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Fixed Point Results Using $F_{\mathcal{R}}$-Contractive Map And $F_{\mathcal{R}}$-Expansive Map In Bipolar $\mathcal{R}$-Metric Space*

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Abstract

This paper investigates fixed point theorems in bipolar $\mathcal{R}$-metric space by introducing the concepts of $F_{\mathcal{R}}$-contractive and $F_{\mathcal{R}}$-expansive maps. We first establish a fixed point theorem for $F_{\mathcal{R}}$-contractive map, then extend our results to $F_{\mathcal{R}}$-expansive map. Our findings reveal that under certain conditions, these results can be reduced to novel fixed point results for expansive maps in bipolar metric space. This aligns with existing literature, underscoring the relevance, and applicability of our findings in broader contexts.

1 Introduction and Preliminaries

In the literature of metric analysis, the idea of a bipolar metric space has gained significant recognition after being introduced by [15]. This framework expands the concept of traditional metric spaces by defining the metric d on the Cartesian product of two distinct abstract spaces, thereby providing a more comprehensive approach to metric spaces. This notable area of research has been extensively studied as well as generalized by various authors (see [1, 3, 6, 13, 16] and references cited therein). Extending the idea of a bipolar metric space further in [7], Malhotra and Kumar introduced the notion of a bipolar $\mathcal{R}$-metric space, with $\mathcal{R}$ as an amorphous binary relation on the bipolar metric space, and discussed fixed point results in the context of a contractive mapping. In the recent literature, many generalizations of metric spaces exist. For further reading, one may refer to [2, 4, 9, 11, 12], and references cited therein.

Additionally, the study of expansive maps has emerged as another interesting area of research, as highlighted by Wang in [18]. These mappings present a different class of functions, opening new possibilities for fixed point results across various types of metric spaces (refer to [5, 8, 17] for further insights). The primary aim of the current manuscript is to present the notions of an $F_{\mathcal{R}}$-contractive map, an $F_{\mathcal{R}}$-expansive map and to establish fixed point results in a bipolar $\mathcal{R}$-metric space under these mappings. With specific constraints, the results proved in this manuscript reduce to a novel fixed point approach in a bipolar metric space for an expansive map, and align with some known results in the literature.

Definition 1 ([19]) *Let $\mathfrak{F}$ be the family of functions $F:(0,+\infty)\to(-\infty,+\infty)$ satisfying the following:*

(F_1) *for $\varrho,\varsigma\in(0,+\infty)$, $\varrho<\varsigma$ implies $F(\varrho)<F(\varsigma)$;*

(F_2) *for each sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of positive real numbers such that*

$$\lim_{n\to+\infty}\varrho_n=0 \text{ if and only if } \lim_{n\to+\infty}F(\varrho_n)=-\infty;$$

(F_3) *there exists $\varpi\in(0,1)$ such that $\lim_{\varsigma\to0^+}\varsigma^{\varpi}F(\varsigma)=0$.*

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1.1 Bipolar Metric Space

Definition 2 ([15]) *For two non-empty sets X and Y, let $d : X \times Y \to [0, +\infty)$ be a function satisfying the following:*

(i) $d(\varrho, \varsigma) = 0$ *if and only if* $\varrho = \varsigma$ *for all* $(\varrho, \varsigma) \in X \times Y$;

(ii) $d(\varrho, \varsigma) = d(\varsigma, \varrho)$ *for all* $\varrho, \varsigma \in X \cap Y$;

(iii) $d(\varrho_1, \varsigma_2) \leq d(\varrho_1, \varsigma_1) + d(\varrho_2, \varsigma_1) + d(\varrho_2, \varsigma_2)$ *for all* $\varrho_1, \varrho_2 \in X$, *and* $\varsigma_1, \varsigma_2 \in Y$.

Then the triplet (X, Y, d) is said to be a bipolar metric space.

Definition 3 ([15]) *For a bipolar metric space (X, Y, d), where X and Y are non-empty sets, we say:*

(i) a point is left, right or central point if it belongs to X, Y or $X \cap Y$ respectively,

(ii) a sequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}}$ on the set $X \times Y$ is a bisequence on (X, Y, d),

(iii) a bisequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}}$ is convergent if both $\{\varrho_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$ are convergent to the respective right and left points. In addition, if both $\{\varrho_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$ converge to the same central point, then the bisequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}}$ is said to be biconvergent.

