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Number Of Zeros Of A Lacunary Type Polynomial In A Specific Region*

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Abstract

This paper seeks to establish a comprehensive analysis of zero-free regions for lacunary type polynomials. These polynomials have coefficients with specific constraints, either concerning their real and imaginary parts. Additionally, the study aims to determine bounds on the number of zeros within a designated annular region.

1 Introduction

The problem of determining zero bounds for real and complex zeros of polynomials is a classical problem that has been proven essential in various disciplines such as engineering, mathematics, and mathematical chemistry. As indicated, there is a large body of literature dealing with the problem of providing discs in the complex plane representing so-called inclusion radii (bounds) where all zeros of a univariate complex polynomial are situated. A review of the location of zeros of polynomials, where the polynomials can be factored over disks in the complex plane, can be found in [3, 6]. In accordance with the following first result, the inclusion radii, which indicate the regions within which all zeros of a univariate complex polynomial are distributed, are due to Cauchy [1].

All the zeros of a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0$$

lie in the disk

$$|z| < 1 + M,$$

where $M = \max\limits_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|$. Cauchy type polynomials have been a subject of extensive study for over a century. This research has led to numerous publications exploring various aspects, as evidenced by works such as (see [4], [5], [6], [8]). Mathematical objects related to these polynomials and the distribution of their zeros have been actively researched for a considerable period. Each year, several research papers are published in different journals, employing diverse approaches to serve different purposes.

This article focuses on zero-free regions and, in particular, the number of zeros in a polynomial within a given disc. The following result demonstrates an improvement on the Cauchy bound, assuming that the coefficients satisfy monotonicity conditions. The Eneström-Kakeya Theorem is an elegant result in mathematics that concerns polynomials with real coefficients. It states that if we have a polynomial of degree n with real coefficients arranged in non-increasing order, i.e., $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$, then all the zeros of this polynomial lie within the closed unit disk, i.e., $|z| \leq 1$.

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Furthermore, there are various extensions and generalizations of the Eneström-Kakeya Theorem found in the literature that have been proposed by different authors (references mentioned in the text). One notable generalization, established by Joyal et al. [4], presents a similar result to the Eneström-Kakeya Theorem. In this extension, if we have a polynomial of degree n with coefficients satisfying $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0$, then all its zeros lie in

$$|z| \leq \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}.$$

In general, the methods and techniques used to establish zero-free regions and regions containing zeros differ, catering to the preferences of the readers. While the theory on zero-free regions for univariate complex polynomials has been extensively developed, this article introduces a novel approach to defining zero-free regions for lacunary-type polynomials, which distinguishes it from previously published materials on the subject. Moving forward, let us explore the number of zeros of a polynomial within a given disc. The following result, which pertains to the counting of zeros of a polynomial inside a closed disk, can be found in Titchmarsh's renowned work "The Theory of Functions" (refer to [9], page 171, 2nd edition).

Theorem 1 *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in $|z| \leq R$ and $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \leq R\delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

By imposing restrictions on the coefficients of a polynomial similar to that of the Eneström-Kakeya Theorem, Mohammad [7] used a special case of Theorem 1 to prove the following results:

Theorem 2 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients such that $a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 > 0$. Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

Dewan [2] generalized Theorem 2 to polynomials with complex coefficients and obtained the following result:

Theorem 3 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real*

$$\beta, \quad |\arg a_i - \beta| \leq \alpha \leq \pi/2, \quad i = 0, 1, 2, ..., n$$

and

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_2| \geq |a_1| \geq |a_0| > 0.$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{|a_n|(1 + \cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Theorem 4 Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_i) = \alpha_i$, $\mathrm{Im}(a_i) = \beta_i$, for $i = 0, 1, 2, \ldots, n$,

$$\alpha_n \geq \alpha_{n-1} \cdots \geq \alpha_2 \geq \alpha_1 \geq \alpha_0 > 0,$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{i=0}^{n} |\beta_i|}{|a_0|}.$$

In this paper, we wish to weaken the hypothesis of the above results by considering a larger class of polynomials and obtain results with a relaxed hypothesis that improves the zero bounds in several ways. Besides, our results generalize several well-known results concerning the number of zeros of polynomials. We prove the main results in Section 2.

