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Fixed Point Results For α-Admissible Mappings Of Integral Type With w-Distance*

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Abstract

Two fixed point theorems for α-admissible mappings satisfying contractive conditions of integral type with w-distance in a complete metric space are demonstrated. The results obtained in this paper improve and generalize some well-known results in the literature. An example is given.

1 Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$. Let

$$\Phi_1 = \Big\{\varphi : \mathbb{R}^+ \to \mathbb{R}^+,\ \varphi \text{ is Lebesgue integrable, summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^{\varepsilon} \varphi(t)dt > 0,\ \forall \varepsilon > 0\Big\},$$

$$\Phi_2 = \left\{\varphi : \varphi \text{ belongs to } \Phi_1 \text{ and satisfies } \int_0^{u+v} \varphi(t)dt \le \int_0^{u} \varphi(t)dt + \int_0^{v} \varphi(t)dt,\ \forall u, v > 0\right\},$$

$$\Phi_3 = \left\{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is nondecreasing in } \mathbb{R}^+ \text{ and } \sum_{n=1}^{\infty} \varphi^n(t) < +\infty,\ \forall t > 0\right\}.$$

It is well known that the Banach fixed point theorem has many generalizations. In 2002, Branciari [7] extended the Banach fixed point theorem by giving contractive mappings of integral type and established a fixed point theorem as follows.

Theorem 1 ([7]) *Let (X, d) be a complete metric space, $\varphi \in \Phi_1$, $c \in (0, 1)$ and $T : X \to X$ be a mapping satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X. \tag{1}$$

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Since then, many fixed point theorems which satisfy different contractive inequalities have been proved in metric spaces. Particularly, in 2012, Samet et al. [16] introduced the concept of α-ψ-contractive mappings and proved the following theorems for such mappings.

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Definition 1 ([16]) *Let (X, d) be a metric space, $\alpha : X \times X \to \mathbb{R}^+$ and $T : X \to X$ be two given mappings. If*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1, \quad \forall x, y \in X,$$

then T is called an α-admissible mapping.

Theorem 2 ([16]) *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}^+$, $\psi \in \Phi_3$ and $T : X \to X$ be an α-ψ-contractive mapping, that is,*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{2}$$

Suppose that

(a1) T is α-admissible, that is, $\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1, \quad \forall x, y \in X$;

(a2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(a3) T is continuous.

Then T has a fixed point.

Theorem 3 ([16]) *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}^+$, $\psi \in \Phi_3$ and $T : X \to X$ be a mapping satisfying (2), (a1), (a2) and*

(a4) if $\{x_n\}_{n\in\mathbb{N}_0}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.

Then T has a fixed point.

In 1996, Kada et al. [12] introduced the concept of w-distance in metric spaces and proved some fixed point theorems for some contractive mappings under w-distance. Later on, in 2016, Lakzian et al. [14] introduced the notion of generalized (α-ψ-p)-contractive mappings and proved fixed point results for these mappings, which extend Theorems 2 and 3.

Definition 2 ([12]) *A function $p : X \times X \to \mathbb{R}^+$ is called a w-distance in X if it satisfies the following:*

(p_1) $p(x, z) \leq p(x, y) + p(y, z), \forall x, y, z \in X$;

(p_2) for each $x \in X$, a mapping $p(x, \cdot) : X \to \mathbb{R}^+$ is lower semicontinuous, that is, if $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\lim_{n\to\infty} y_n = y \in X$, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$;

(p_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Theorem 4 ([14]) *Let p be a w-distance on a complete metric space (X, d), $\alpha : X \times X \to \mathbb{R}^+$, $\psi \in \Phi_3$ and $T : X \to X$ be an (α-ψ-p)-contractive mapping, that is,*

$$\alpha(x, y)p(Tx, Ty) \leq \psi(p(x, y)), \quad \forall x, y \in X.$$

Suppose that

(b1) T is α-admissible, that is, $\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1, \quad \forall x, y \in X$;

(b2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(b3) either T is continuous or, for any sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X if $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.

Then there exists a point $u \in X$ such that $Tu = u$. Moreover, if $\alpha(u, u) \geq 1$, then $p(u, u) = 0$.

Inspired by the ideas in the literature [1-16], especially those in [7] and [14], we give two fixed point theorems for α-admissible mappings of integral type with w-distance in metric spaces. The results presented herein extend Theorems 1–4. An example is constructed to support the obtained main results.

