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Some Special Helices In Myller Configuration*

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Abstract

Some new types of special curves, such as $\bar{\xi}$-helix, $\bar{\xi}_1$-helix, $\bar{\mu}$-helix, $\bar{\nu}$-helix and W_k-Darboux helices in the Myller configuration $M(C,\bar{\xi},\pi)$ are defined and studied where $k \in \{n,r,o\}$. The necessary and sufficient conditions for a curve in $M(C,\bar{\xi},\pi)$ to be classified as a special helix are established. Additionally, the axes of these helices are presented, and the relationships between them are discussed.

1 Introduction

In the Euclidean 3-space E^3, the geometric properties of a curve C are investigated by the aid of orthonormal frames defined along the curve. The most well-known of such frames is the Serret-Frenet frame $\{t,n,b\}$, where the unit vector fields t,n,b denote the tangent, principal normal, and binormal of C, respectively. Myller considered some more general frames along a curve C. By considering a unit vector $\bar{\xi}$ and a plane π along a curve C, and calling them a versor field $(C,\bar{\xi})$ and a plane field (C,π) such that $\bar{\xi}\in\pi$, he defined a configuration in E^3 called the Myller configuration, denoted by $M(C,\bar{\xi},\pi)$ [7]. If the planes π are tangent to C, then this configuration is called a tangent Myller configuration, denoted by $M_t(C,\bar{\xi},\pi)$. Specifically, if the curve C is a surface curve lying on a surface $S\subset E^3$ with arclength parameter s, where $\bar{\xi}(s)$ is the tangent versor field to S along C, $\pi(s)$ is the tangent plane field to S along C, then $M_t(C,\bar{\xi},\pi)$ is called the tangent Myller configuration intrinsic associated with the geometric objects S, C, and $\bar{\xi}$. Thus, the geometry of the versor field $(C,\bar{\xi})$ on a surface S is the geometry of the associated Myller configurations $M_t(C,\bar{\xi},\pi)$. Moreover, $M_t(C,\bar{\xi},\pi)$ represents a particular case of $M(C,\bar{\xi},\pi)$. In the special case where the tangent Myller configuration $M_t(C,\bar{\xi},\pi)$ is the associated Myller configuration to a curve C on a surface S, the classical theory of surface curves (curves lying on a surface) is obtained.

The parallelism of the versor field $(C,\bar{\xi})$ in the plane field (C,π) was studied by Alexandru Myller [8]. He obtained a generalization of Levi-Civita parallelism on the curved surfaces. Later, Mayer introduced new invariants of $M(C,\bar{\xi},\pi)$ and $M_t(C,\bar{\xi},\pi)$ [6]. Miron extended the notion of Myller configuration to Riemannian geometry [7]. Vaisman considered the Myller configuration in symplectic geometry [12]. Furthermore, Myller configurations were studied in different spaces [1, 7]. Moreover, Macsim et al. have studied rectifying-type curves in a Myller configuration [5].

In the present paper, we study certain special helices in a Myller configuration M. We provide characterizations for these curves and describe their axes. Additionally, we introduce the relations between these special helices in M.

2 Preliminaries

This section provides a brief summary of curves in the Myller configuration $M(C,\bar{\xi},\pi)$. For more detailed information, we refer to [7].

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Let $(C,\bar{\xi})$ be a versor field and (C,π) be a plane field such that $\bar{\xi}\in\pi$ in E^3. Then, the pair $((C,\bar{\xi}),(C,\pi))$ is called *Myller configuration* in E^3 and is denoted by $M(C,\bar{\xi},\pi)$, or briefly M. Let $R=(O;\bar{i}_1,\bar{i}_2,\bar{i}_3)$ be an orthonormal frame. Then, $(C,\bar{\xi})$ can be expressed as follows

$$\bar{r}=\bar{r}(s),\ \bar{\xi}=\bar{\xi}(s),\ s\in I=(s_1,s_2),$$
$$\bar{r}(s)=x(s)\bar{i}_1+y(s)\bar{i}_2+z(s)\bar{i}_3=\overrightarrow{OP}(s),$$
$$\bar{\xi}(s)=\xi_1(s)\bar{i}_1+\xi_2(s)\bar{i}_2+\xi_3(s)\bar{i}_3=\overrightarrow{PQ},$$

where s is the arclength parameter of the curve C, $\bar{r}=\bar{r}(s)$ is the position vector of C and

$$\left\|\bar{\xi}(s)\right\|^2=\left\langle\bar{\xi}(s),\bar{\xi}(s)\right\rangle=1.$$

Defining $\bar{\xi}_1(s)=\bar{\xi}(s)$ and considering $\frac{d\bar{\xi}_1(s)}{ds}$, we define versor field $\bar{\xi}_2(s)$ as follows,

$$\frac{d\bar{\xi}_1(s)}{ds}=K_1(s)\bar{\xi}_2(s),$$

where $K_1(s)=\left\|\frac{d\bar{\xi}}{ds}\right\|$ is called *curvature (or K_1-curvature)* of $(C,\bar{\xi})$. Clearly, $\bar{\xi}_2(s)$ is orthogonal to $\bar{\xi}_1(s)$ and exists when $K_1(s)\neq 0$. If we define $\bar{\xi}_3(s)=\bar{\xi}_1(s)\times\bar{\xi}_2(s)$, the frame $R_F\left(P(s);\bar{\xi}_1(s),\bar{\xi}_2(s),\bar{\xi}_3(s)\right)$ is positively oriented and orthonormal, and is called the *invariant frame of Frenet-type* of the versor field $(C,\bar{\xi})$ [7].