1.2 Bipolar $\mathcal{R}$-Metric Space

Definition 4 ([7]) *Two non-empty sets X and Y together with a metric $d : X \times Y \to [0, +\infty)$ and a binary relation $\mathcal{R} \subseteq X \times Y$ are called a bipolar $\mathcal{R}$-metric space (denoted by $(X, Y, d, \mathcal{R})$) if (X, Y, d) is a bipolar metric space, and $\mathcal{R}$ is an arbitrary binary relation on $X \times Y$.*

Definition 5 ([7]) *For a bipolar $\mathcal{R}$-metric space $(X, Y, d, \mathcal{R})$, we say a bisequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}}$ is an $\mathcal{R}$-bisequence if $(\varrho_n, \varsigma_{n+1}) \in \mathcal{R}$ or $(\varrho_{n+1}, \varsigma_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$.*

Definition 6 ([7]) *For a bipolar $\mathcal{R}$-metric space $(X, Y, d, \mathcal{R})$, we say a bisequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}}$ is convergent $\mathcal{R}$-bisequence if both $\{\varrho_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$ are convergent to their respective right and left points.*

Definition 7 ([7]) *For a bipolar $\mathcal{R}$-metric space $(X, Y, d, \mathcal{R})$, we say a bisequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}}$ is a biconvergent $\mathcal{R}$-bisequence if both $\{\varrho_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$ are convergent to the same central point.*

Definition 8 ([7]) *For a bipolar $\mathcal{R}$-metric space $(X, Y, d, \mathcal{R})$, a map $\varphi : X \cup Y \to X \cup Y$ is said to be bipolar $\mathcal{R}$-continuous if for any convergent $\mathcal{R}$-bisequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}} \in X \times Y$ such that for some $(\varrho, \varsigma) \in X \times Y$, we have*

$$\begin{aligned} &\varrho_n \to \varsigma, \text{ and } \varsigma_n \to \varrho \text{ as } n \to +\infty, \\ \text{implies} \quad &\varphi\varrho_n \to \varphi\varsigma, \text{ and } \varphi\varsigma_n \to \varphi\varrho \text{ as } n \to +\infty. \end{aligned}$$

Definition 9 ([7]) *A bipolar $\mathcal{R}$-metric space $(X, Y, d, \mathcal{R})$ is said to be a complete bipolar $\mathcal{R}$-metric space if every Cauchy $\mathcal{R}$-bisequence is a convergent $\mathcal{R}$-bisequence.*

Lemma 1 ([7]) *In a bipolar $\mathcal{R}$-metric space, every convergent Cauchy $\mathcal{R}$-bisequence is a biconvergent $\mathcal{R}$-bisequence.*

2 Main Result

Definition 10 *Let $(X,\ Y,\ d,\ \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space, then $\varphi : X \cup Y \to X \cup Y$ is said to be an $F_{\mathcal{R}}$-contractive map, where $F \in \mathfrak{F}$, if there exists $\tau > 0$ such that for $(\varrho,\ \varsigma) \in \mathcal{R}$, we have*

$$\tau + F(d(\varphi\varrho,\ \varphi\varsigma)) \leqslant F(d(\varrho,\ \varsigma)).$$

Definition 11 *Let $(X,\ Y,\ d,\ \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space, then $\varphi : X \cup Y \to X \cup Y$ is said to be an $F_{\mathcal{R}}$-expansive map, where $F \in \mathfrak{F}$, if there exists $\tau > 0$ such that for $(\varrho,\ \varsigma) \in \mathcal{R}$, we have*

$$F(d(\varphi\varrho,\ \varphi\varsigma)) \geqslant F(d(\varrho,\ \varsigma)) + \tau.$$

Example 1 *Let $X = \mathbb{N}$ and $Y = \mathbb{Z}$ with metric d as usual metric space. Define $\varphi : X \cup Y \to X \cup Y$ as $\varphi(\varrho) = 29\varrho$ and let $F(\mu) = \ln \mu$. Let the binary relation $\mathcal{R}$ on $X \times Y$ be defined as $(\varrho,\ \varsigma) \in \mathcal{R}$ implies $\varrho = 1$, $\varsigma \in \mathbb{Z}$. Then $(X,\ Y,\ d,\ \mathcal{R})$ is a bipolar $\mathcal{R}$-metric space. Now, for all $(\varrho,\ \varsigma) \in \mathcal{R}$, we have*

$$F(d(\varphi\varrho,\ \varphi\varsigma)) = F(d(\varphi 1,\ \varphi\varsigma)) = \ln(29) + \ln(1-\varsigma), \tag{1}$$

and

$$F(d(\varrho,\ \varsigma)) + \tau = \ln(1-\varsigma) + \tau. \tag{2}$$

By (1), and (2) we can conclude that φ is an $F_{\mathcal{R}}$-expansive map for $0 < \tau \leqslant \ln(29)$.

