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Repelling Points For Polynomial Operators*

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Abstract

In this paper, we explore a property of polynomial operators related to a question first posed by Walter Rudin, a distinguished mathematician famous for his textbooks on Mathematical Analysis. The original question was whether surjective bilinear operators on complex spaces, like their linear counterparts, are necessarily open at the origin. This question took just over 50 years to be completely answered (see Introduction), but similar questions remain for the more general class of polynomial operators. In particular, we show that polynomial operators, unlike their bilinear cousins, do not have an open mapping theorem, except for the case where the range has dimension 1. We also consider the topic of Repelling Points for polynomial operators, which is related to the existence or nonexistence of an open mapping theorem, and see how this notion further distinguishes the general class of polynomial operators from multilinear operators.

1 Introduction

Walter Rudin was a well-known 20th-century mathematician renowned for his influential textbooks on Mathematical Analysis. In 1969, in his book "Function Theory in Polydiscs" [5], he posed an open question concerning bilinear operators on complex spaces. In particular, he asked if every surjective bilinear operator must be open at the origin. Several counterexamples were subsequently given [1], [4], but these relied on the range space having at least 4 dimensions. The question was finally resolved in 2020 [3], where a positive result was obtained conditional upon the dimension of the range space. In particular, it is shown that surjective bilinear operators onto spaces of dimension 3 or smaller must be open at the origin.

Also in [3] was an example showing that the positive result cannot be extended to more general polynomials. The example is that of a polynomial onto $\mathbb{C}^3$ which has the origin (in the range) as a repelling point, a certain pathology which excludes the possibility of openness. In fact, the notions of repelling points and openness at the origin are equivalent for multilinear operators but not for polynomials. We include here the definition of repelling point and the example mentioned above for convenience. We also mention that while open and open at the origin are equivalent for linear operators, they are not for bilinear or more general operators, and we are primarily concerned here with openness at the origin (as in [5]), and so by an open mapping theorem we mean that continuous surjective operators in that class must be open at the origin.

Definition 1 *Let $f : \mathbb{X} \to \mathbb{Y}$ be a function between topological vector spaces. A point $y \in \mathbb{Y}$ is a repelling point for f if no neighborhood of y is the image of a bounded subset of $\mathbb{X}$ under f.*

We mention that functions which have repelling points are easy to construct if the functions are not assumed to be continuous or surjective. However, imposing these two conditions makes the existence of repelling points much more restrictive. In fact, a function with a repelling point cannot be open at any preimage of that point. In particular, the Open Mapping Theorem for linear operators guarantees that no surjective linear operator can have a repelling point. Also in [3], it is shown that no surjective bilinear operator with range of dimension 3 or smaller can have a repelling point either. What made the notion of repelling points interesting initially was that for multilinear operators (more generally n-homogeneous), having the origin be a repelling point is equivalent to the operator not being open at the origin. We note this in the following lemma :

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Lemma 1 *Let $f : \mathbb{X} \to \mathbb{Y}$ be a surjective n-homogeneous operator between normed vector spaces. Then f is open at the origin if and only if 0 is not a repelling point for f.*

Proof. Since f is n-homogeneous, $f(0_{\mathbb{X}}) = 0_{\mathbb{Y}}$. If f is open at $0_{\mathbb{X}}$, then $0_{\mathbb{Y}}$ is interior to $f(B_{\mathbb{X}}(0,1))$ and so $0_{\mathbb{Y}}$ is not repelling for f. Now assume that $0_{\mathbb{Y}}$ is not repelling for f. Let W be any neighborhood of $0_{\mathbb{X}}$. Since $0_{\mathbb{Y}}$ is not repelling, there exist $M > 0$ and $\epsilon > 0$ such that $B_{\mathbb{Y}}(0,\epsilon) \subseteq f(B_{\mathbb{X}}(0,M))$. Let $B_{\mathbb{X}}(0,\delta) \subseteq W$. Assume that $\delta < 1$. Then

$$f(B_{\mathbb{X}}(0,\delta)) = f(\frac{\delta}{M} B_{\mathbb{X}}(0,M)) = \frac{\delta^n}{M^n} f(B_{\mathbb{X}}(0,M)) \supseteq \frac{\delta^n}{M^n} B_{\mathbb{Y}}(0,\epsilon).$$

■

The example mentioned above of a surjective polynomial which has the origin as a repelling point is given below. Note that the complement of the set of repelling points in the range of an operator is easily seen to be an open set, so that the set of repelling points is closed.