2 Main Results

Theorem 5 *Let*

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n, \ a_0 \neq 0$$

be a polynomial of degree n with real coefficients such that some $k_j \geq 1$, $j = \mu, \mu+1, \ldots, \mu+r-1$, where $\mu \leq r \leq n$,

$$k_{\mu-1}a_n \geq k_\mu a_{n-1} \geq k_{\mu+1}a_{n-2} \geq \cdots \geq k_{\mu+r-2}a_{n-r+1} \geq k_{\mu+r-1}a_{n-r} \geq a_{n-r-1} \geq \cdots \geq a_\mu \geq a_0.$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|},$$

where

$$M = k_{\mu-1}(|a_n| + a_n) + 2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)|a_{n+\mu-j-1}| - a_0 + |a_0|.$$

Taking $k_j = 1$, $j = \mu, \mu+1, \ldots, \mu+r-1$, in Theorem 5, we obtain the following result:

Corollary 1 *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, $a_0 \neq 0$ be a polynomial of degree n with real coefficients such that $a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_\mu \geq a_0$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|a_n| + a_n - a_0 + |a_0|}{|a_0|}.$$

Remark 1 *On setting $a_0 > 0$ and $\delta = 1/2$, Corollary 1 reduces to Theorem 2.*

Theorem 6 *Let*

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n, a_0 \neq 0$$

be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$ and $k_j \geq 1$, $j = \mu-1, \mu, \mu+1, \ldots, \mu+r-1$, where $\mu \leq r \leq n$

$$\begin{aligned} |k_{\mu-1}a_n| &\geq |k_\mu a_{n-1}| \geq |k_{\mu+1}a_{n-2}| \geq \cdots \geq |k_{\mu+r-2}a_{n-r+1}| \\ &\geq |k_{\mu+r-1}a_{n-r}| \geq \cdots \geq |a_\mu| \geq |a_0|. \end{aligned} \tag{1}$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{K}{|a_0|},$$

where

$$K = k_{\mu-1}|a_n|\left(1 + \cos\alpha + \sin\alpha\right) + 2\sin\alpha \left\{ \sum_{j=\mu}^{\mu+r-1} k_j |a_{n+\mu-j-1}| + \sum_{j=r+1}^{n} |a_{n-j}| \right\} + 2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)|a_{n+\mu-j}|.$$

Remark 2 *If we take $k_j = 1$, $j = \mu-1, \mu, \mu+1, \ldots, \mu+r-1$, and $\delta = 1/2$, Theorem 6 reduces to Theorem 3.*

For $\alpha = \beta = 0$, we obtain the following result:

Corollary 2 Let

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \le \mu \le n, \ a_0 \ne 0$$

be a polynomial of degree n with real coefficients such that for some $k_j \ge 1$, $j = \mu, \mu+1, \dots, \mu+r-1$, where $\mu \le r \le n$,

$$k_{\mu-1}a_n \ge k_\mu a_{n-1} \ge k_{\mu+1}a_{n-2} \ge \cdots \ge k_{\mu+r-2}a_{n-r+1} \ge k_{\mu+r-1}a_{n-r} \ge a_{n-r-1} \ge \cdots \ge a_\mu \ge a_0.$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \le \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2k_{\mu-1}a_n + 2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)a_{n+\mu-j}}{a_0}.$$

Theorem 7 Let

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \le \mu \le n, \ a_0 \ne 0$$

be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_i) = \alpha_i$, $\mathrm{Im}(a_i) = \beta_i$, for $i = 0, 1, 2, \dots, n$ and $k_j \ge 1$, $j = \mu, \mu+1, \dots, \mu+r-1$, where $\mu \le r \le n-1$,

$$k_{\mu-1}\alpha_n \ge k_\mu \alpha_{n-1} \ge \cdots \ge k_{\mu+r-2}\alpha_{n-r+1} \ge k_{\mu+r-1}\alpha_{n-r} \ge \alpha_{n-r-1} \ge \cdots \ge \alpha_2 \ge \alpha_\mu \ge \alpha_0,$$

then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \le \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{N}{|a_0|},$$

where

$$N = k_{\mu-1}(|\alpha_n| + \alpha_n) + 2\sum_{i=\mu}^{\mu+r-1}(k_i - 1)|\alpha_{n+\mu-i-1}| + 2\left(\sum_{i=\mu}^{n}|\beta_i| + |\beta_o|\right) - \alpha_\mu + |\alpha_\mu| + 2|\alpha_0|.$$