The following lemmas play a key role in this paper.

Lemma 1 **([15])** *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = 0 \Longleftrightarrow \lim_{n\to\infty} r_n = 0.$$

Lemma 2 **([16])** *Let $\varphi \in \Phi_3$. Then for each $t > 0$, $\lim_{n\to\infty} \varphi^n(t) = 0$ implies $\varphi(t) < t$.*

2 Main Results

In this section, we prove the existence of fixed points and iterative approximations for two new α-admissible contractive mappings of integral type in a complete metric space with w-distance. The results obtained in this paper improve and extend Theorems 1–4 mentioned in the previous section.

Our main results are as follows.

Definition 3 *A mapping T from a complete metric space (X, d) into itself is said to be a generalized $(\alpha$-ψ-$p)$-contraction of type (A) if there exist $\alpha : X \times X \to \mathbb{R}^+$ and a w-distance p on X and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ satisfying*

$$\alpha(x, y) \int_0^{p(Tx,Ty)} \varphi(t)dt \le \psi \left(\int_0^{M_1(x,y)} \varphi(t)dt \right), \quad \forall x, y \in X, \tag{3}$$

where

$$M_1(x, y) = \max \left\{ p(x, y), p(x, Tx), \frac{p(x, y)[1 + p(x, Tx)]}{1 + p(x, y)}, \frac{p(x, Tx)[1 + p(x, y)]}{1 + p(x, Tx)} \right\}. \tag{4}$$

Theorem 5 *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized $(\alpha$-ψ-$p)$-contraction of type (A) such that*

(c1) T is α-admissible, that is, $\alpha(x, y) \ge 1 \Longrightarrow \alpha(Tx, Ty) \ge 1, \quad \forall x, y \in X$;

(c2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;

if one of the following conditions holds:

(c3) T is continuous;

(c4) for any sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X if $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}_0$.

Then there exists a point $u \in X$ such that $Tu = u$. Moreover, if $\alpha(u, u) \ge 1$, then $p(u, u) = 0$.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$, where x_0 satisfies $(c2)$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$. Clearly, $x_{n_0} = Tx_{n_0}$ and $\lim_{n\to\infty} T^n x_0 = x_{n_0}$.

Now, we suppose that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}_0$. In light of $(c1)$ and $(c2)$, we obtain that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \ge 1 \Longrightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \ge 1. \tag{5}$$

It is obvious that

$$\alpha(x_n, x_{n+1}) \ge 1, \quad \forall n \in \mathbb{N}. \tag{6}$$

On the basis of (3), (4), (6) and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$, we infer that

$$\begin{aligned} M_1(x_{n-1}, x_n) &= \max\Big\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), \frac{p(x_{n-1}, x_n)[1 + p(x_{n-1}, Tx_{n-1})]}{1 + p(x_{n-1}, x_n)}, \\ &\quad \frac{p(x_{n-1}, Tx_{n-1})[1 + p(x_{n-1}, x_n)]}{1 + p(x_{n-1}, Tx_{n-1})}\Big\} \\ &= \max\Big\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), \frac{p(x_{n-1}, x_n)[1 + p(x_{n-1}, x_n)]}{1 + p(x_{n-1}, x_n)}, \\ &\quad \frac{p(x_{n-1}, x_n)[1 + p(x_{n-1}, x_n)]}{1 + p(x_{n-1}, x_n)}\Big\} \\ &= p(x_{n-1}, x_n) \end{aligned}$$

and

$$\begin{aligned} 0 &\le \int_0^{p(x_n, x_{n+1})} \varphi(t)dt = \int_0^{p(Tx_{n-1}, Tx_n)} \varphi(t)dt \\ &\le \alpha(x_{n-1}, x_n) \int_0^{p(Tx_{n-1}, Tx_n)} \varphi(t)dt \le \psi\Big(\int_0^{M_1(x_{n-1}, x_n)} \varphi(t)dt\Big) \\ &= \psi\Big(\int_0^{p(x_{n-1}, x_n)} \varphi(t)dt\Big) \\ &\le \psi^n\Big(\int_0^{p(x_0, x_1)} \varphi(t)dt\Big) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \end{aligned} \tag{7}$$

which yields that

$$\lim_{n\to\infty} \int_0^{p(x_n, x_{n+1})} \varphi(t)dt = 0. \tag{8}$$

On account of (8) and Lemma 1, we obtain that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{9}$$