Example 1 *Define $B : \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^3$ by $B(x,y,z) = (x_1z_1, x_1z_2, x_1z_2y + x_2z_1)$. To see that B is surjective, let $(a,b,c) \in \mathbb{C}^3$ and first assume that $a = b = 0$. Then we choose $x_1 = 0$, $x_2 = 1$ and $z_1 = c$. If $a = 0$ and $b \neq 0$, then we choose $z_1 = 0$, $x_1 = 1$, $z_2 = b$ and $y = \frac{c}{b}$. Similarly, if $a \neq 0$ and $b = 0$, then we choose $z_2 = 0$, $x_1 = 1$, $z_1 = a$ and $x_2 = \frac{c}{a}$. Finally, if $a \neq 0$ and $b \neq 0$, then we choose $x_1 = 1$, $z_1 = a$, $z_2 = b$, $x_2 = 0$ and $y = \frac{c}{b}$. Now, we claim that $(0,0,1)$ is a repelling point for B. To see this, we consider the path of points $(t^2,t,1) \to (0,0,1)$ as $t \to 0$. Then $B(x,y,z) = (t^2,t,1)$. It implies that $z_1 = \frac{t^2}{x_1}$, $z_2 = \frac{t}{x_1}$ and $ty + x_2\frac{t^2}{x_1} = 1$. Hence, $y + x_2\frac{t}{x_1} = \frac{1}{t}$ and so $y + x_2z_2 = \frac{1}{t}$, which shows that at least one of y, x_2, z_2 must be large in norm, and $(0,0,1)$ is a repelling point for B. Moreover, using the same argument as above, $(0,0,\alpha)$ is a repelling point as well for any fixed nonzero α. Finally, as the set of repelling points is closed, letting $\alpha \to 0$ we see in fact that $(0,0,0)$ is a repelling point for B and B is not open at the origin.*

Note that the construction in Example 1 would be impossible if the operator were bilinear, but the $'y'$ coordinate makes the operator not bilinear and this somewhat subtle change allows a pathology like a repelling point to exist. So we know now that surjective bilinear operators cannot have 0 as a repelling point if the range is of dimension 3 or smaller, but general polynomials can. Two natural questions thus arise. The first is what about polynomials whose range is of dimension 2 or smaller and the second is whether Lemma 1 extends to general polynomials, namely, can a polynomial fail to be open at the origin without having a repelling point there. We address both of these below.

2 Main Results

Lemma 2 below gives an example of a surjective polynomial which is not open at the origin but for which 0 is not a repelling point, showing that unlike their multilinear cousins, these two properties are not equivalent for general polynomials.

In what follows, for two polynomials P and Q in the variables $x, y_1, \ldots, y_n$, we will use the notation $\mathrm{Res}^x(P,Q)$ for the resultant of P and Q when viewed as polynomials in x, so that $\mathrm{Res}^x(P,Q)$ is a polynomial in $y_1, \ldots, y_n$. Also, we will use P^x and Q^x to refer to P and Q as polynomials in x with polynomial coefficients in the other variables and $P^x(y_1, \ldots, y_k)$ to mean P^x with coefficients evaluated at the values $y_1, \ldots, y_k$.

Lemma 2 *There exists a continuous surjective polynomial onto $\mathbb{C}^2$ which is not open at the origin and for which 0 is not a repelling point.*

Proof. Define $f : \mathbb{C}^2 \to \mathbb{C}^2$ by $f(x,y) = (y^2x + y^3, (y+1)x + y^2 + y)$. We claim that f is surjective, not open at the origin but 0 is not a repelling point of f. To see that f is onto, let $(z,w) \in \mathbb{C}^2$.