The derivative formulas for R_F are

$$\frac{d\bar{r}(s)}{ds}=\bar{\alpha}(s)=a_1(s)\bar{\xi}_1(s)+a_2(s)\bar{\xi}_2(s)+a_3(s)\bar{\xi}_3(s),$$

with

$$\frac{d\bar{\xi}_1(s)}{ds}=K_1(s)\bar{\xi}_2(s),$$
$$\frac{d\bar{\xi}_2(s)}{ds}=-K_1(s)\bar{\xi}_1(s)+K_2(s)\bar{\xi}_3(s),$$

and

$$\frac{d\bar{\xi}_3(s)}{ds}=-K_2(s)\bar{\xi}_2(s),$$

where $a_1^2(s)+a_2^2(s)+a_3^2(s)=1$, $K_1(s)>0$ and

$$K_2(s)=-\left\langle\frac{d\bar{\xi}_3(s)}{ds},\bar{\xi}_2(s)\right\rangle$$

is called *torsion* (or K_2-torsion) of $(C,\bar{\xi})$. One can consider the geometrical interpretation of functions $K_1(s)$ and $K_2(s)$ as the same as the curvature and torsion of a curve in E^3 and the functions $a_1(s)$, $a_2(s)$, $a_3(s)$, $K_1(s)$, $K_2(s)$ $(s\in I)$ are invariants of the versor field $(C,\bar{\xi})$. Obviously, if $a_1(s)=1,\ a_2(s)=0$, and $a_3(s)=0$, one obtains the Frenet equations of a curve in E^3 [7].

The following theorem is the fundamental theorem for the versor field (C,ξ):

Theorem 1 ([7]) *If the functions $K_1(s)$, $K_2(s)$, $a_1(s)$, $a_2(s)$, $a_3(s)$ of class C^∞ are given a priori for $s\in[a,b]$ with*

$$a_1^2(s)+a_2^2(s)+a_3^2(s)=1,$$

then there exists a curve $C:[a,b]\to E^3$ with arclength parameter s and a versor field $\bar{\xi}(s)$, such that the functions $a_i(s)$, $i=1,2,3$, $K_1(s)$ and $K_2(s)$ are the invariants of $(C,\bar{\xi})$. Any two such versor fields $(C,\bar{\xi})$ differ by a proper Euclidean motion.

We define the $\bar{\xi}_i$, $i = 1, 2, 3$, helices as follows:

Definition 1 *Let the curve C with the invariant type Frenet frame $R_F\left(P(s); \bar{\xi}_1(s), \bar{\xi}_2(s), \bar{\xi}_3(s)\right)$ in M be a helix in E^3 with the unit axis $\bar{d}_\xi$, that is, $\left\langle \bar{\alpha}, \bar{d}_\xi \right\rangle$ is constant. Then, the curve C is called $\bar{\xi}_i$-helix if the versor field $\bar{\xi}_i$ makes a constant angle with the same fixed direction $\bar{d}_\xi$, where $i = 1, 2, 3$.*

Let $\bar{v}(s)$ be the unit normal to the oriented plane π. Define $\bar{\mu}(s) = \bar{v}(s) \times \bar{\xi}(s)$. Then the positively oriented orthonormal frame, denoted by $R_D\left(P(s); \bar{\xi}(s), \bar{\mu}(s), \bar{v}(s)\right)$, is called the *Darboux frame* of the curve C. This frame is geometrically associated with the Myller configuration M [7]. The following theorem gives the fundamental equations of M.

Theorem 2 ([7]) *The moving equations of Darboux frame for M are as follows*

$$\frac{d\bar{r}}{ds} = \bar{\alpha}(s) = c_1(s)\bar{\xi}(s) + c_2(s)\bar{\mu}(s) + c_3(s)\bar{v}(s);\ c_1^2 + c_2^2 + c_3^2 = 1,$$

$$\frac{d\bar{\xi}}{ds} = G(s)\bar{\mu}(s) + K(s)\bar{v}(s),$$

$$\frac{d\bar{\mu}}{ds} = -G(s)\bar{\xi}(s) + T(s)\bar{v}(s),$$

and

$$\frac{d\bar{v}}{ds} = -K(s)\bar{\xi}(s) - T(s)\bar{\mu}(s),$$

where $c_1(s),\ c_2(s),\ c_3(s);\ G(s),\ K(s)$ and $T(s)$ are invariants and are uniquely determined.

The functions $G(s),\ K(s)$ and $T(s)$ are called *the geodesic curvature, the normal curvature and the geodesic torsion* of the versor field $(C, \bar{\xi})$, respectively, in M.

The fundamental theorem for the Darboux frame R_D is stated as follows:

Theorem 3 ([7]) *Let C^∞-functions $c_1(s),\ c_2(s),\ c_3(s)$, with $c_1^2 + c_2^2 + c_3^2 = 1$, $G(s),\ K(s)$, and $T(s)$, for $s \in [a, b]$, be given a priori. Then, there exists a Myller configuration $M(C, \bar{\xi}, \pi)$ such that the given functions and parameter s are the invariants and arclength of curve C, respectively. Two such configurations differ by a proper Euclidean motion.*

The relations between the invariants of the field $(C, \bar{\xi})$ and the invariants of $(C, \bar{\xi})$ in M are given as follows

$$\frac{d\bar{r}}{ds} = \bar{\alpha}(s) = a_1\bar{\xi}_1 + a_2\bar{\xi}_2 + a_3\bar{\xi}_3 = c_1\bar{\xi} + c_2\bar{\mu} + c_3\bar{v},$$

$$\bar{\xi}(s) = \bar{\xi}_1(s),\ \bar{\xi}_2(s) = (\sin\psi)\bar{\mu}(s) + (\cos\psi)\bar{v}(s),$$

$$\bar{\xi}_3(s) = -(\cos\psi)\bar{\mu}(s) + (\sin\psi)\bar{v}(s),$$

$$c_1(s) = a_1(s),\ c_2(s) = (\sin\psi)a_1(s) - (\cos\psi)a_3(s),$$

$$c_3(s) = (\cos\psi)a_2(s) + (\sin\psi)a_3(s),$$

and,

$$G(s) = (\sin\psi)K_1(s),\ K(s) = (\cos\psi)K_1(s), T(s) = K_2(s) + \frac{d\psi}{ds}, \quad (1)$$

where $\psi = \measuredangle(\bar{\xi}_2, \bar{v})$ [7].