On setting $\beta_i = 0$, Theorem 7 reduces to Theorem 5. Taking $k_j = 1$, $j = \mu-1$, μ, $\mu+1, \dots, \mu+r-1$, in Theorem 7, we obtain the following result:

Corollary 3 Let

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \le \mu \le n, \ a_0 \ne 0$$

be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_i) = \alpha_i$, $\mathrm{Im}(a_i) = \beta_i$, for $i = 0, 1, 2, \dots, n$ and $j = \mu, \mu+1, \dots, \mu+r-1$, where $\mu \le r \le n-1$,

$$\alpha_n \ge \alpha_{n-1} \cdots \ge \alpha_2 \ge \alpha_\mu \ge \alpha_0,$$

then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \le \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|\alpha_n| + \alpha_n + 2\sum_{i=\mu}^{n}(|\beta_i| + |\beta_o|) - \alpha_\mu + |\alpha_\mu| + 2|\alpha_0|}{|a_0|}.$$

Remark 3 *If we assume $\alpha_0 > 0$ and $\delta = 1/2$, Corollary 3 reduces to Theorem 4.*

3 Lemmas

For the proof of these theorems, we require the following lemma which is due to Govil and Rahman [3].

Lemma 1 *If for some real* β, $|\arg a_j - \beta| \leq \alpha \leq \pi/2$, $a_j \neq 0$, *and if for some positive real numbers* t_1 *and* t_2, $t_1|a_j| \geq t_2|a_{j-1}|$, *then*

$$|t_1 a_j - t_2 a_{j-1}| \leq (t_1|a_j| - t_2|a_{j-1}|)\cos\alpha + (t_1|a_j| + t_2|a_{j-1}|)\sin\alpha.$$

4 Proof of Main Results

Proof of Theorem5. Let

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n, \; a_0 \neq 0$$

be a polynomial of degree n with real coefficients such that some $k_j \geq 1$, $j = \mu, \mu+1, \ldots, \mu+r-1$, where $\mu \leq r \leq n$. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_{n-r} - a_{n-r-1})z^{n-r} + \cdots + a_\mu z^\mu - a_0 z + a_0 \\
&= -a_n z^{n+1} + (k_{\mu-1}a_n - k_\mu a_{n-1} - (k_{\mu-1}-1)a_n + (k_\mu - 1)a_{n-1})z^n \\
&\quad + (k_\mu a_{n-1} - k_{\mu+1}a_{n-2} - (k_\mu - 1)a_{n-1} + (k_{\mu+1}-1)a_{n-2})z^{n-1} \\
&\quad + \cdots + (k_{\mu+r-2}a_{n-r+1} - a_{n-r} - (k_{\mu+r-2}-1)a_{n-r+1})z^{n-r+1} \\
&\quad + (k_{\mu+r-1}a_{n-r} - a_{n-r-1} - (k_{\mu+r-1}-1)a_{n-r})z^{n-r} \\
&\quad + \cdots + (a_{\mu+1} - a_\mu)z^{\mu+1} + a_\mu z^\mu - a_0 z + a_0.
\end{aligned}$$

It follows that

$$\begin{aligned}
|F(z)| &= |-a_n z^{n+1} + (k_{\mu-1}a_n - k_\mu a_{n-1} - (k_{\mu-1}-1)a_n + (k_\mu - 1)a_{n-1})z^n \\
&\quad + (k_\mu a_{n-1} - k_{\mu+1}a_{n-2} - (k_\mu - 1)a_{n-1} + (k_{\mu+1}-1)a_{n-2})z^{n-1} \\
&\quad + \cdots + (k_{\mu+r-2}a_{n-r+1} - a_{n-r} - (k_{\mu+r-2}-1)a_{n-r+1})z^{n-r+1} \\
&\quad + (k_{\mu+r-1}a_{n-r} - a_{n-r-1} - (k_{\mu+r-1}-1)a_{n-r})z^{n-r} \\
&\quad + \cdots + (a_{\mu+1} - a_\mu)z^{\mu+1} + a_\mu z^\mu - a_0 z + a_0|.
\end{aligned}$$