Theorem 1 *Let $(X,\ Y,\ d,\ \mathcal{R})$ be a complete bipolar $\mathcal{R}$-metric space, and let $\varphi : X \cup Y \to X \cup Y$ be a map such that for some $F \in \mathfrak{F}$, following holds:*

(i) $\varphi(X) \subseteq X$, and $\varphi(Y) \subseteq Y$;

(ii) φ is an $F_{\mathcal{R}}$-contractive map;

(iii) there exists $(\varrho_0,\ \varsigma_0) \in X \times Y$ such that $(\varrho_0,\ \varsigma_0) \in \mathcal{R}$, and $(\varrho_0,\ \varphi\varsigma_0) \in \mathcal{R}$;

(iv) φ is bipolar $\mathcal{R}$-continuous;

(v) for each $(\varrho,\ \varsigma) \in \mathcal{R}$, we have $(\varphi\varrho,\ \varphi\varsigma) \in \mathcal{R}$.

Then, φ possesses a fixed point. Furthermore, if there exist two fixed points $\varrho,\ \varrho^*$, then $(\varrho,\ \varrho^*) \in \mathcal{R}$, and in such case φ possesses a unique fixed point.

Proof. Let $(\{\varrho_n\},\ \{\varsigma_n\})_{n\in\mathbb{N}\cup\{0\}}$ be a bisequence in $X \times Y$ such that

$$\varphi\varrho_{n-1} = \varrho_n \text{ and } \varphi\varsigma_{n-1} = \varsigma_n.$$

Since $(\varrho_0,\ \varsigma_0) \in X \times Y$, by condition (iii), we obtain

$$(\varrho_0,\ \varsigma_0) \in \mathcal{R} \text{ and } (\varrho_0,\ \varphi\varsigma_0) \in \mathcal{R}.$$

On using condition (v), we obtain

$$(\varphi\varrho_0,\ \varphi\varsigma_0) \in \mathcal{R} \text{ and } (\varphi\varrho_0,\ \varphi\varsigma_1) \in \mathcal{R}.$$

By repeated use of condition (v), we get

$$(\varrho_n,\ \varsigma_n) \in \mathcal{R} \text{ and } (\varrho_n,\ \varsigma_{n+1}) \in \mathcal{R} \ \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Thus $(\{\varrho_n\},\ \{\varsigma_n\})_{n\in\mathbb{N}\cup\{0\}}$ is an $\mathcal{R}$-bisequence. Now,

$$\begin{aligned} F(d(\varrho_{n+1},\ \varsigma_{n+1})) &= F(d(\varphi\varrho_n,\ \varphi\varsigma_n)) \leqslant F(d(\varrho_n,\ \varsigma_n)) - \tau \\ &\leqslant F(d(\varrho_{n-1},\ \varsigma_{n-1})) - 2\tau \leqslant \ldots \leqslant F(d(\varrho_0,\ \varsigma_0)) - (n+1)\tau. \end{aligned} \tag{3}$$

Letting $n \to +\infty$ in (3), and using F_2 property of F, we have

$$\lim_{n\to+\infty} d(\varrho_{n+1}, \varsigma_{n+1}) = 0.$$

Further, using the F_3 property of F, we obtain that there exists $\varpi \in (0,\, 1)$ such that

$$\lim_{n\to+\infty} \big(d(\varrho_{n+1},\ \varsigma_{n+1})\big)^{\varpi} F(d(\varrho_{n+1},\ \varsigma_{n+1})) = 0. \tag{4}$$

Using (3) in (4), we obtain

$$\big(d(\varrho_{n+1},\ \varsigma_{n+1})\big)^{\varpi}\big(F(d(\varrho_{n+1},\varsigma_{n+1})) - F(d(\varrho_0,\ \varsigma_0))\big) \leqslant -(n+1)\big(d(\varrho_{n+1},\ \varsigma_{n+1})\big)^{\varpi}\tau. \tag{5}$$