Let $\varepsilon > 0$ and δ be defined by (p_3). Note that $\sum_{n=1}^{\infty} \psi^n\big(\int_0^{p(x_0, x_1)} \varphi(t)dt\big) < +\infty$, which implies that there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} \psi^k\Big(\int_0^{p(x_0, x_1)} \varphi(t)dt\Big) < \int_0^{\delta} \varphi(t)dt. \tag{10}$$

Next, we show that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in X. Using (3), (4), (7), (10), (p_1) and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$, we get that

$$\begin{aligned} \int_0^{p(x_n, x_m)} \varphi(t)dt &\le \int_0^{\sum_{k=n}^{m-1} p(x_k, x_{k+1})} \varphi(t)dt \le \sum_{k=n}^{m-1} \int_0^{p(x_k, x_{k+1})} \varphi(t)dt \\ &\le \sum_{k=n}^{m-1} \psi^k\Big(\int_0^{p(x_0, x_1)} \varphi(t)dt\Big) \\ &< \int_0^{\delta} \varphi(t)dt, \quad \forall m, n \in \mathbb{N} \text{ with } m > n \ge n_0, \end{aligned} \tag{11}$$

which means that

$$p(x_n, x_m) < \delta, \quad \forall m, n \in \mathbb{N} \text{ with } m > n \ge n_0, \tag{12}$$

and

$$p(x_{n_0}, x_n) < \delta \ \text{ and } \ p(x_{n_0}, x_m) < \delta, \quad \forall m, n \in \mathbb{N} \text{ with } m > n \geq n_0. \tag{13}$$

In virtue of (13) and (p_3), we deduce that

$$d(x_n, x_m) < \varepsilon, \quad \forall m, n \in \mathbb{N} \text{ with } m > n \geq n_0. \tag{14}$$

Therefore, $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists a point $u \in X$ with

$$\lim_{n\to\infty} x_n = u. \tag{15}$$

Suppose that $(c3)$ holds. By (15) and $(c3)$, we conclude that

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\big(\lim_{n\to\infty} x_n\big) = Tu. \tag{16}$$

Assume that $(c4)$ holds. By means of (6), (15), and $(c4)$, we have

$$\alpha(x_n, u) \geq 1, \quad \forall n \in \mathbb{N}_0. \tag{17}$$

Arguing similarly to the proofs of (10)–(12), we get that for arbitrary $\varepsilon_1 > 0$ there exists $n_1 \in \mathbb{N}$ satisfying

$$0 \leq p(x_n, x_m) < \varepsilon_1, \quad \forall m, n \in \mathbb{N}_0 \text{ with } m > n \geq n_1. \tag{18}$$

Combining with (p_2) and (15), we obtain that

$$0 \leq p(x_n, u) \leq \liminf_{m\to\infty} p(x_n, x_m) \leq \varepsilon_1, \quad \forall n \in \mathbb{N}_0 \text{ with } n \geq n_1, \tag{19}$$

that is,

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{20}$$

In view of (3), (4), (9), (17), (20), $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ and Lemma 2, we receive that

$$\begin{aligned}
\lim_{n\to\infty} M_1(x_n, u) &= \lim_{n\to\infty} \max\Big\{p(x_n, u), p(x_n, Tx_n), \frac{p(x_n, u)[1 + p(x_n, Tx_n)]}{1 + p(x_n, u)}, \\
&\qquad \frac{p(x_n, Tx_n)[1 + p(x_n, u)]}{1 + p(x_n, Tx_n)}\Big\} \\
&= \lim_{n\to\infty} \max\Big\{p(x_n, u), p(x_n, x_{n+1}), \frac{p(x_n, u)[1 + p(x_n, x_{n+1})]}{1 + p(x_n, u)}, \\
&\qquad \frac{p(x_n, x_{n+1})[1 + p(x_n, u)]}{1 + p(x_n, x_{n+1})}\Big\} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \int_0^{p(x_{n+1}, Tu)} \varphi(t)dt = \int_0^{p(Tx_n, Tu)} \varphi(t)dt \\
&\leq \alpha(x_n, u) \int_0^{p(Tx_n, Tu)} \varphi(t)dt \leq \psi\Big(\int_0^{M_1(x_n, u)} \varphi(t)dt\Big) \\
&\leq \int_0^{M_1(x_n, u)} \varphi(t)dt \\
&= \int_0^{\max\{p(x_n, u), p(x_n, x_{n+1})\}} \varphi(t)dt \longrightarrow 0 \text{ as } n \longrightarrow \infty,
\end{aligned} \tag{21}$$

