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The Omega Calculus Of Variations*†

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Abstract

We prove a necessary optimality condition of Euler-Lagrange type for the calculus of variations with Omega derivatives, which turns out to be sufficient under joint convexity of the Lagrangian.

1 Introduction

The Calculus of Variations (CoV) is a classical branch of mathematical analysis that focuses on finding functions that optimize certain quantities expressed as integrals, known as functionals. Applications of this field are widespread, including physics, engineering, and economics, particularly in problems involving optimal paths. For the sake of motivation, and for a gentle introduction to the CoV, we refer the readers to classical books [4, 7]. The fundamental problem of the CoV, also known as the basic variational problem with fixed endpoints, consists of finding minimizers $y \in C^2[a,b]$ of the problem.

$$\begin{cases} \min \mathcal{L}[y(\cdot)] = \int_a^b L(x, y(x), y'(x))\, dx \\ y(a) = y_a, \quad y(b) = y_b, \end{cases} \tag{1}$$

where $\mathcal{L} : C^2[a,b] \longrightarrow \mathbb{R}$ is a given functional, the Lagrangian L is a function that has at least continuous partial derivatives of the second order, and a, b, y_a, $y_b \in \mathbb{R}$ are fixed. The central result of the CoV is a necessary optimality condition for a smooth function to be a solution to (1), known as the Euler-Lagrange equation.

Theorem 1 *Let $\tilde{y}$ be a solution to problem (1). Then, $\tilde{y}$ satisfies the Euler-Lagrange equation*

$$\frac{d}{dx}\left[\partial_3 L(x, y(x), y'(x))\right] = \partial_2 L(x, y(x), y'(x))$$

for all $x \in [a,b]$, where $\partial_2 L$ and $\partial_3 L$ denote the partial derivatives of $L(x,y,y')$ with respect to y and y', respectively.

Over the years, various different versions and extensions of the CoV have emerged, see, e.g., [1, 6]. Here, our main goal is to extend the classical CoV to the concept of Omega derivatives (see Section 2).

The paper is organized as follows. In the next section, we provide a brief introduction to Omega derivatives, presenting some basic results that will be crucial throughout our work. In Section 3, we introduce a version of the variational problem (1) that incorporates Omega derivatives and derive the corresponding Euler-Lagrange equation. Additionally, we derive a sufficient condition for a smooth function to be a solution to the problem. We end by illustrating our results with an example.

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2 Omega Derivative

Over time, we have observed the development of new differential and integral operators, including both fractional and generalized types. Here, we are interested in exploring the notion of Omega derivatives, a concept that generalizes the classical derivative and was first introduced in [5]. In this section, we only give the notions and results that will be useful in the sequel. For more details we refer to [2, 3], where all such definitions and results are stated and proved.

We begin by presenting the definition of Ω-derivative.

Definition 1 *Let I be an open interval (bounded or unbounded), and let $f : I \longrightarrow \mathbb{R}$ and $\Omega : I \longrightarrow \mathbb{R}$ be functions such that Ω is continuous and strictly increasing on I. For $x_0 \in I$, the function f is termed Ω-differentiable at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{\Omega(x) - \Omega(x_0)}$$

exists. In this case, we denote its value by $D_\Omega f(x_0)$, which we call the Ω-derivative of f at x_0.

Note that, when $\Omega(x) = x$, we end up with the classical ordinary derivative of a function f. Moreover, if $f'(x_0)$ and $\Omega'(x_0)$ both exist and $\Omega'(x_0) \neq 0$, then

$$D_\Omega f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{\Omega(x) - \Omega(x_0)} = \lim_{x \to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{\Omega(x)-\Omega(x_0)}{x-x_0}} = \frac{f'(x_0)}{\Omega'(x_0)}.$$

Definition 2 *We say that f belongs to the class f_Ω, i.e., f is admissible, if*

$$f_\Omega(I) = \left\{ f : I \longrightarrow \mathbb{R}, \ \frac{f'(x)}{\Omega(x)} \neq k, \ k \in \mathbb{R} \right\}.$$

Naturally, there are some properties regarding Ω-derivatives that are worth mentioning.

Theorem 2 (See, e.g., [3]) *Let f and g be admissible, Ω-differentiable at a point $t > 0$, and $\alpha \in (0, 1]$. Then,*

- $D_\Omega(af + bg)(t) = aD_\Omega(f)(t) + bD_\Omega(g)(t)$;
- $D_\Omega(t^p) = \frac{pt^{p-1}}{\Omega'(t)}$, $p \in \mathbb{R}$;
- $D_\Omega(p) = 0$, $p \in \mathbb{R}$;
- $D_\Omega(fg)(t) = fD_\Omega(g)(t) + gD_\Omega(f)(t)$;
- $D_\Omega\left(\frac{f}{g}\right)(t) = \frac{gD_\Omega(f)(t) - fD_\Omega(g)(t)}{g^2(t)}$;
- $D_\Omega(f \circ g)(t) = D_\Omega(f(g)(t)) = f'(g(t))D_\Omega g(t)$.

