\|_1 = \|1-\mathbf{c}_A\|_1$, *then either $s < \infty$ or if $s = \infty$, then* $\mathrm{card}\,\big(\mathrm{supp}(1-\mathbf{r}_A)\big) = \mathrm{card}\,\big(\mathrm{supp}(1-\mathbf{c}_A)\big)$.

As the following corollary states, it follows from Theorems 11 and 12 that the equality $\mathrm{ssd}(A) = \mathrm{ssd}(A^t)$ holds for all increasable d.s.s. matrices:

**Theorem 13** *If $A \in \mathcal{DSS}(I)$ is increasable, then* $\mathrm{ssd}(A) = \mathrm{ssd}(A^t)$.

**Acknowledgment.** We are very grateful to the referee for a careful reading of the paper and for his/her valuable comments.

## References

[1] A. B. Eshkaftaki, Doubly (sub) stochastic operators on $\ell^p$ spaces, J. Math. Anal. Appl., 498(2021), 124923, 14 pp.

[2] A. B. Eshkaftaki, Increasable doubly substochastic matrices with application to infinite linear equations, Linear Multilinear Algebra, 70(2021), 5902–5912.

[3] A. B. Eshkaftaki, Minimal completion of $I\times I$ doubly substochastic matrices, Linear Multilinear Algebra, 70(2020), 2065–2077.

[4] A. B. Eshkaftaki, Sub-defect of product of $I \times I$ finite sub-defect matrices, Linear Multilinear Algebra, 72(2024), 1581–1589.

[5] F. Bahrami, A. B. Eshkaftaki and S. M. Manjegani, Linear preservers of majorization on $\ell^p(I)$, Linear Algebra Appl. 436(2012), 3177–3195.

[6] F. Bahrami, S. M. Manjegani and S. Moein, Semi doubly stochastic operators and majorization of integrable functions, Bull. Malays. Math. Sci. Soc., 44(2021), 693–703.

[7] L. Cao, A minimal completion of $(0, 1)$-matrices without total support, Linear Multilinear Algebra, 67(2019), 1522–1530.

[8] L. Cao and Z. Chen, Partitions of the polytope of doubly substochastic matrices, Linear Algebra Appl., 563(2019), 98–122.

[9] L. Cao and S. Koyuncu, Sub-defect of product of doubly substochastic matrices, Linear Multilinear Algebra, 65(2017), 653–657.

[10] L. Cao, S. Koyuncu and T. Parmer, A minimal completion of doubly substochastic matrix, Linear Multilinear Algebra, 64(2016), 2313–2334.

[11] M. Ljubenović and D. S. Djordjević, Linear preservers of weak majorization on $\ell^1(I)^+$, when $I$ is an infinite set, Linear Algebra Appl., 517(2017), 177–198.

[12] S. Y. T. Lin and Y. F. Lin, Set Theory With Applications, Sec. edition, Mariner Publishing Company, 1981.

[13] S. M. Manjegani and S. Moein. Majorization and semi doubly stochastic operators on $L^1(X)$, J. Inequal. Appl., 27(2023), 1–20.