Case 1: If $z = 0$, then we choose $y = 0$ and $x = w$.

Case 2: If $w = 0$, then we choose $y = -1$ and $x = z + 1$.

Case 3: If z and w are both nonzero, a preimage of (z, w) will be a pair (x, y) so that x is a common root of the two polynomials $P - z = y^2x + y^3 - z$ and $Q - w = (y + 1)x + y^2 + y - w$, where we view P and Q as polynomials in the variable x with coefficients that are polynomial functions in y. Such a root x exists if y is a root of $\mathrm{Res}^x(P - z, Q - w) = -wy^2 + zy + z$, which has roots $y = \frac{-z \pm \sqrt{z^2+4zw}}{-2w}$. Moreover, since z and w are nonzero, y is nonzero, and one can check that either of these choices of y along with $x = \frac{z-y^3}{y^2}$ gives a preimage of (z, w), and we see that f is onto.

We now show that f is not open at the origin. To see this, consider points of the form $(z, 0)$ where $z \neq 0$. Then

$$f(x, y) = (z, 0) \implies y \neq 0 \implies (y + 1)(\frac{z - y^3}{y^2}) + y^2 + y = 0 \implies z(y + 1) = 0 \implies y = -1.$$

In particular, for $\delta < 1$, $f(B_{\mathbb{C}^2}(0, \delta))$ does not contain $B_{\mathbb{C}^2}(0, \epsilon)$ for any $\epsilon > 0$ and we see that f is not open at the origin.

Finally, we show that preimages of (z, w), as (z, w) ranges over $B_{\mathbb{C}^2}(0, 1)$, can be chosen in a uniformly bounded way, which will show that f does not have 0 as a repelling point. For $(z, w) \in B_{\mathbb{C}^2}(0, 1)$, if $w = 0$ then we can let $y = -1$ and $x = z + 1$ and thus $\max\{|x|, |y|\} \leq 2$. Thus we assume $w \neq 0$, so that $y \neq -1$ and choose y to be the root

$$y_{z,w} = \frac{-z + \sqrt{z^2 + 4wz}}{-2w} \quad \text{if } z = \sqrt{z^2}$$

and choose

$$y_{z,w} = \frac{-z - \sqrt{z^2 + 4wz}}{-2w} \quad \text{if } z = -\sqrt{z^2}.$$

First assume that $|\frac{w}{z}| \geq \frac{1}{4}$, and let M_1 be a uniform upper bound on $|\sqrt{t^2 + 4t}|$ for $|t| \leq \frac{1}{4}$ in $\mathbb{C}^2$. Then

$$|y_{z,w}| = \frac{1}{2} \times |\frac{z}{w} \pm \sqrt{\frac{z^2}{w^2} + 4\frac{z}{w}}| \leq \frac{1}{2}(4 + M_1).$$

If $|\frac{w}{z}| < \frac{1}{4}$, then we express $y_{z,w}$ as $y_{z,w} = 2 \times \frac{1-\sqrt{1+D}}{D}$ where $D = \frac{4w}{z}$. Then $|D| < 1$ and so $y_{z,w}$ has series representation

$$y_{z,w} = -\frac{1}{2} + \frac{D}{8} - \frac{D^2}{16} + \frac{5D^3}{128} - \dots,$$

which clearly has upper bound

$$M_2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{5}{128} \dots$$

and we see that, in either case,

$$|y_{z,w}| < M = \max\{\frac{1}{2}(4 + M_1), M_2\}.$$

Finally, for any preimage (x, y) of (z, w), $w \neq 0 \implies y \neq -1$, and

$$x = \frac{z}{y^2} - y = \frac{w}{y + 1} - y \implies \frac{z}{y^2} = \frac{w}{y + 1}.$$

If $|y_{z,w}| < \frac{1}{2}$, then

$$|y_{z,w} + 1| > 1 - |y_{z,w}| > \frac{1}{2} \implies \frac{|w|}{|y_{z,w} + 1|} < 2 \implies |x| < 2 + \frac{1}{2} = \frac{3}{2}.$$