which together with Lemma 1 yields that

$$\lim_{n\to\infty} p(x_{n+1}, Tu) = 0. \tag{22}$$

According to (9), (22) and (p_1), we obtain that

$$0 \le p(x_n, Tu) \le p(x_n, x_{n+1}) + p(x_{n+1}, Tu) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

that is,

$$\lim_{n\to\infty} p(x_n, Tu) = 0. \tag{23}$$

Let $\varepsilon_2 > 0$. It follows from (p_3) that there exists $\delta_1 > 0$ such that $p(u, v) \le \delta_1$ and $p(u, w) \le \delta_1$ imply that $d(v, w) \le \varepsilon_2$. By combining (20) and (23), we find that there exists $n_2 \in \mathbb{N}$ such that $p(x_n, u) \le \delta_1$ and $p(x_n, Tu) \le \delta_1$ for all $n \ge n_2$. Therefore, $d(u, Tu) \le \varepsilon_2$. Taking $\varepsilon_2 \to 0^+$, we arrive at

$$u = Tu. \tag{24}$$

Lastly, we certify that $p(u, u) = 0$ if $\alpha(u, u) \ge 1$. Assume that $p(u, u) > 0$. In terms of (3), (4), (24), $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ and Lemma 2, we obtain that

$$\begin{aligned} M_1(u,u) &= \max\left\{p(u,u), p(u,Tu), \frac{p(u,u)[1+p(u,Tu)]}{1+p(u,u)}, \frac{p(u,Tu)[1+p(u,u)]}{1+p(u,Tu)}\right\} \\ &= \max\left\{p(u,u), p(u,u), \frac{p(u,u)[1+p(u,u)]}{1+p(u,u)}, \frac{p(u,u)[1+p(u,u)]}{1+p(u,u)}\right\} \\ &= p(u,u) \end{aligned}$$

and

$$\begin{aligned} 0 &< \int_0^{p(u,u)} \varphi(t)dt = \int_0^{p(Tu,Tu)} \varphi(t)dt \\ &\le \alpha(u,u) \int_0^{p(Tu,Tu)} \varphi(t)dt \le \psi\left(\int_0^{M_1(u,u)} \varphi(t)dt\right) \\ &= \psi\left(\int_0^{p(u,u)} \varphi(t)dt\right) < \int_0^{p(u,u)} \varphi(t)dt, \end{aligned}$$

which is a contradiction. Consequently, $p(u, u) = 0$. This completes the proof. ■

Definition 4 *A mapping T from a complete metric space (X, d) into itself is said to be a generalized $(\alpha$-ψ-$p)$-contraction of type (B) if there exist $\alpha : X \times X \to \mathbb{R}^+$, a w-distance p in X and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ such that*

$$\alpha(x,y) \int_0^{p(Tx,Ty)} \varphi(t)dt \le \psi\left(\int_0^{M_2(x,y)} \varphi(t)dt\right), \quad \forall x, y \in X,$$

where

$$M_2(x,y) = \max\left\{p(x,y), p(x,Tx), \frac{p(x,y)+p(x,Tx)}{2[1+p(x,y)]}, \frac{p(x,Tx)+p(x,y)}{2[1+p(x,Tx)]}\right\}.$$

Similarly to the proof of Theorem 5, we have the following result and omit its proof.

Theorem 6 *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized $(\alpha$-ψ-$p)$-contraction of type (B) such that (c1)–(c4), defined in Theorem 5, hold. Then there exists a point $u \in X$ such that $Tu = u$. Moreover, if $\alpha(u, u) \ge 1$, then $p(u, u) = 0$.*

Remark 1 *It is easy to see that Theorems 5 and 6 extend Theorems 1–4. Example 1 below shows that Theorem 5 is a proper generalization of Theorem 1.*

Example 1 *Let* $X = [0, \frac{1}{2}] \cup \{\frac{3}{4}\} \cup \{1\}$ *with the usual metric* $d(x, y) = |x - y|$ *for all* $x, y \in X$*. Define* $p : X \times X \to \mathbb{R}^+$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$ *and* $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$p(x, y) = y, \quad \forall x, y \in X,$$

$$\varphi(t) = 1, \quad \psi(t) = \frac{5}{6}t, \quad \forall t \in \mathbb{R}^+,$$

and

$$Tx = \begin{cases} \frac{3}{4}x^2, & \forall x \in [0, \frac{1}{2}], \\ \frac{1}{2}, & x = \frac{3}{4}, \\ \frac{3}{4}, & x = 1, \end{cases} \qquad \text{and} \quad \alpha(x, y) = \begin{cases} 1, & \forall x, y \in [0, \frac{1}{2}] \cup \{\frac{3}{4}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, p is a w-distance in X and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$*. Let* $x, y \in X$*. In order to demonstrate (3), we need to consider three cases as follows.*