3 $\bar{\xi}$-helices in Myller Configuration $M(C, \bar{\xi}, \pi)$

In the Euclidean 3-space E^3, special curves for which their orthonormal frame vectors make a constant angle with some constant directions play an important role. Well-known examples of such curves, studied using the Frenet frame, are helices, slant helices, and Darboux helices [3, 11, 13]. Moreover, when the curve is a surface curve, meaning it lies entirely on a surface, one can also consider the Darboux frame $\{T(s), V(s), U(s)\}$ of the curve to study the same special conditions where the vector fields of the Darboux frame make a constant angle with some fixed directions. Here, $T(s)$ is the unit tangent of the curve, $U(s)$ is the unit surface normal along the curve and $V(s)$ is a unit vector defined by $V(s) = U(s) \times T(s)$. A surface curve is called *surface helix* (or, respectively, a *relatively normal slant helix* and an *isophote curve*) if the unit vector T (or, respectively, V and U) makes a constant angle with a fixed direction. In this case, surface helices, relatively normal slant helices and isophote curves are the well-known examples of such curves [2, 4, 9]. In this section, we consider the same conditions in the Myller configuration M and introduced $\bar{\xi}$-helices in M.

Definition 2 *Let C be a unit speed curve with Darboux frame $R_D\left(P(s); \bar{\xi}(s), \bar{\mu}(s), \bar{v}(s)\right)$ in the Myller configuration M. Let C be a helix in E^3 with a unit axis $\bar{d}_\xi$, such that $\left\langle \bar{\alpha}, \bar{d}_\xi \right\rangle$ is constant. The curve C is called a $\bar{\xi}$-helix in M if the versor field $\bar{\xi}$ makes a constant angle with the same fixed unit direction $\bar{d}_\xi$, i.e., there exists a constant angle θ such that $\left\langle \bar{\xi}, \bar{d}_\xi \right\rangle = \cos\theta$.*

Theorem 4 *The curve C with Darboux frame R_D and $(G, K) \neq (0, 0)$ in M is a $\bar{\xi}$-helix iff the following functions are constant*

$$\begin{cases} \sigma_\xi = \cot\theta = \pm\frac{K^2\left(\frac{G}{K}\right)' - (G^2+K^2)T}{(G^2+K^2)^{\frac{3}{2}}}, \\ \lambda_\xi = c_1\cos\theta \mp c_2\sin\theta\frac{K}{(G^2+K^2)^{\frac{1}{2}}} \pm c_3\sin\theta\frac{G}{(G^2+K^2)^{\frac{1}{2}}}. \end{cases} \tag{2}$$

Proof. From Definition 2, there exist a unit constant vector $\bar{d}_\xi$ and a constant function θ such that $\left\langle \bar{\xi}, \bar{d}_\xi \right\rangle = \cos\theta$. The unit vector $\bar{d}_\xi$ can be given in the form

$$\bar{d}_\xi = (\cos\theta)\bar{\xi} + x_2\bar{\mu} + x_3\bar{v}, \tag{3}$$

where $x_2 = x_2(s)$ and $x_3 = x_3(s)$ are smooth functions of s. Differentiating (3) with respect to s gives

$$\frac{d\bar{d}_\xi}{ds} = (-x_2G - x_3K)\,\bar{\xi} + (G\cos\theta + x_2' - x_3T)\,\bar{\mu} + (K\cos\theta + x_2T + x_3')\,\bar{v}.$$

Since $\bar{d}_\xi$ is constant, we have the system

$$\begin{cases} -x_2G - x_3K = 0, \\ G\cos\theta + x_2' - x_3T = 0, \\ K\cos\theta + x_2T + x_3' = 0. \end{cases} \tag{4}$$

From the first equation in (4), it follows $x_3 = -x_2\frac{G}{K}$. Writing that in the second and third equations in (4) gives the following differential equation

$$x_2'\left(1 + \left(\frac{G}{K}\right)^2\right) + x_2\frac{G}{K}\left(\frac{G}{K}\right)' = 0. \tag{5}$$

The solution of (5) is $x_2 = \rho\frac{K}{\sqrt{G^2+K^2}}$, where ρ is integration constant. Hence, $x_3 = -\rho\frac{G}{\sqrt{G^2+K^2}}$. Since $\left\|\bar{d}_\xi\right\| = 1$, from (3) we have $\rho = \mp\sin\theta$. Then, (3) becomes

$$\bar{d}_\xi = (\cos\theta)\bar{\xi} \mp \sin\theta\frac{K}{\sqrt{G^2+K^2}}\bar{\mu} \pm \sin\theta\frac{G}{\sqrt{G^2+K^2}}\bar{v}. \tag{6}$$

By differentiating $\left\langle\bar{\xi}, \bar{d}_\xi\right\rangle = \cos\theta$ two times, it follows

$$-\cos\theta\left(G^2+K^2\right) \mp \left[\frac{(G'K-GK') - \left(G^2+K^2\right)T}{\sqrt{G^2+K^2}}\right]\sin\theta = 0,$$

which gives

$$\sigma_\xi = \cot\theta = \mp\frac{K^2\left(\frac{G}{K}\right)' - \left(G^2+K^2\right)T}{\left(G^2+K^2\right)^{\frac{3}{2}}}. \tag{7}$$

It follows that σ_ξ is constant. Furthermore, using that $\left\langle\bar{\alpha}, \bar{d}_\xi\right\rangle = const.$, from (6), we have that λ_ξ is constant.