Taking limit as $n \to +\infty$ in (5), we get

$$\lim_{n\to+\infty} (n+1)\big(d(\varrho_{n+1},\ \varsigma_{n+1})\big)^{\varpi} = 0.$$

Thus, there exists $n^* \in \mathbb{N}\cup\{0\}$ such that for all $n \geqslant n^*$,

$$d(\varrho_{n+1},\ \varsigma_{n+1}) \leqslant \frac{1}{(n+1)^{1/\varpi}}. \tag{6}$$

Then

$$\begin{aligned} F(d(\varrho_n,\ \varsigma_{n+1})) = F(d(\varphi\varrho_{n-1},\ \varphi\varsigma_n)) &\leqslant F(d(\varrho_{n-1},\ \varsigma_n)) - \tau \\ &\leqslant F(d(\varrho_{n-2},\ \varsigma_{n-1})) - 2\tau \\ &\leqslant \cdots \leqslant F(d(\varrho_0,\ \varsigma_1)) - n\tau. \end{aligned} \tag{7}$$

Taking limit as $n \to +\infty$ in (7), we get

$$\lim_{n\to+\infty} d(\varrho_n,\ \varsigma_{n+1}) = 0.$$

By using F_3 property of F, we obtain that there exists $\varpi^* \in (0,\, 1)$ such that

$$\lim_{n\to+\infty} \big(d(\varrho_n,\ \varsigma_{n+1})\big)^{\varpi^*} F(d(\varrho_n,\ \varsigma_{n+1})) = 0. \tag{8}$$

Using (8) in (7), we have

$$\Big(d(\varrho_n,\ \varsigma_{n+1})\Big)^{\varpi^*}\Big(F(d(\varrho_n,\ \varsigma_{n+1})) - F(d(\varrho_0,\ \varsigma_1))\Big) \leqslant -n\big(d(\varrho_n,\ \varsigma_{n+1})\big)^{\varpi^*}\tau. \tag{9}$$

Taking limit as $n \to +\infty$ in (9), we get

$$\lim_{n\to+\infty} n\big(d(\varrho_n,\ \varsigma_{n+1})\big)^{\varpi^*} = 0.$$

Thus, there exists some $n^{**} \in \mathbb{N}\cup\{0\}$ such that for all $n \geqslant n^{**}$, we obtain

$$d(\varrho_n,\ \varsigma_{n+1}) \leqslant \frac{1}{n^{1/\varpi^*}}. \tag{10}$$

Consider $n,\, n',\, m \in \mathbb{N}\cup\{0\}$ where $n' = \max\{n^*,\, n^{**}\}$, and $m > n > n'$ we have

$$\begin{aligned} d(\varrho_n,\ \varsigma_m) &\leqslant d(\varrho_n,\ \varsigma_{n+1}) + d(\varrho_{n+1},\ \varsigma_{n+1}) + d(\varrho_{n+1},\ \varsigma_{n+2}) + \ldots + d(\varrho_{m-1},\ \varsigma_m) \\ &= \Big(d(\varrho_{n+1},\ \varsigma_{n+1}) + d(\varrho_{n+2},\ \varsigma_{n+2}) + \ldots + d(\varrho_{m-1},\ \varsigma_{m-1})\Big) \\ &\quad + \Big(d(\varrho_n,\ \varsigma_{n+1}) + d(\varrho_{n+1},\ \varsigma_{n+2}) + \ldots + d(\varrho_{m-1},\ \varsigma_m)\Big) \\ &\leqslant \sum_{\gamma=1}^{+\infty} d(\varrho_\gamma,\ \varsigma_\gamma) + \sum_{\delta=1}^{+\infty} d(\varrho_\delta,\ \varsigma_{\delta+1}). \end{aligned} \tag{11}$$

Using (6) and (10) in (11), we obtain

$$d(\varrho_n,\ \varsigma_n) \leqslant \sum_{\gamma=1}^{+\infty} \frac{1}{(\gamma+1)^{\frac{1}{\varpi}}} + \sum_{\delta=1}^{+\infty} \frac{1}{(\delta)^{\frac{1}{\varpi^*}}}. \tag{12}$$

Since (12) is a convergent series, $(\{\varrho_n\},\ \{\varsigma_n\})_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy $\mathcal{R}$-bisequence on complete bipolar $\mathcal{R}$-metric space. Using Lemma 1, we conclude that there exists $\sigma \in X \cap Y$ such that

$$\lim_{n\to\infty} \varrho_n = \sigma, \text{ and } \lim_{n\to\infty} \varsigma_n = \sigma.$$

Using the $\mathcal{R}$-continuity of φ, we get

$$\lim_{n\to+\infty} \varphi\varrho_n = \lim_{n\to+\infty} \varrho_{n+1} = \varphi\sigma, \text{ and } \lim_{n\to\infty} \varphi\varsigma_n = \lim_{n\to\infty} \varsigma_{n+1} = \varphi\sigma,$$

that is, $\sigma = \varphi\sigma$. Thus, φ possesses a fixed point.