By using the hypothesis, we have for $|z| \leq 1$,

$$\begin{aligned}
|F(z)| &\leq |a_n| + (k_{\mu-1}-1)|a_n| + k_{\mu-1}a_n - k_\mu a_{n-1} + (k_\mu - 1)|a_{n-1}| + k_\mu a_{n-1} - k_{\mu+1}a_{n-2} \\
&\quad + (k_\mu - 1)|a_{n-1}| + (k_{\mu+1}-1)|a_{n-2}| + \cdots + k_{\mu+r-2}a_{n-r+1} - k_{\mu+r-1}a_{n-r} \\
&\quad + (k_{\mu+r-2}-1)|a_{n-r+1}| + (k_{\mu+r-1}-1)|a_{n-r}| + k_{\mu+r-1}a_{n-r} - a_{n-r-1} + (k_{\mu+r-1}-1)|a_{n-r}| \\
&\quad + a_{n-r-1} - a_{n-r-2} + \cdots + a_{\mu+1} - a_\mu + a_\mu - a_0 + |a_0| \\
&= |a_n| + (k_{\mu-1}-1)|a_n| + k_{\mu-1}a_n + 2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)|a_{n+\mu-j-1}| - a_0 + |a_0| \\
&= k_{\mu-1}(|a_n| + a_n) + 2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)|a_{n+\mu-j-1}| - a_0 + |a_0| \\
&= M(say).
\end{aligned}$$

Since $F(z)$ is analytic in $|z| \leq 1$, $F(0) = a_0 \neq 0$ and $|F(z)| \leq M$. By Theorem 1, the number of zeros of $F(z)$ in $|z| \leq \delta$ does not exceed $\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|F(0)|}$. As the number of zeros of $P(z)$ in $|z| \leq \delta$ is equal to the number of zeros of $F(z)$ in $|z| \leq \delta$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed $\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$. ■

Proof of Theorem6. Consider

$$\begin{aligned}
G(z) &= (1-z)P(z)\\
&= -a_nz^{n+1}+(a_n-a_{n-1})z^n+\cdots+(a_{n-r}-a_{n-r-1})z^{n-r}+\cdots+a_\mu z^\mu-a_0z+a_0\\
&= -a_nz^{n+1}+(k_{\mu-1}a_n-k_\mu a_{n-1}-(k_{\mu-1}-1)a_n+(k_\mu-1)a_{n-1})z^n\\
&\quad+(k_\mu a_{n-1}-k_{\mu+1}a_{n-2}-(k_\mu-1)a_{n-1}+(k_{\mu+1}-1)a_{n-2})z^{n-1}\\
&\quad+\cdots+(k_{\mu+r-2}a_{n-r+1}-a_{n-r}-(k_{\mu+r-2}-1)a_{n-r+1})z^{n-r+1}\\
&\quad+(k_{\mu+r-1}a_{n-r}-a_{n-r-1}-(k_{\mu+r-1}-1)a_{n-r})z^{n-r}\\
&\quad+\cdots+(a_{\mu+1}-a_\mu)z^{\mu+1}+a_\mu z^\mu-a_0z+a_0.
\end{aligned}$$

It follows that

$$\begin{aligned}
|G(z)| &= |-a_nz^{n+1}+(k_{\mu-1}a_n-k_\mu a_{n-1}-(k_{\mu-1}-1)a_n+(k_\mu-1)a_{n-1})z^n\\
&\quad+(k_\mu a_{n-1}-k_{\mu+1}a_{n-2}-(k_\mu-1)a_{n-1}+(k_{\mu+1}-1)a_{n-2})z^{n-1}\\
&\quad+\cdots+(k_{\mu+r-2}a_{n-r+1}-a_{n-r}-(k_{\mu+r-2}-1)a_{n-r+1})z^{n-r+1}\\
&\quad+(k_{\mu+r-1}a_{n-r}-a_{n-r-1}-(k_{\mu+r-1}-1)a_{n-r})z^{n-r}\\
&\quad+\cdots+(a_{\mu+1}-a_\mu)z^{\mu+1}+a_\mu z^\mu-a_0z+a_0|.
\end{aligned}$$