Conversely, let the functions σ_ξ and λ_ξ given in (2) be constants. Let $\bar{d}_\xi$ be a unit vector defined by

$$\bar{d}_\xi = \cos\theta\bar{\xi} \mp \sin\theta\frac{K}{\sqrt{G^2+K^2}}\bar{\mu} \pm \sin\theta\frac{G}{\sqrt{G^2+K^2}}\bar{v},$$

where θ is constant. Differentiating last equality one obtains

$$\begin{aligned}\frac{d\bar{d}_\xi}{ds} &= \left(\mp G\sin\theta\left[\frac{-(G'K-GK') + \left(G^2+K^2\right)T}{\left(G^2+K^2\right)^{\frac{3}{2}}}\right] + G\cos\theta\right)\bar{\mu} \\ &+ \left(\mp K\sin\theta\left[\frac{\left(G^2+K^2\right)T - (G'K-GK')}{\left(G^2+K^2\right)^{\frac{3}{2}}}\right] + K\cos\theta\right)\bar{v}.\end{aligned}$$

Now, writing (2) in this result, we have $\frac{d\bar{d}_\xi}{ds} = 0$, i.e., $\bar{d}_\xi$ is a constant vector field and $\left\langle\bar{\xi}, \bar{d}_\xi\right\rangle$ is constant. Moreover, we have $\left\langle\bar{\alpha}, \bar{d}_\xi\right\rangle = \lambda_\xi$. Thus, we obtain that C is a $\bar{\xi}$-helix in Myller configuration M. ■

From Theorem 4, the following corollaries are obtained:

Corollary 1 *The axis of $\bar{\xi}$-helix C in the Myller configuration M is given by*

$$\bar{d}_\xi = (\cos\theta)\bar{\xi} \mp \sin\theta\frac{K}{\sqrt{G^2+K^2}}\bar{\mu} \pm \sin\theta\frac{G}{\sqrt{G^2+K^2}}\bar{v},$$

where θ is the constant angle defined by $\left\langle\bar{\xi}, \bar{d}_\xi\right\rangle = \cos\theta$.

Corollary 2 *i) The curve C with $K = 0$ in the Myller configuration M is $\bar{\xi}$-helix iff $\sigma_\xi = \pm\frac{T}{G}$ and $\lambda_\xi = c_1\cos\theta \pm c_3\sin\theta$ are constants.*

ii) The curve C with $G = 0$ in the Myller configuration M is $\bar{\xi}$-helix iff $\sigma_\xi = \pm\frac{T}{K}$ and $\lambda_\xi = c_1\cos\theta \mp c_2\sin\theta$ are constants.

iii) The curve C with $T = 0$ in the Myller configuration M is $\bar{\xi}$-helix iff $\sigma_\xi = \mp\frac{G'K-GK'}{\left(G^2+K^2\right)^{\frac{3}{2}}}$ and λ_ξ given in (2) are constants.

Theorem 5 *The curve C in M is a $\bar{\xi}_1$-helix according to the Frenet-type frame R_F iff $\frac{K_2}{K_1}$ and $a_1 \cos\theta \pm a_3 \sin\theta \mp (a_1 - a_2)\sin\theta \sin\psi \cos\psi$ are constants.*

Proof. *By taking into account the relations between the invariants of the field $(C, \bar{\xi})$ and the invariants of $(C, \bar{\xi})$ in M, we have $\bar{\xi} = \bar{\xi}_1$. Thus, it is clear that C is a $\bar{\xi}$-helix if and only if it is a $\bar{\xi}_1$-helix. Now, substituting (1) into (2), it follows that $\sigma_\xi = \pm\frac{K_2}{K_1}$, $\lambda_\xi = a_1 \cos\theta \pm a_3 \sin\theta \mp (a_1 - a_2)\sin\theta \sin\psi \cos\psi$, which concludes the proof.* ■

Moreover, from Corollary 2 and Theorem 5, we obtain the following corollary.

Corollary 3 *The axis of a $\bar{\xi}_1$-helix C according to Frenet-type frame R_F is given by $\bar{d}_{\xi_1} = (\cos\theta)\bar{\xi}_1 \pm (\sin\theta)\bar{\xi}_3$, where θ is constant.*

4 W_n-Darboux Helices in the Myller Configuration $M(C, \bar{\xi}, \pi)$

Darboux vector, which is the angular velocity vector of the Frenet-Serret frame of a space curve, is an important tool for studying the differential geometry of space curves. This vector is directly proportional to angular momentum, which is why it is also called angular momentum vector [10]. If we consider another frame along a curve that is different from the Frenet frame, new forms of the Darboux vector can be defined. In this section, we define the Darboux vector of the Darboux frame R_D of a curve C in a Myller configuration M. Furthermore, we define some new forms of the Darboux vector and introduce some special Darboux helices in the Myller configuration M.