Next, we claim that σ^* is another fixed point of φ, and hence $(\sigma,\ \sigma^*) \in \mathcal{R}$. Now,

$$F(d(\sigma,\ \sigma^*)) \leqslant \tau + F(d(\varphi\sigma,\ \varphi\sigma^*)) \leqslant F(d(\sigma,\ \sigma^*)),$$

if $\sigma = \sigma^*$. Hence, φ possesses a unique fixed point. ■

Theorem 2 *Let $(X,\ Y,\ d,\ \mathcal{R})$ be a complete bipolar $\mathcal{R}$-metric space, and let $\varphi : X \cup Y \to X \cup Y$ be a surjective map with φ^* as the inverse of φ such that for some $F \in \mathfrak{F}$, the following holds:*

(i) $\varphi(X),\ \varphi^*(X) \subseteq X$, and $\varphi(Y),\ \varphi^*(Y) \subseteq Y$;

(ii) φ is an $F_{\mathcal{R}}$-expansive map;

(iii) there exists $(\varrho_0,\ \varsigma_0) \in X \times Y$ such that $(\varrho_0,\ \varsigma_0) \in \mathcal{R}$, $(\varrho_0,\ \varphi\varsigma_0) \in \mathcal{R}$, and $(\varrho_0,\ \varphi^*\varsigma_0) \in \mathcal{R}$;

(iv) $\varphi,\ \varphi^*$ are $\mathcal{R}$-continuous;

(v) for each $(\varrho,\ \varsigma) \in \mathcal{R}$, we have $(\varphi\varrho,\ \varphi\varsigma) \in \mathcal{R}$, and $(\varphi^*\varrho,\ \varphi^*\varsigma) \in \mathcal{R}$.

Then, φ possesses a fixed point. Furthermore, if there exist two fixed points ϱ and ϱ^*, then $(\varrho,\ \varrho^*) \in \mathcal{R}$. In this case, φ possesses a unique fixed point.

Proof. Let $(\{\varrho_n\},\ \{\varsigma_n\})_{n\in\mathbb{N}\cup\{0\}}$ be a bisequence in $X \times Y$ such that

$$\varphi\varrho_{n-1} = \varrho_n, \text{ and } \varphi\varsigma_{n-1} = \varsigma_n \quad \text{for all } n \in \mathbb{N}.$$

Now, proceeding on the lines of Theorem 1, we obtain that $(\{\varrho_n\},\ \{\varsigma_n\})_{n\in\mathbb{N}\cup\{0\}}$ is an $\mathcal{R}$-bisequence. Since $(\varrho_n,\ \varsigma_n) \in \mathcal{R}$ for $n \in \mathbb{N}$ and φ is surjective, we have $\varphi^* : X \cup Y \to X \cup Y$ such that

$$\varphi^*\varrho_n = \varrho_{n-1}, \text{ and } \varphi^*\varsigma_{n+1} = \varsigma_n \text{ for all } n \in \mathbb{N}\cup\{0\}.$$

Next,

$$\begin{aligned} F(d(\varrho_n,\ \varsigma_{n+1})) &= F(d(\varphi\varrho_{n-1},\ \varphi\varsigma_n)) \\ &\geqslant F(d(\varrho_{n-1},\ \varsigma_n)) + \tau \\ &= F(d(\varphi^*\varrho_n,\ \varphi^*\varsigma_{n+1})) + \tau. \end{aligned} \tag{13}$$