Otherwise,

$$|y_{z,w}| \geq \frac{1}{2} \implies |x| = \left|\frac{z}{y_{z,w}^2} - y_{z,w}\right| < \frac{1}{\frac{1}{4}} + M.$$

Thus, we see that for any $(z,w) \in B_{\mathbb{C}^2}(0,1)$ there is a preimage (x,y) with $|x| \le \max\{\frac{3}{2}, 4M\}$ and $|y| < M$ and 0 is not a repelling point of f. ■

Polynomials onto $\mathbb{C}$ are open mappings and therefore cannot have repelling points. Example 1 shows that if the range is 3-dimensional (or greater) then polynomials can have repelling points, so the only dimension unaccounted for is dimension 2. Theorem 1 below gives conditions under which a polynomial onto $\mathbb{C}^2$ cannot have 0 as a repelling point. We will need the following definition.

Definition 2 *Two polynomials P and Q in the variables $x, y, y_2, \ldots, y_n$ are xy-robust if, after fixing $y_2, \ldots y_n$ (if $n > 2$) at some values, we have*

$$P^x = a_n x^n + \ldots + a_1 x + a_0 \quad \text{and} \quad Q^x = b_m x^m + \ldots + b_1 x + b_0$$

and they are such that $\min\{n,m\} \ge 1$ and a_n, b_m have no roots y in common.

Example 2 *Let $P(x,y) = yx^2 + y = (x^2+1)y$ and $Q(x,y) = y^2 x$. Then P and Q are not xy-robust as y and y^2 share the root $y = 0$, but P and Q are yx-robust as x^2+1 and x have no root in common.*

Before stating our main theorem we note here three facts which will be used in the proof. By $\|\cdot\|$ we mean the max norm both for polynomials and vectors.

Note 1: For nonconstant polynomials, their roots change continuously with their coefficients. In particular, if h is a nonzero polynomial with a finite set of roots $r_1, \ldots, r_k$, then for any $\delta > 0$, there is an $\epsilon > 0$ such that $\|h - g\| < \epsilon$. Then g has roots within δ of $r_1, \ldots, r_k$ after ordering appropriately.

Note 2: For two polynomials P and Q of two variables x and y, y_0 is a root of $\text{Res}^x(P,Q)$ if and only if either there exists an x_0 such that $P(x_0,y_0) = Q(x_0,y_0) = 0$, or $\text{Deg}P^x(y_0) < \text{Deg}P^x$ and $\text{Deg}Q^x(y_0) < \text{Deg}Q^x$.

Note 3: For two polynomials P and Q in the variables x and y, and $(z,w) \in \mathbb{C}^2$, $\text{Res}^x(P-z, Q-w) = \text{Res}^x(P,Q) + \phi(z,w)$ where $\phi(z,w)$ is a polynomial in y with coefficients in z and w so that $\|\phi(z,w)\| \to 0$ as $(z,w) \to (0,0)$.

While the proof of Note 3 is elementary (and can be done by induction) we offer the following simple example to illustrate:

Example 3 *Let $P(x,y) = y^2x^2 + yx$ and $Q(x,y) = y^3x + y + 1$. Then*

$$Res^x(P,Q) = \begin{vmatrix} y^2 & y^3 & 0 \\ y & y+1 & y^3 \\ 0 & 0 & y+1 \end{vmatrix} = -y^5 + 2y^3 + y^2$$

and

$$Res^x(P-z, Q-w) = \begin{vmatrix} y^2 & y^3 & 0 \\ y & y+1-w & y^3 \\ -z & 0 & y+1-w \end{vmatrix} = -y^5 + 2y^3 + y^2 + \phi(z,w)$$

where

$$\phi(z,w) = y^2w^2 + (y^4 - 2y^3 - 2y^2)w - y^6 z.$$

Theorem 1 *Let $f = (P,Q)$ be a surjective polynomial onto $\mathbb{C}^2$. If f has variables x and y such that $Res^x(P,Q)$ is nonconstant and P, Q are xy-robust, then f does not have 0 as a repelling point.*