Definition 3 *Let C be a curve with Darboux frame $R_D\left(P(s); \bar{\xi}(s), \bar{\mu}(s), \bar{v}(s)\right)$ in the Myller configuration M. The vector $W = T\bar{\xi} - K\bar{\mu} + G\bar{v}$ is called Darboux vector field of the Darboux frame R_D . The vectors*

$$W_n = -K\bar{\mu} + G\bar{v}, \quad W_r = T\bar{\xi} + G\bar{v} \quad \text{and} \quad W_o = T\bar{\xi} - K\bar{\mu},$$

are called the normal-type Darboux vector (or ND-vector), the rectifying-type Darboux vector (or RD-vector) and the osculating-type Darboux vector (or OD-vector) of R_D, respectively.

Considering versor field $(C, \bar{W}_n)$ with $\bar{W}_n = \frac{W_n}{\|W_n\|}$, we can give the following:

Definition 4 *Let C be a curve with unit ND-vector field $\bar{W}_n$ in the Myller configuration M. Let C also be a helix in E^3 with a unit axis $\bar{l}_n$, such that $\left\langle \bar{\alpha}, \bar{l}_n \right\rangle$ is constant. The curve C is called W_n-helix in M if $\bar{W}_n$ makes a constant angle with the same fixed unit direction $\bar{l}_n$.*

Theorem 6 *The curve C with a unit ND-vector field $\bar{W}_n$ and $(G, K) \neq (0, 0)$ in the Myller configuration M is a W_n-helix iff C is a $\bar{\xi}$-helix in M.*

Proof. From Definition 4, we have $\left\langle \bar{W}_n, \bar{l}_n \right\rangle = \cos\varphi$, where φ is the constant angle between unit versor fields $\bar{W}_n$and $\bar{l}_n$. Differentiating, one obtains

$$\frac{K^2\left(\frac{G}{K}\right)' - \left(G^2 + K^2\right)T}{\left(G^2 + K^2\right)^{\frac{3}{2}}} \left\langle G\bar{\mu} + K\bar{v}, \bar{l}_n \right\rangle = 0.$$

Since $\bar{\xi}' = G\bar{\mu} + K\bar{v}$, the last equality becomes $\sigma_\xi \left\langle \bar{\xi}', \bar{l}_n \right\rangle = 0$. If we assume that $\sigma_\xi = 0$, then σ_ξ is constant. Let us investigate the case $\sigma_\xi \neq 0$. In this case, $\left\langle \bar{\xi}', \bar{l}_n \right\rangle = 0$. On the other hand, by taking into account

$\left\langle \bar{W}_n, \bar{\xi}' \right\rangle = 0$, we have $\bar{\xi}' = \bar{W}_n \times \bar{l}_n$. Then, it follows that the vectors $\bar{l}_n$, $\bar{W}_n$ and $\bar{\xi}$ lie on the same plane, i.e., $\bar{l}_n \in sp\left\{\bar{W}_n, \bar{\xi}\right\}$. So, we can write $\bar{l}_n = \pm(\sin\varphi)\bar{\xi} + (\cos\varphi)\bar{W}_n$ or, equivalently,

$$\bar{l}_n = \pm(\sin\varphi)\bar{\xi} + \cos\varphi\left(\frac{-K}{\sqrt{G^2+K^2}}\mu + \frac{G}{\sqrt{G^2+K^2}}\bar{\upsilon}\right).$$

Differentiating $\left\langle \bar{\xi}', \bar{l}_n \right\rangle = 0$ yields

$$\mp\sin\varphi\left(G^2+K^2\right) + \cos\varphi\left(\frac{-\left(G'K - GK'\right) + \left(G^2+K^2\right)T}{\left(G^2+K^2\right)^{\frac{3}{2}}}\right) = 0.$$

Hence, it follows that

$$\cot\varphi = \mp\frac{1}{\frac{K^2\left(\frac{G}{K}\right)' - (G^2+K^2)T}{(G^2+K^2)^{\frac{3}{2}}}},$$

or, equivalently, $\cot\varphi = \mp\frac{1}{\sigma_\xi}$. Furthermore, we have

$$\left\langle \bar{\alpha}, \bar{l}_n \right\rangle = \pm c_1(\sin\varphi) + c_2\cos\varphi\frac{-K}{\sqrt{G^2+K^2}} + c_3\cos\varphi\frac{G}{\sqrt{G^2+K^2}}.$$

Since $\theta + \varphi = \pi/2$, we see that $\left\langle \bar{\alpha}, \bar{l}_n \right\rangle = \pm\lambda_\xi$, which completes the proof. ■

From Theorem 6, the following corollaries are given.

Corollary 4 *The curve C in M is a $\bar{\xi}_1$-helix according to Frenet-type frame R_F iff it is a W_n-helix.*

Corollary 5 *The axis $\bar{l}_n$ of W_n-helix in M is defined by*

$$\bar{l}_n = \pm(\sin\varphi)\bar{\xi} + \cos\varphi\left(\frac{-K}{\sqrt{G^2+K^2}}\bar{\mu} + \frac{G}{\sqrt{G^2+K^2}}\bar{\upsilon}\right),$$

where φ is the constant angle given by $\left\langle \bar{W}_n, \bar{l}_n \right\rangle = \cos\varphi$.