By (13) and Theorem 1, φ^* has a unique fixed point. Then there exists a unique $\sigma \in X \cap Y$ such that

$$\varphi^*\sigma = \sigma,$$

that is, $\varphi\sigma = \sigma$. Thus, φ possesses a unique fixed point. ■

Example

Example 2 *Let $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$ together with metric $d : X \times Y \to [0, +\infty)$ defined as $d(\varrho, \varsigma) = |\varrho - \varsigma|$, and a binary relation $\mathcal{R} \subset X \times Y$ defined as*

$$\mathcal{R} = \{(1,\ 3),\ (2,\ 3),\ (3,\ 3)\}.$$

Define a map $\varphi : X \cup Y \to X \cup Y$ by $\varphi(1) = 2$, $\varphi(2) = 3$, $\varphi(3) = 3$, $\varphi(4) = 4$ and $\varphi(5) = 4$. Clearly $\varphi(X) \subseteq X$, $\varphi(Y) \subseteq Y$, and for any $(\varrho, \varsigma) \in \mathcal{R}$, we have $(\varphi(\varrho), \varphi(\varsigma)) \in \mathcal{R}$. Also, for any convergent $\mathcal{R}$-bisequence $(\{\varrho_n\}, \{\varsigma_n\})_{n\in\mathbb{N}} \in X \times Y$, we have $\lim_{n\to+\infty} \varrho_n = 3$, and $\lim_{n\to+\infty} \varsigma_n = 3$. Then $\lim_{n\to+\infty} \varphi\varrho_n = \varphi 3 = 3$, and $\lim_{n\to+\infty} \varphi\varsigma_n = \varphi 3 = 3$.

Next, we show that φ is an $F_{\mathcal{R}}$-contractive map. Here, $F(\mu) = \ln(\mu) + \mu$. We consider the following cases:

***Case (i)**: Let $(\varrho, \varsigma) = (1, 3)$. Then,*

$$\tau + F(d(\varphi\varrho,\ \varphi\varsigma)) = \tau + \ln(d(2,\ 3)) + d(2,\ 3) = \tau + 1$$

and

$$F(d(\varrho, \varsigma)) = \ln(d(1, 3)) + d(1, 3) = \ln(2) + 2.$$

So the $F_{\mathcal{R}}$-contractive condition holds in this case for any $\tau \in (0, \ln(2) + 1)$.

***Case (ii)**: Let $(\varrho, \varsigma) = (2, 3)$. Then,*

$$\tau + F(d(\varphi\varrho,\ \varphi\varsigma)) \to -\infty \text{ and } F(d(\varrho,\ \varsigma)) = \ln(d(2,\ 3)) + d(2,\ 3) = 1.$$

So the $F_{\mathcal{R}}$-contractive condition holds in this case for any $\tau > 0$.

***Case (iii)**: Let $(\varrho, \varsigma) = (3, 3)$. Then,*

$$\tau + F(d(\varphi\varrho,\ \varphi\varsigma)) \to -\infty \text{ and } F(d(\varrho,\ \varsigma)) \to -\infty.$$

So the $F_{\mathcal{R}}$-contractive condition holds in this case for any $\tau > 0$.
Since the conditions (i)–(v) of Theorem 1 hold, φ possesses fixed points which are $\varrho = 3$ and $\varrho = 4$.

Comparison with Result in the Literature

In this section of the manuscript, we reduce the result proved in the previous section for bipolar metric space (X, Y, d) under a contractive map, and obtain some fixed point result available in the literature.

Theorem 3 ([10]) *Let (X, Y, d) be a complete bipolar metric space, and let $\varphi : X \cup Y \to X \cup Y$ be a map where φ is continuous map with $\varphi(X) \subseteq X$, $\varphi(Y) \subseteq Y$, and for some $F \in \mathfrak{F}$, there exists $\tau > 0$ such that*

$$\tau + F(d(\varphi\varrho,\ \varphi\varsigma)) \leqslant F(d(\varrho,\ \varsigma)).$$

Then φ possesses a unique fixed point.

Proof. In Theorem 1, if we consider $\mathcal{R}$ to be the universal relation on $X \times Y$, that is, $\mathcal{R} = X \times Y$, then the above result is obtained. ∎

Conclusion

In this manuscript, we have presented fixed point results in bipolar $\mathcal{R}$-metric spaces by introducing the novel concepts of $F_{\mathcal{R}}$-contractive and $F_{\mathcal{R}}$-expansive maps. The results obtained further substantiate that, under specific conditions, these results can be reduced to some well-known results in the literature, thereby generalizing existing ideas.

Additionally, if in Theorem 2, we consider the binary relation $\mathcal{R}$ to be the universal relation, we obtain a fixed point result in bipolar metric spaces in conjunction with expansive maps. This particular case illustrates the utility of the results proved in the present manuscript, which further opens up broader potential directions for research in the literature on fixed point theory.

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