Proof. By the hypothesis, we assume that f is a function of just x and y. Let

$$P^x = a_n x^n + \ldots + a_1 x + a_0 \quad \text{and} \quad Q^x = b_m x^m + \ldots + b_1 x + b_0$$

where a_i and b_j are polynomials in y. As $\text{Res}^x(P,Q)$ is nonconstant, it has a finite set of roots which change continuously with its coefficients. Let y_0 be such a root, and assume w.l.o.g that y_0 is not a root of a_n (P, Q are xy-robust). Accordingly, $P^x(y_0)$ is a non-constant polynomial ($n \geq 1$) with a finite set of roots $x_1^0, x_2^0, \ldots, x_k^0$. Let ϵ_1 be such that $\|P^x(y_0) - g\| < \epsilon_1$. Then the roots of g are within 1 of these (after ordering appropriately). By the continuity of the coefficients of P^x, there is an ϵ_0 small enough so that $|y' - y_0| < \epsilon_0$. Then both $\|P^x(y_0) - P^x(y')\| < \frac{\epsilon_1}{2}$ and y' is also not a root of a_n. Let ϵ_2 be such that $\|Res^x(P,Q) - h\| < \epsilon_2$. Then their roots are within ϵ_0, and Let ϵ_3 be such that $\|(z,w)\| < \epsilon_3$. Then $\|\phi(z,w)\| < \epsilon_2$, where $\phi(z,w)$ is as in Note 3.

Finally, let $\epsilon = \min\{\epsilon_0, \frac{\epsilon_1}{2}, \epsilon_2, \epsilon_3\}$. Then for $\|(z,w)\| < \epsilon$, we have that

$$\|\text{Res}^x(P,Q) - \text{Res}^x(P-z, Q-w)\| = \|\text{Res}^x(P,Q) - (\text{Res}^x(P,Q) - \phi(z,w))\| = \|\phi(z,w)\| < \epsilon_2.$$

Then Res $^x(P-z, Q-w)$ has a root $y_{z,w}$ with $\|y_{z,w} - y_0\| < \epsilon_0$. It follows that

$$\|P^x(y_0) - (P^x(y_{z,w}) - z)\| \leq \|P^x(y_0) - P^x(y_{z,w})\| + |z| < \epsilon_1.$$

So $P^x(y_{z,w}) - z$ has a root $x_{z,w}$ within 1 of one of $x_1^0, x_2^0, \ldots, x_k^0$. Moreover, as $y_{z,w}$ is not a root of a_n, by Note 2, $x_{z,w}$ corresponds to a common root of $P^x(y_{z,w}) - z$ and $Q^x(y_{z,w}) - w$. In particular, $f(x_{z,w}, y_{z,w}) = (z,w)$ and

$$\|(x_{z,w}, y_{z,w})\| < \max\{\max\{|x_1^0|, |x_2^0|, \ldots, |x_k^0|\} + 1, |y_0| + \epsilon_0\}.$$

As these bounds are uniform for $\|(z,w)\| < \epsilon$, f cannot have 0 as a repelling point. ■

3 Known Results

As of the writing of this article, Table 1 below summarizes what is known for classes of operators regarding open mapping theorems and or repelling points:

Table 1

Operator Class Continuous And Surjective	Largest Dimension Of Range For Open Mapping Theorem	Smallest Dimension Of Range Origin Can Be Repelling	Open at 0 equivalent to Non-Repelling At 0
Linear	∞	N/A	Yes
Bilinear	3	4	Yes
Multilinear	2 or 3	3 or 4	Yes
Polynomial	1	2 or 3	No
General	Unknown	unknown	No

4 Conclusion

The operator in Lemma 2 settles the question of open mapping theorems for polynomials according to the dimension of the range space. Moreover, it is interesting to note that this operator does not meet the hypothesis of Theorem 1 as both $\text{Res}^x(P,Q)$ and $\text{Res}^y(P,Q)$ are identically zero, and so the existence of a surjective polynomial onto $\mathbb{C}^2$ for which 0 is a repelling point is still in question. Moreover, as seen in Table 1, there are also unresolved questions for multilinear operators regarding the dimension of the range space.

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