Example 1 *Let consider the versor fields*

$$\bar{\xi} = \frac{1}{\sqrt{2}}\left(1, \cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}\right),\ \bar{\mu} = \frac{1}{\sqrt{2}}\left(1, -\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}\right),\ \bar{\upsilon} = \left(0, -\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}\right),$$

and invariants $K = \frac{1}{2}$, $G = 0$ and $T = -\frac{1}{2}$. We have that $\sigma_\xi = \cot\theta = \pm1$, that is, $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$. By choosing $\theta = \frac{\pi}{4}$, $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{3}$, we get

$$c_3 = \frac{\sqrt{23}}{6} \quad \text{and} \quad \lambda_\xi = \frac{\sqrt{2}}{2}(c_1 + c_2) = \frac{5\sqrt{2}}{12},$$

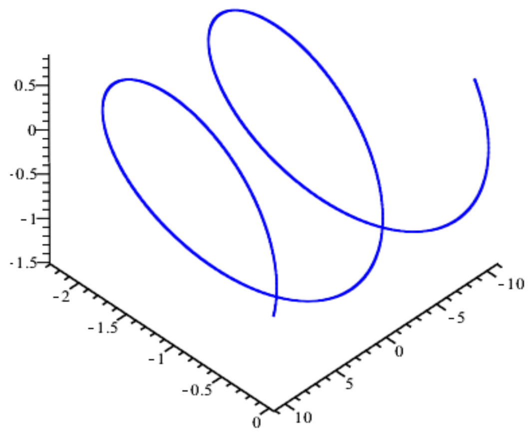
i.e., σ_ξ and λ_ξ are constants. Thus, the curve C_1 given by the parametrization

$$r(s) = \left(\frac{5s}{6\sqrt{2}}, \frac{1}{3}\sin\frac{\sqrt{2}s}{2} + \frac{\sqrt{46}}{6}\cos\frac{\sqrt{2}s}{2}, -\frac{1}{3}\cos\frac{\sqrt{2}s}{2} - \frac{\sqrt{46}}{6}\sin\frac{\sqrt{2}s}{2}\right)$$

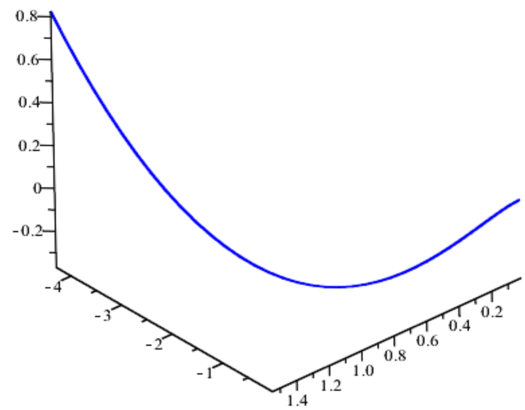
is a $\bar{\xi}$-helix (and, also W_n-helix) in M. (Figure 1(a)). If we choose $\theta = \frac{\pi}{4}$, $c_1 = -\sin\frac{s}{\sqrt{2}}$ and $c_2 = 1 + \sin\frac{s}{\sqrt{2}}$, we see that

$$c_3 = \sqrt{-2\sin\frac{s}{\sqrt{2}}\left(1 + \sin\frac{s}{\sqrt{2}}\right)} \quad \text{and} \quad \lambda_\xi = \frac{\sqrt{2}}{2}(c_1 + c_2) = \frac{\sqrt{2}}{2},$$

i.e., σ_ξ and λ_ξ are constants. Using the Maple program, we obtain the $\bar{\xi}$-helix C_2 in M shown in Figure 1(b).



(a) $\bar{\xi}$-helix C_1



(b) $\bar{\xi}$-helix C_2

Figure 1: $\bar{\xi}$ helices C_1 and C_2 with curvatures $K = \frac{1}{2}$, $G = 0$, and $T = -\frac{1}{2}$.

5 $\bar{\mu}$-Helices and $\bar{\upsilon}$-Helices in the Myller Configuration $M(C, \bar{\xi}, \pi)$

In this section, we introduce $\bar{\mu}$-helices, $\bar{\upsilon}$-helices, W_r-Darboux helices, and W_o-Darboux helices in the Myller configuration M. The proofs of theorems presented below can be carried out in a manner similar to those presented in previous sections.

Definition 5 *Let C be a unit speed curve with Darboux frame $R_D\left(P(s); \bar{\xi}(s), \bar{\mu}(s), \bar{\upsilon}(s)\right)$ in the Myller configuration M. Let C also be a helix in E^3 with a unit axis $\bar{d}_\mu$, such that $\left\langle \bar{\alpha}, \bar{d}_\mu \right\rangle$ is constant. The curve C is called a $\bar{\mu}$-helix in M if the versor field $\bar{\mu}$ makes a constant angle with the same fixed unit direction $\bar{d}_\mu$, i.e., there exists a constant angle η such that $\left\langle \bar{\mu}, \bar{d}_\mu \right\rangle = \cos\eta$.*

Theorem 7 *The curve C with Darboux frame R_D and $(G, T) \neq (0, 0)$ in M is a $\bar{\mu}$-helix iff the following functions are constant*

$$\sigma_\mu = \mp \frac{G^2 \left(\frac{T}{G}\right)' - \left(G^2 + T^2\right) K}{\left(G^2 + T^2\right)^{\frac{3}{2}}}$$

and

$$\lambda_\mu = \mp c_1 \sin\eta \frac{T}{\sqrt{G^2 + T^2}} + c_2 \cos\eta \mp c_3 \sin\eta \frac{G}{\sqrt{G^2 + T^2}}.$$

Corollary 6 *The axis of a $\bar{\mu}$-helix C in M is given by*

$$\bar{d}_\mu = \mp \sin\eta \frac{T}{\sqrt{G^2 + T^2}} \bar{\xi} + (\cos\eta)\bar{\mu} \mp \sin\eta \frac{G}{\sqrt{G^2 + T^2}} \bar{\upsilon},$$

where η is the constant angle defined by $\left\langle \bar{\mu}, \bar{d}_\mu \right\rangle = \cos\eta$.