There is also an extension of the Fermat theorem regarding minimum and maximum values of a function f. Before stating this result, we present a basic definition.

Definition 3 *We say that f is Ω-differentiable on an open interval if it is Ω-differentiable at every point in the interval.*

Theorem 3 (The Ω-Fermat theorem [2]) *Suppose f is Ω-differentiable on the interval (a, b). If f has a relative maximum or a relative minimum at $x_0 \in (a, b)$, then $D_\Omega f(x_0) = 0$.*

Definition 4 *Let f be an admissible function defined on an open interval I. We say that F is an Ω-antiderivative of f in I if $D_\Omega F(x) = f(x)$ for all $x \in I$.*

Remark 1 *Let I be an open interval (bounded or unbounded), and let $f : I \longrightarrow \mathbb{R}$ and $\Omega : I \longrightarrow \mathbb{R}$ be C^1 functions such that Ω is strictly increasing on I, that is, $\Omega'(x) > 0$ for all $x \in I$. If we define the indefinite integral operator as*

$$J_\Omega(f)(x) = \int f(x)\Omega'(x)\,dx,$$

then one has

$$J_\Omega(D_\Omega f(x)) = \int D_\Omega f(x)\Omega'(x)\,dx = \int \frac{f'(x)}{\Omega'(x)}\Omega'(x)\,dx = f(x).$$

For the following definition, we denote by $\mathcal{R}(\Omega)$ the set of all Riemann-Stieltjes integrable functions with respect to Ω, where Ω is a continuous, strictly increasing function on a closed and bounded interval $[a, b]$.

Definition 5 *Let I be an interval $I \subseteq \mathbb{R}$, a, $t \in I$ and $\alpha \in \mathbb{R}$. The integral operator J_Ω is defined for every locally integrable function f on I and Ω in the class $\mathcal{R}(\Omega)$ as*

$$J_{\Omega,a+}(f)(t) = \int_a^t f(s)\Omega'(s)\,ds = \int_a^t f(s)d\Omega(s), \quad t \geq a,$$

and ($\Omega'(t)$ is a function of constant sign over I)

$$J_\Omega(f)(t) = \int_t^b f(s)\Omega'(s)\,ds = -J_{\Omega,b+}(f)(t),$$
$$J_{\Omega,a}(f)(b) = \int_a^b f(s)\Omega'(s)\,ds = J_{\Omega,b-}(f)(t) + J_{\Omega,a+}(f)(t).$$

Next, we present two propositions that relate the notion of Ω-derivatives to the integral operator J_Ω.

Proposition 1 (See, e.g., [3]) *Let $I \subseteq \mathbb{R}$, $a \in I$, and f be a Ω-differentiable function on I such that f' is a locally integrable function on I. Then $J_{\Omega,a}(D_\Omega(f))(t) = f(t) - f(a)$ for all $t \in I$.*

Proposition 2 (See, e.g., [3]) *Let $I \subseteq \mathbb{R}$ and $a \in I$. Then $D_\Omega(J_{\Omega,a}(f))(t) = f(t)$ for every continuous function f on I and a, $t \in I$.*

We now recall some important properties of the integral operator $J_{\Omega,a}$.

Theorem 4 (See, e.g., [3]) *Let $I \subseteq \mathbb{R}$ and a, $b \in I$. Suppose that f, g are locally integrable functions on I, and k_1, $k_2 \in \mathbb{R}$. Then,*

- $J_{\Omega,a}(k_1 f + k_2 g)(t) = k_1 J_{\Omega,a} f(t) + k_2 J_{\Omega,a} g(t)$;
- *if $f \geq g$, then $J_{\Omega,a} f(t) \geq J_{\Omega,a} g(t)$ for every $t \in I$ with $t \geq a$;*
- $|J_{\Omega,a} f(t)| \leq J_{\Omega,a}|f|(t)$ *for every $t \in I$ with $t \geq a$.*

Theorem 5 (Integration by parts [3]) *Let f, $g : [a, b] \longrightarrow \mathbb{R}$ be Ω-differentiable functions. Then,*

$$J_{\Omega,a}((f)(D_\Omega g(t))) = [f(t)g(t)]_a^b - J_{\Omega,a}((g)(D_\Omega f(t))).$$

3 The Ω-Variational Problem

Before posing our problem, we introduce the notion of Omega partial derivative.