Case 1. $x \in [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$ *and* $y \in [0, \frac{1}{2}]$*. It is apparent that*

$$\begin{aligned} \alpha(x, y) \int_0^{p(Tx,Ty)} \varphi(t)dt &= \int_0^{\frac{3}{4}y^2} dt = \frac{3}{4}y^2 \leq \frac{5}{6}y = \psi(y) \\ &= \psi\Big(\int_0^{p(x,y)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big). \end{aligned}$$

Case 2. $x \in [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$ *and* $y = \frac{3}{4}$*. It is easy to see that*

$$\begin{aligned} \alpha(x, y) \int_0^{p(Tx,Ty)} \varphi(t)dt &= \int_0^{\frac{1}{2}} dt = \frac{1}{2} \leq \frac{5}{6} \cdot \frac{3}{4} = \psi(\frac{3}{4}) \\ &= \psi\Big(\int_0^{p(x,y)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big). \end{aligned}$$

Case 3. $x \notin [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$ *or* $y \notin [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$*. It is obvious that*

$$\alpha(x, y) \int_0^{p(Tx,Ty)} \varphi(t)dt = 0 \leq \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big).$$

Therefore, (3) holds. Let $x, y \in X$ *such that* $\alpha(x, y) \geq 1$*. It follows that* $x, y \in [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$ *and*

$$Tx = \begin{cases} \frac{3}{4}x^2, & \forall x \in [0, \frac{1}{2}], \\ \frac{1}{2}, & x = \frac{3}{4}, \end{cases} \qquad Ty = \begin{cases} \frac{3}{4}y^2, & \forall y \in [0, \frac{1}{2}], \\ \frac{1}{2}, & y = \frac{3}{4}, \end{cases}$$

which mean that $Tx \in [0, \frac{1}{2}]$ *and* $Ty \in [0, \frac{1}{2}]$*, that is,* $\alpha(Tx, Ty) = 1$*. Thus, T is an* α*-admissible mapping. Taking* $x_0 = \frac{3}{4} \in X$*. It is evident that* $\alpha(x_0, Tx_0) = \alpha(\frac{3}{4}, \frac{1}{2}) = 1$*.*

At last, let $\{x_n\}_{n \in \mathbb{N}_0}$ *be a sequence in X satisfying* $\alpha(x_n, x_{n+1}) \geq 1, \forall n \in \mathbb{N}_0$ *and* $x_n \to x \in X$ *as* $n \to \infty$*. Apparently,* $x_n \in [0, \frac{1}{2}] \cup \{\frac{3}{4}\}, \forall n \in \mathbb{N}_0$*. Because* $\{x_n\}_{n \in \mathbb{N}_0}$ *is a sequence in the closed subset* $[0, \frac{1}{2}] \cup \{\frac{3}{4}\}$ *of X, we can obtain that the point x belongs to* $[0, \frac{1}{2}] \cup \{\frac{3}{4}\}$*. Consequently,* $\alpha(x_n, x) = 1$ *for each* $n \in \mathbb{N}_0$*.*

That is, the conditions of Theorem 5 are fulfilled. It follows from Theorem 5 that T has a fixed point $0 \in X$*.*

However, Theorem 1 cannot be applied to testify the existence of the fixed point for the mapping T in X. Assume that the conditions of Theorem 1 are fulfilled. Put $y^* = \frac{3}{4}$. It is clear that

$$0 < \int_0^{\frac{5}{16}} \varphi(t)dt = \int_0^{d(T\frac{1}{2},T\frac{3}{4})} \varphi(t)dt \leq c\int_0^{d(\frac{1}{2},\frac{3}{4})} \varphi(t)dt < \int_0^{\frac{1}{4}} \varphi(t)dt \leq \int_0^{\frac{5}{16}} \varphi(t)dt,$$

which leads to a contradiction. Hence, we cannot invoke Theorem 1 to verify that the mapping T has a fixed point in X.

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