Corollary 7 *i) The curve C with $K = 0$ in M is $\bar{\mu}$-helix iff $\sigma_\mu = \mp \frac{T'G - TG'}{(G^2+T^2)^{\frac{3}{2}}}$ and*

$$\lambda_\mu = \mp c_1 \sin\eta \frac{T}{\sqrt{G^2 + T^2}} + c_2 \cos\eta \mp c_3 \sin\eta \frac{G}{\sqrt{G^2 + T^2}}$$

are constants.

ii) *The curve C with $G=0$ in M is $\bar{\mu}$-helix iff $\sigma_\mu = \pm\frac{K}{T}$ and $\lambda_\mu = \mp c_1 \sin\eta + c_2\cos\eta$ are constants.*

iii) *The curve C with $T=0$ in M is $\bar{\mu}$-helix iff $\sigma_\mu = \pm\frac{K}{G}$ and $\lambda_\mu = c_2\cos\eta \mp c_3\sin\eta$ are constants.*

Considering versor field $(C, \bar{W}_r)$ with $\bar{W}_r = \frac{W_r}{\|W_r\|}$, we can give the following:

Definition 6 *Let C be a curve with unit RD-vector field $\bar{W}_r$ in the Myller configuration M. Let C also be a helix in E^3 with a unit axis $\bar{l}_r$, such that $\left\langle \bar{\alpha}, \bar{l}_r \right\rangle$ is constant. The curve C is called a W_r-helix in M if $\bar{W}_r$ makes a constant angle with the same fixed unit direction $\bar{l}_r$.*

Theorem 8 *The curve C with unit RD-vector field $\bar{W}_r$ and $(G,T) \neq (0,0)$ in M is a W_r-helix iff C is a $\bar{\mu}$-helix in M.*

Corollary 8 *The axis $\bar{l}_r$ of W_r-helix in M is defined by*

$$\bar{l}_r = \cos\vartheta \frac{T}{\sqrt{G^2+T^2}}\bar{\xi} \mp (\sin\vartheta)\bar{\mu} + \cos\vartheta\frac{G}{\sqrt{G^2+T^2}}\bar{v},$$

where ϑ is the constant angle defined by $\left\langle \bar{W}_r, \bar{l}_r \right\rangle = \cos\vartheta$.

Definition 7 *Let C be a unit speed curve with Darboux frame R_D in the Myller configuration M. Let C also be a helix in E^3 with a unit axis $\bar{d}_v$, such that $\left\langle \bar{\alpha}, \bar{d}_v \right\rangle$ is constant. The curve C is called a $\bar{v}$-helix in M if the versor field $\bar{v}$ makes a constant angle with the same fixed unit direction $\bar{d}_v$, i.e. there exists a constant angle ω such that $\left\langle \bar{v}, \bar{d}_v \right\rangle = \cos\omega$.*

Theorem 9 *The curve C with Darboux frame R_D and $(T,K) \neq (0,0)$ in M is a $\bar{v}$-helix iff the following functions are constant*

$$\sigma_v = \cot\omega = \mp\frac{K^2\left(\frac{T}{K}\right)' + \left(T^2+K^2\right)G}{\left(T^2+K^2\right)^{\frac{3}{2}}}$$

and

$$\lambda_v = \mp c_1 \sin\omega \frac{T}{\sqrt{T^2+K^2}} \pm c_2\sin\omega\frac{K}{\sqrt{T^2+K^2}} + c_3\cos\omega.$$

Corollary 9 *The axis of a $\bar{v}$-helix C in M is given by*

$$\bar{d}_v = \mp\sin\omega\frac{T}{\sqrt{T^2+K^2}}\bar{\xi} \pm \sin\omega\frac{K}{\sqrt{T^2+K^2}}\bar{\mu} + (\cos\omega)\bar{v},$$

where ω is the constant angle defined by $\left\langle \bar{v}, \bar{d}_v \right\rangle = \cos\omega$.

Corollary 10 *i)* *The curve C with $K=0$ in M is $\bar{v}$-helix iff $\sigma_v = \mp\frac{G}{T}$ and $\lambda_v = \mp c_1\sin\omega + c_3\cos\omega$ are constants.*

ii) *The curve C with $G=0$ in M is $\bar{v}$-helix iff*

$$\sigma_v = \mp\frac{T'K - TK'}{\left(T^2+K^2\right)^{\frac{3}{2}}} \quad \text{and} \quad \lambda_v = \mp c_1\sin\omega\frac{T}{\sqrt{T^2+K^2}} \pm c_2\sin\omega\frac{K}{\sqrt{T^2+K^2}} + c_3\cos\omega$$

are constants.

iii) The curve C with $T=0$ in M is $\bar{v}$-helix iff $\sigma_v = \mp\frac{G}{K}$ and $\lambda_v = \pm c_2 \sin\omega + c_3\cos\omega$ are constants.

Considering versor field $(C, \bar{W}_o)$ with $\bar{W}_o = \frac{W_o}{\|W_o\|}$, we can give the following:

Definition 8 *Let C be a curve with unit OD-vector field $\bar{W}_o$ in the Myller configuration M. Let C also be a helix in E^3 with a unit axis $\bar{l}_o$, such that $\left\langle \bar{\alpha}, \bar{l}_o \right\rangle$ is constant. The curve C is called W_o-helix in M if $\bar{W}_o$ makes a constant angle with the same fixed unit direction $\bar{l}_o$.*

Theorem 10 *The curve C with unit OD-vector field $\bar{W}_o$ and $(K,T) \neq (0,0)$ in M is a W_o-helix iff C is a $\bar{v}$-helix in M.*

Corollary 11 *The axis $\bar{l}_o$ of W_o-helix in M is defined by*

$$\bar{l}_o = \cos\varepsilon \frac{T}{\sqrt{T^2+K^2}}\bar{\xi} - \cos\varepsilon\frac{K}{\sqrt{T^2+K^2}}\bar{\mu} \mp (\sin\varepsilon)\bar{v},$$

where ε is the constant angle defined by $\left\langle \bar{W}_o, \bar{l}_o \right\rangle = \cos\varepsilon$.