Definition 6 *Let $n \in \mathbb{N}$, I_i be open intervals (bounded or unbounded), $i = i, \ldots, n$, and $f : I_1 \times \cdots \times I_n \longrightarrow \mathbb{R}$ and $\Omega : I_i \longrightarrow \mathbb{R}$ be C^1 functions such that Ω is strictly increasing on I_i, $i = i, \ldots, n$. We define the Omega partial derivative of f with respect to x_i by*

$$D^i_\Omega f(x_1, \ldots, x_n) = \frac{f_{x_i}(x_1, \ldots, x_n)}{\Omega'(x_i)},$$

where $f_{x_i}(x_1, \ldots, x_n)$ denotes the usual partial derivative $\frac{\partial f}{\partial x_i}$ and $i \in \{1, \ldots, n\}$.

Definition 7 *Let $f(x_1, x_2, \ldots, x_n) \in C^1$, $n \in \mathbb{N}$. The integral operator $J^i_{\Omega,a}(f(x_1, \ldots, x_n))(b)$ is defined by*

$$J^i_{\Omega,a}(f(x_1, \ldots, x_n))(b) = \int_a^b f(x_1, \ldots, x_n)\Omega'(x_i)\, dx_i, \quad i \in \{1, \ldots, n\}.$$

Now let us consider the variational problem

$$\begin{cases} \min \mathcal{L}(y(\cdot)) = J^1_{\Omega,a}\, L\left(x, y(x), D_\Omega y(x)\right)(b) \\ y(a) = y_a, \quad y(b) = y_b. \end{cases} \tag{2}$$

Note that in the particular case that Ω is the identity function, then our problem (2) reduces to the classical problem of the calculus of variations (1).

Proposition 3 *If $f(x) \geq 0$, then $J_{\Omega,a}(f(x))(b) \geq 0$ for all $x \in [a, b]$.*

Proof. By definition, the following relation holds:

$$J_{\Omega,a}(f(x))(b) = \int_a^b f(x)\Omega'(x)dx.$$

Since $\Omega(x)$ is a continuous and strictly increasing function in $[a, b]$, $\Omega'(x) > 0$ for all $x \in [a, b]$. By hypothesis, $f(x) \geq 0$ and it follows that

$$\int_a^b f(x)\Omega'(x)dx \geq 0.$$

The result is proved. ■

Lemma 1 (Fundamental lemma of The Omega Calculus of Variations) *If $g(x)$ is a continuous function in $a \leq x \leq b$ and if*

$$J_{\Omega,a}[g(x)\eta(x)](b) = 0,$$

where $\eta(x)$ is an arbitrary function with $\eta(a) = \eta(b) = 0$, then $g(x) = 0$ at every point in the interval $a \leq x \leq b$.

Proof. Let us suppose that $g(x) > 0$ in some subinterval $\alpha \leq x \leq \beta$ within the interval $a \leq x \leq b$. Since $\eta(x)$ is an arbitrary function, let us choose it as

$$\eta(x) = \begin{cases} 0, & \text{if } x < \alpha, \\ (x-\alpha)^2(\beta - x)^2, & \text{if } \alpha \leq x \leq \beta, \\ 0, & \text{if } x > \beta. \end{cases}$$

Note that $(x - \alpha)^2 > 0$ and $(\beta - x)^2 > 0$ for $\alpha < x < \beta$. By Proposition 3, the integral

$$J_{\Omega,a}[g(x)\eta(x)](b) = J_{\Omega,\alpha}[g(x)(x-\alpha)^2(\beta - x)^2](\beta)$$

is always positive since $g(x)(x-\alpha)^2(\beta-x)^2 > 0$ throughout the interval, except at $x=\alpha$ and $x=\beta$, where it is zero. Thus, the only way for the integral over the entire interval to be zero is the function $g(x)=0$ everywhere in the interval. ■

In Lemma 2, we present an extension of the Leibniz integral rule. To prove such result we recall the following definition.