For $|z|\le 1$, we have

$$\begin{aligned}
|G(z)| &\le |a_n|+(k_{\mu-1}-1)|a_n|+|k_{\mu-1}a_n-k_\mu a_{n-1}|+(k_\mu-1)|a_{n-1}|+|k_\mu a_{n-1}-k_{\mu+1}a_{n-2}|\\
&\quad+(k_\mu-1)|a_{n-1}|+(k_{\mu+1}-1)|a_{n-2}|+\cdots+k_{\mu+r-2}a_{n-r+1}-k_{\mu+r-1}a_{n-r}\\
&\quad+(k_{\mu+r-2}-1)|a_{n-r+1}|+(k_{\mu+r-1}-1)|a_{n-r}|+|k_{\mu+r-1}a_{n-r}-a_{n-r-1}|+(k_{\mu+r-1}-1)|a_{n-r}|\\
&\quad+|a_{n-r-1}-a_{n-r-2}|+\cdots+|a_{\mu+1}-a_\mu|+|a_\mu-a_0|+|a_0|.
\end{aligned}$$

In view of (1), applying Lemma 1, we have for $|z|\le 1$,

$$\begin{aligned}
|G(z)| &\le k_{\mu-1}|a_n|\left(1+\cos\alpha+\sin\alpha\right)+2\sin\alpha\left\{\sum_{j=\mu}^{\mu+r-1}k_j|a_{n+\mu-j-1}|+\sum_{j=r+1}^{n}|a_{n-j}|\right\}\\
&\quad+2\sum_{j=\mu}^{\mu+r-1}(k_j-1)|a_{n+\mu-j-1}|-|a_0|(\cos\alpha+\sin\alpha-1)\\
&\le k_{\mu-1}|a_n|\left(1+\cos\alpha+\sin\alpha\right)+2\sin\alpha\left\{\sum_{j=\mu}^{\mu+r-1}k_j|a_{n-j}|+\sum_{j=r+1}^{n}|a_{n-j}|\right\}+2\sum_{j=\mu}^{\mu+r-1}(k_j-1)|a_{n-j}|\\
&= K(say).
\end{aligned}$$

Since $G(z)$ is analytic in $|z|\le 1$, $G(0)=a_0\neq 0$ and $|G(z)|\le K$. By Theorem 1, the number of zeros of $G(z)$ in $|z|\le\delta$ does not exceed $\frac{1}{\log\frac{1}{\delta}}\log\frac{K}{|G(0)|}$. As the number of zeros of $P(z)$ in $|z|\le\delta$ is equal to the number of zeros of $G(z)$ in $|z|\le\delta$, the number of zeros of $P(z)$ in $|z|\le\delta$ does not exceed $\frac{1}{\log\frac{1}{\delta}}\log\frac{K}{|a_0|}$. ■

Proof of Theorem7. Consider

$$\begin{aligned}
T(z) &= (1-z)P(z)\\
&= -a_nz^{n+1}+\sum_{i=1}^{n}(a_i-a_{i-1})z^i+a_\mu z^\mu-a_0z+a_0.
\end{aligned}$$