Example 2 *Let consider the versor fields*

$$\bar{\xi} = \left(\frac{3}{4}\sin\frac{s}{2} + \frac{1}{4}\sin\frac{3s}{2}, -\frac{3}{4}\cos\frac{s}{2} - \frac{1}{4}\cos\frac{3s}{2}, -\frac{\sqrt{3}}{2}\sin\frac{s}{2}\right),$$

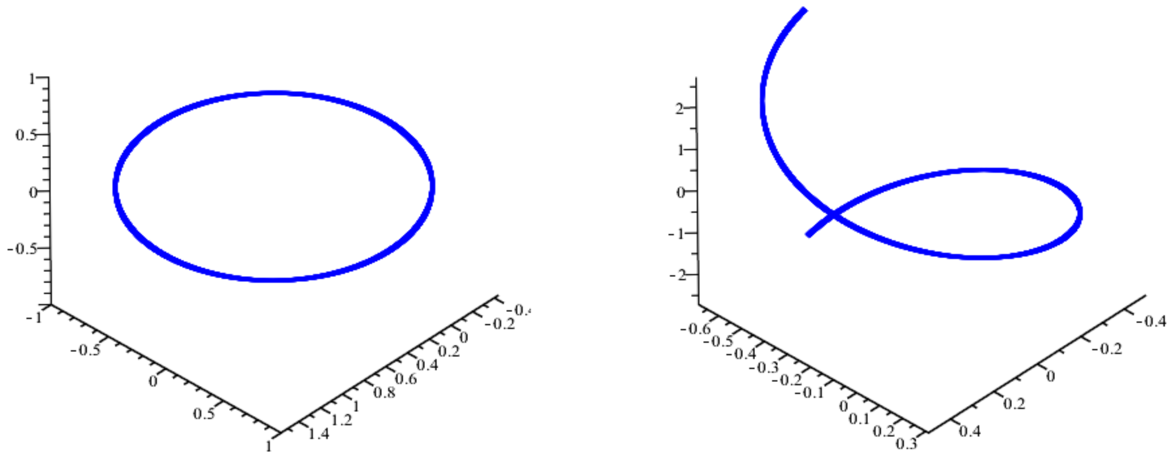
$$\bar{\mu} = \left(\frac{1}{2}\cos\frac{s}{2}\left(2\cos^2\frac{s}{2} - 3\right), -\sin^3\frac{s}{2}, -\frac{\sqrt{3}}{2}\cos\frac{s}{2}\right),$$

$$\bar{v} = \left(\left(\frac{-\sqrt{3}}{4\cos(s/2)}\right)\left(\cos\frac{s}{2} + \cos\frac{3s}{2}\right), \left(\frac{-\sqrt{3}}{4\cos(s/2)}\right)\left(\sin\frac{s}{2} + \sin\frac{3s}{2}\right), \frac{1}{2}\right),$$

and invariants $K = -\frac{\sqrt{3}}{2}\cos\frac{s}{2}$, $G=0$ and $T = \frac{\sqrt{3}}{2}\sin\frac{s}{2}$. We have that $\sigma_v = \cot\omega = \mp\frac{\sqrt{3}}{3}$, that is, $\omega = \frac{2\pi}{3}$ or $\omega = \frac{\pi}{3}$. By choosing $\omega = \frac{2\pi}{3}$, $c_1 = -\cos\frac{s}{2}$ and $c_2 = \sin\frac{s}{2}$, we get $c_3 = 0$ and $\lambda_v = 0$, i.e., σ_v and λ_v are constants. Thus, the curve C_3 given by the parametrization $r(s) = \left(2\cos^2\frac{s}{2} - \frac{7}{16}, 2\cos\frac{s}{2}\sin\frac{s}{2}, 0\right)$ is a $\bar{v}$-helix (and, also W_0-helix) in M. (Figure 2(a)). If we choose $\omega = \frac{\pi}{3}$, $c_1 = \sin\frac{s}{2}$ and $c_2 = \cos\frac{s}{2}$, we see that $c_3 = 0$ and $\lambda_v = \frac{\sqrt{3}}{2}$, i.e., σ_v and λ_v are constants. Using the Maple program, we obtain the $\bar{v}$-helix C_4 in M shown in Figure 2(b).

6 Conclusions

Some new types of special curves in the Myller configuration $M(C, \bar{\xi}, \pi)$ are defined and studied. The definitions and characterizations of $\bar{\xi}$-helices, $\bar{\mu}$-helices, $\bar{v}$-helices, W_k-Darboux helices ($k \in \{n, r, o\}$), and $\bar{\xi}_1$-helices are introduced and presented. The axes of these helices are determined, and the relations between these special curves are also established.



(a) $\bar{\xi}$-helix C_3 (b) $\bar{\xi}$-helix C_4

Figure 2: $\bar{v}$ helices C_3 and C_4 with curvatures $K = -\frac{\sqrt{3}}{2}\cos\frac{s}{2},\ G = 0$ and $T = \frac{\sqrt{3}}{2}\sin\frac{s}{2}$.

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