Definition 8 (See, e.g., [1]) *A function f defined on $[a,b]\times\mathbb{R}$ is called continuous in the second variable, uniformly in the first variable, if for each $\varepsilon>0$ there exists $\delta>0$ such that $|x_1-x_2|<\delta$ implies*

$$|f(t,x_1)-f(t,x_2)|<\varepsilon \text{ for all } t\in[a,b].$$

Lemma 2 *Suppose that $F(x)=J^1_{\Omega,a}[f(t,x)](b)$ is well-defined. If $D^2_\Omega f(t,x)$ is continuous in x, uniformly in t, then $D_\Omega F(x)=J^1_{\Omega,a}[D^2_\Omega\ f(t,x)](b)$.*

Proof. First note that $D_\Omega F(x)=\frac{F'(x)}{\Omega'(x)}$. By Definitions 6 and 7 we have

$$J^1_{\Omega,a}[D^2_\Omega\ f(t,x)](b)=\int_a^b \frac{f_x(t,x)}{\Omega'(x)}\Omega'(t)\,dt.$$

If $\varepsilon>0$, then there exists $\delta>0$ such that $|f_x(t,x_1)-f_x(t,x_2)|<\varepsilon$ whenever $|x_1-x_2|<\delta$ and $t\in[a,b]$. Let $h\in\mathbb{R}$ with $|h|<\delta$. Thus,

$$\begin{aligned}
&\left|\frac{F(x+h)-F(x)}{h\,\Omega'(x)}-\int_a^b\frac{f_x(t,x)}{\Omega'(x)}\Omega'(t)dt\right|\\
=&\left|\frac{J^1_{\Omega,a}(f(t,x+h)-f(t,x))(b)}{h\,\Omega'(x)}-\frac{1}{\Omega'(x)}\int_a^b f_x(t,x)\Omega'(t)dt\right|\\
=&\left|\frac{1}{\Omega'(x)}\int_a^b\frac{f(t,x+h)-f(t,x)}{h}\Omega'(t)dt-\frac{1}{\Omega'(x)}\int_a^b f_x(t,x)\Omega'(t)dt\right|\\
=&\left|\frac{1}{\Omega'(x)}\int_a^b[f_x(t,x+\theta h)-f_x(t,x)]\Omega'(t)dt\right|\\
\leq&\ \frac{1}{\Omega'(x)}\int_a^b|f_x(t,x+\theta h)-f_x(t,x)|\Omega'(t)dt=\frac{\varepsilon\,(\Omega(b)-\Omega(a))}{\Omega'(x)},
\end{aligned}$$

where $\theta=\theta(t,x)\in(0,1)$. This completes the proof. ■

With the following result we derive a necessary condition for a smooth function to be a solution to the variational problem (2).

Theorem 6 (The Omega Euler-Lagrange Equation) *If $\tilde{y}$ is a local minimizer of the variational problem (2), then it satisfies*

$$D_\Omega(D^3_\Omega L(x,y(x),D_\Omega y(x)))=D^2_\Omega L(x,y(x),D_\Omega y(x)) \tag{3}$$

for all $x\in[a,b]$.

Proof. Consider an arbitrary admissible variation $\eta(x)\in C^1$ such that $\eta(a)=\eta(b)=0$ and define the function $\Phi(\varepsilon):\mathbb{R}\longrightarrow\mathbb{R}$ by $\Phi(\varepsilon)=\mathcal{L}(y+\varepsilon\eta)$. The first variation of the variational problem (2) is defined by

$$\mathcal{L}_1(y,\eta)=D_\Omega\Phi(\varepsilon)\Big|_{\varepsilon=0}.$$

Let $f(x,\varepsilon) = L(x, y(x)+\varepsilon\eta(x), D_\Omega y(x)+\varepsilon D_\Omega\eta(x))$. By Theorem 2, if $D^2_\Omega f(x,\varepsilon)$ is continuous in ε, uniformly in x, then

$$\mathcal{L}_1(y,\eta) = J^1_{\Omega,a}[D^2_\Omega L(x,y(x),D_\Omega y(x))\eta(x) + D^3_\Omega L(x,y(x),D_\Omega y(x))D_\Omega\eta(x)](b).$$

Now, note that $\mathcal{L}_1(y,\eta) = \mathcal{L}(y+\varepsilon\eta)$. Additionally, suppose that $\tilde{y}$ is a local minimizer of (2). Following Theorem 3, $\mathcal{L}_1(\tilde{y},\eta) = 0$ for all admissible variations η. Thus, using integration by parts, we get

$$\begin{aligned} 0 &= \mathcal{L}_1(\tilde{y},\eta) = J^1_{\Omega,a}[D^2_\Omega L(x,y(x),D_\Omega y(x))\eta(x) + D^3_\Omega L(x,y(x),D_\Omega y(x))D_\Omega\eta(x)](b) \\ &= J^1_{\Omega,a}\left[(D^2_\Omega L(x,y(x),D_\Omega y(x)) - D_\Omega(D^3_\Omega L(x,y(x),D_\Omega y(x))))\eta(x)\right](b) \\ &\quad + \left[D^3_\Omega L(x,y(x),D_\Omega y(x))\eta(x)\right]_a^b \\ &= J^1_{\Omega,a}\left[(D^2_\Omega L(x,y(x),D_\Omega y(x)) - D_\Omega(D^3_\Omega L(x,y(x),D_\Omega y(x))))\eta(x)\right](b) \end{aligned}$$

and, by Lemma 1, it follows that $D^2_\Omega L(x,y(x),D_\Omega y(x)) - D_\Omega(D^3_\Omega L(x,y(x),D_\Omega y(x))) = 0$. ■