We have for $|z| \leq 1$,

$$\begin{aligned}
|T(z)| &\leq |a_n| + \sum_{i=\mu+1}^{n} |a_i - a_{i-1}| + a_\mu + 2|a_0| \\
&\leq |\alpha_n| + |\beta_n| + \sum_{i=\mu+1}^{n} |\alpha_i - \alpha_{i-1}| + \sum_{i=\mu+1}^{n} |\beta_i - \beta_{i-1}| + |\alpha_\mu| + |\beta_\mu| + 2(|\alpha_0| + |\beta_o|) \\
&\leq |\alpha_n| + |\beta_n| + \sum_{i=\mu+1}^{n} |\alpha_i - \alpha_{i-1}| + \sum_{i=\mu+1}^{n} |\beta_i + \beta_{i-1}| + |\alpha_\mu| + |\beta_\mu| + 2(|\alpha_0| + |\beta_o|) \\
&\leq |\alpha_n| + |\beta_n| + \sum_{i=\mu}^{n-1} |\alpha_{n+\mu-i} - \alpha_{n+\mu-i-1}| + \sum_{i=\mu+1}^{n} \big(|\beta_i| + |\beta_{i-1}|\big) + |\alpha_\mu| + |\beta_\mu| + 2(|\alpha_0| + |\beta_o|) \\
&= |\alpha_n| + \sum_{i=\mu}^{\mu+r} |k_{i-1}\alpha_{n+\mu-i} - k_i\alpha_{n+\mu-i-1} - (k_{i-1} - 1)\alpha_{n+\mu-i} + (k_i - 1)\alpha_{n+\mu-i-1}| \\
&\quad + \sum_{i=\mu+r+1}^{n-1} |\alpha_{n+\mu-i} - \alpha_{n+\mu-i-1}| + 2\sum_{i=\mu}^{n} (|\beta_i| + |\beta_0|) + |\alpha_\mu| + 2|\alpha_0|,\ k_{r+1} = 1 \\
&\leq |\alpha_n| + \sum_{i=\mu}^{\mu+r} |k_{i-1}\alpha_{n+\mu-i} - k_i\alpha_{n+\mu-i-1}| + \sum_{i=\mu}^{\mu+r} |(k_{i-1} - 1)\alpha_{n+\mu-i}| + \sum_{i=\mu}^{\mu+r} |(k_i - 1)\alpha_{n+\mu-i-1}| \\
&\quad + \sum_{i=\mu+r+1}^{n-1} |\alpha_{n+\mu-i} - \alpha_{n+\mu-i-1}| + 2\sum_{i=\mu}^{n} (|\beta_i| + |\beta_0|) + |\alpha_\mu| + 2|\alpha_0|,\ k_{r+1} = 1.
\end{aligned}$$

By using the given hypothesis, we get

$$\begin{aligned}
|T(z)| &= |\alpha_n| + \sum_{i=\mu}^{\mu+r} (k_{i-1}\alpha_{n+\mu-i} - k_i\alpha_{n+\mu-i-1}) + (k_\mu - 1)|\alpha_n| + 2\sum_{i=\mu+1}^{\mu+r} (k_i - 1)|\alpha_{n+\mu-i}| \\
&\quad + \sum_{i=\mu+r+1}^{n-1} (\alpha_{n+\mu-i} - \alpha_{n+\mu-i-1}) + 2\sum_{i=\mu}^{n} (|\beta_i| + |\beta_0|) + |\alpha_\mu| + 2|\alpha_0|,\ k_{\mu+r} = 1 \\
&= |\alpha_n| + k_{\mu-1}\alpha_n + (k_{\mu-1} - 1)|\alpha_n| + 2\sum_{i=\mu}^{\mu+r-1} (k_i - 1)|\alpha_{n+\mu-i-1}| - \alpha_\mu \\
&\quad + 2\left(\sum_{i=\mu}^{\mu+r-1} |\beta_i| + |\beta_0|\right) + |\alpha_\mu| + 2|\alpha_0| \\
&= k_{\mu-1}(|\alpha_n| + \alpha_n) + 2\sum_{i=\mu}^{\mu+r-1} (k_i - 1)|\alpha_{n+\mu-i-1}| + 2\left(\sum_{i=\mu}^{n} |\beta_i| + |\beta_o|\right) - \alpha_\mu + |\alpha_\mu| + 2|\alpha_0| \\
&= N(say).
\end{aligned}$$

Since $T(z)$ is analytic in $|z| \leq 1$, $T(0) = a_0 \neq 0$ and $|T(z)| \leq N$. By Theorem 1, the number of zeros of $T(z)$ in $|z| \leq \delta$ does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{N}{|T(0)|}$. As the number of zeros of $P(z)$ in $|z| \leq \delta$ is equal to the number of zeros of $T(z)$ in $|z| \leq \delta$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{N}{|a_0|}$. ■

5 Conclusion

This paper delivers a rigorous and comprehensive analysis of zero-free regions for lacunary-type polynomials, shedding light on their intricate zero-distribution patterns under specific coefficient constraints. By deriving sharp bounds on the number of zeros within defined annular regions, the study not only deepens theoretical understanding but also lays a foundation for advancing research in complex polynomial dynamics. These findings hold significant potential for applications in various mathematical and applied fields, emphasizing the relevance and impact of exploring lacunary polynomials.

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