Example 1 *Consider the problem*

$$\begin{cases} \min \mathcal{L}(y(\cdot)) = J^1_{\Omega,0}(D_\Omega y(x)^2)(1) \\ y(0) = 0, \quad y(1) = 1, \end{cases} \tag{4}$$

with $\Omega(x) = 2e^{\frac{1}{2}x}$. *Then,* $\Omega'(x) = e^{\frac{1}{2}x}$ *and* $L(x,y(x),D_\Omega y(x)) = (D_\Omega y(x))^2$. *It follows that the Euler-Lagrange equation* (3) *is given by*

$$D_\Omega\left(\frac{2D_\Omega y(x)}{e^{\frac{1}{2}x}}\right) = 0.$$

Therefore, applying the integral operator in both sides of this equation, we have

$$\frac{2D_\Omega y(x)}{e^{\frac{1}{2}x}} = k \Leftrightarrow D_\Omega y(x) = k_1 e^{\frac{1}{2}x}, \quad k,\ k_1 \in \mathbb{R}.$$

Following the definition of the Ω*-derivative, the previous equation is equivalent to* $y'(x) = k_1 e^x$, *that is,* $y(x) = k_1 e^x + k_2$, k_1, $k_2 \in \mathbb{R}$. *Using the initial conditions* $y(0) = 0$ *and* $y(1) = 1$, *we get that* $k_1 = -\frac{1}{1-e}$ *and* $k_2 = \frac{1}{1-e}$, *leading to*

$$y(x) = \frac{1-e^x}{1-e}, \tag{5}$$

which is a candidate to be the minimizer of problem (4).

Now we derive a sufficient condition for a smooth function to be a solution to (2). To this end, we first need to introduce the notion of joint convexity.

Definition 9 *Given a function* $f(x,y,z) \in C^1$, *we say that* f *is jointly convex in* $S \subseteq \mathbb{R}^3$ *for the variables* y *and* z *if*

$$f(x,y+y_1,z+z_1) - f(x,y,z) \geq (D^2_\Omega f(x,y,z))y_1 + (D^3_\Omega f(x,y,z))z_1$$

holds for every (x,y,z), $(x,y+y_1,z+z_1) \in S$.

Theorem 7 *Let* $L(x,y(x),D_\Omega y(x))$ *be jointly convex in* $(y(x),D_\Omega y(x))$. *If* $\tilde{y}$ *satisfies the Euler-Lagrange equation* (3) *with* $y(a) = y_a$ *and* $y(b) = y_b$, *then* $\tilde{y}$ *is a global minimizer of the variational problem* (2).

Proof. Consider the variational problem (2) and suppose $\tilde{y}$ satisfies the Euler-Lagrange equation (3). Then, applying Definition 9 and integration by parts,

$$\begin{aligned} \mathcal{L}(y) - \mathcal{L}(\tilde{y}) &= J^1_{\Omega,a}\left[L(x,y,D_\Omega y) - L(x,\tilde{y},D_\Omega\tilde{y})\right](b) \\ &\geq J^1_{\Omega,a}\left[D^2_\Omega L(x,\tilde{y},D_\Omega\tilde{y})(y-\tilde{y}) + D^3_\Omega L(x,\tilde{y},D_\Omega\tilde{y})(D_\Omega y - D_\Omega\tilde{y})\right](b) \\ &= J^1_{\Omega,a}\left[(D^2_\Omega L(x,\tilde{y},D_\Omega\tilde{y}) - D_\Omega(D^3_\Omega L(x,\tilde{y},D_\Omega\tilde{y})))(y-\tilde{y})\right](b) = 0. \end{aligned}$$

Therefore, it follows that $\mathcal{L}(\tilde{y}) \leq \mathcal{L}(y)$ and so $\tilde{y}$ is a global minimizer of problem (2). ■

Using Theorem 7, it follows that the candidate (5) we have found in Example 1 is indeed the global minimizer for problem (4).

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